KINEMATICS OF MACHINERY

BY

C. D. ALBERT, M.E.

Professor of Machine Design, Emeritus, Cornell University
Member of the American Society of
Mechanical Engineers

AND

F. S. ROGERS, B.S., M.E.

Professor of Machine Design, Cornell University
Member of the American Society of
Mechanical Engineers

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PREFACE

THIS book is the outgrowth of the experience of the authors in teaching kinematics to engineering students in the College of Engineering, Cornell University. Although the book was written primarily for a recitation course of forty-five periods, more subject matter than required was intentionally introduced. The purpose of this was to furnish the additional kinematics needed for a course in Advanced Kinematics and Kinetics. In further justification, it was felt that a work on kinematics should, in these days of high speed machinery, emphasize certain aspects of cam and gear design and should include a general method of determining the linear velocity and acceleration of any point in a mechanism and the angular velocity and acceleration of any link.

The principal omissions suggested for a forty-five period recitation course are: articles 72 to 83, inclusive; 96 to 99 inclusive; 105; 132 to 135, inclusive; 144; 146 to 148, inclusive; and 151. To these omissions may be added, as may appear expedient, some or all of the following articles: 63; 105; 113; 118; 169; 177; 181 to 184, inclusive; 198; and 199. These omissions, and others if necessary, can be made without interrupting the continuity of treatment of those topics usually associated with undergraduate courses in kinematics. It is hoped that the necessary omissions for such courses will dispel the notion that more kinematics than necessary is required of undergraduates, and it is hoped also that, noting the omissions, more students will be encouraged to take advanced work in the subject.

The book presupposes a course in engineering mechanics. Since kinematics is primarily an application and graphical extension of the laws of motion of a particle, a valuable review of these fundamental laws from mechanics is given in the first chapter. An attempt has been made throughout to treat each topic thoroughly and at the same time concisely. To this end considerable attention has been given to the form of mathematical developments, to the convenience of the notation used, and to the suitability of the illustrations, many of which are to scale.

A number of questions and problems which completely cover the subject matter of the text are given at the end of the book. The system of numbering is such that each question or problem is readily

associated with the particular article or part thereof to which it relates. The purpose of these questions and problems is to assist the student in preparing each assignment and in reviewing for examinations and also to facilitate the conduct of the recitations and the making out of examinations.

Particular thanks are due Mr. Allen H. Candee of the Gleason Works, Rochester, New York, who gave generously of his time and expert knowledge in critically reviewing the manuscript of the four chapters on gears. As a result of his helpful criticism these chapters were quite extensively revised, and his many valuable suggestions were incorporated.

In the preparation of the manuscript, catalogues, bulletins, magazine articles, and books on kinematics and related subjects were freely consulted. Direct use of any information from such sources is duly acknowledged in the text. A list of general references is given at the close of the book.

One or two half-tone illustrations were kindly furnished by each of the following firms: Jones and Lamson Machine Company, Springfield, Vermont; Newton Machine Tool Works, Philadelphia, Pennsylvania; W. A. Jones Foundry and Machine Company, Chicago, Illinois; Fellows Gear Shaper Company, Springfield, Vermont; and the Gleason Works, Rochester, New York.

C. D. ALBERT

ITHACA, N. Y.   F. S. ROGERS

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NOTATION

Numbers *1*, *2*, *3*, etc., have been used throughout the book to designate the members of a mechanism, the fixed member being marked *O*, or zero. Numbers also have been used to designate successive positions of a point, line, or member. Capital letters, *P*, *B*, *Q*, etc., have been used to designate definite points in a mechanism. Small letters have been used to designate construction points and the termini of vectors in velocity vector diagrams and, when primed to designate the termini of vectors in acceleration vector diagrams.

A Linear acceleration.

A_{BC} Linear acceleration of B relative to C.

A^n_{DG} Normal component of linear acceleration of D relative to G.

A^t_{DG} Tangential component of linear acceleration of D relative to G.

A^x_{BC} X component of linear acceleration of B relative to C.

A^y_{BC} Y component of linear acceleration of B relative to C.

B Radius of base circle of gears or width of belt.

D Diameter of pitch circle of gears and sprockets or diameter of pulleys, etc.

E External radius or modulus of elasticity.

N Number of instant centers, normal force, number of teeth, **or** revolutions per minute.

O Fixed member of mechanism.

O_{13} Instant center of member *1* relative to *3*, or of *3* relative to *1*.

P Point, especially contact point of curves and pitch point of gears.

R Radius of pitch circle of gears and sprockets or radius of pulleys, etc.

V Linear velocity.

V_{BC} Linear velocity of B relative to C.

V^n_{DG} Normal component of linear velocity of D relative to G.

V^t_{DG} Tangential component of linear velocity of D relative to G.

V^x_{BC} X component of linear velocity of B relative to C.

V^y_{BC} Y component of linear velocity of B relative to C.

a Addendum of gear teeth.

b Backlash of gear teeth or width of face of gear.

c Clearance of gear teeth.

d Dedendum of gear teeth.

f Cam factor or feed of cutting tool.

g Acceleration of gravity.

h Total displacement of cam follower or whole depth of gear teeth.

k Ratio or product.

k_s Space scale.

k_t Time scale.

k_v Velocity scale.

k_a Acceleration scale.

l Length or lead of screw or gear.

n Ratio, number of links, speed, or number of threads per inch.

p Pitch of chain.

p_c Circular pitch; p_c', diametral pitch.

p_b Base pitch.

p_n Normal pitch.

p_a Axial pitch.

r Radius.

s Linear displacement.

t Time, tooth thickness, or belt thickness.

α Angular acceleration, angle of obliquity, pressure angle, etc.

β Angle of action or other angle.

δ Angle.

θ Angular displacement, angle between shafts, or other angle.

λ Lead angle.

μ Coefficient of friction.

ρ Radius of curvature or radius of gyration.

ϕ Helix angle or other angle.

ω Angular velocity.

ω_{13} Angular velocity of member 1 relative to 3.

$\twoheadrightarrow$ Vector addition.

$\rightarrow$ Vector subtraction.

KINEMATICS OF MACHINERY

CHAPTER I

FUNDAMENTAL CONCEPTIONS

1. Kinematics. Kinematics may be defined as that branch of mechanics which treats of the relative motions of bodies. It is sometimes called the geometry of motion. *Kinematics of Machinery* treats of the relative motions of the parts of mechanisms and machines. The fact that the motions of the parts of mechanisms and machines can be treated quite apart from the forces acting and quite apart also from the exact form and weight of the parts, has made the development of kinematics as a separate subject possible, and frees its application from the many complex considerations and decisions necessary to the complete design of any mechanism or machine. The possibilities and advantages of such a separation are quite apparent and of great importance.

2. Place of Kinematics in the Design of a Machine. The design of any machine may be considered in four steps:

(a) Determination of the kinematic scheme;
(b) Determination of the forces acting on each part;
(c) Proportioning the various parts;
(d) Specification and drawing.

(a) *Determination of the Kinematic Scheme.* By this is meant the determination of such a combination of kinematic elements as will transmit and transform the motions as required by the source of energy and of motion and by the useful work to be done. This may involve merely the selection of some known mechanism or combination of mechanisms, or the solution may require the invention of a new mechanism or combination. A knowledge of kinematics and of how the various kinematic elements such as shafts, bearings, pulleys, cams, gears, links, etc., have been used in combination, is of great assistance in arriving at a solution. Having found a suitable or promising combination, a complete analysis of the motions of the various parts can be made. As will be shown, graphical rather than analytical methods

are usually employed. By means of skeleton figures and diagrams, the displacements, velocities, and accelerations of the moving parts or points on the same can be determined.

(b) *Determination of the Forces Acting on Each Part.* The magnitude and direction of the forces acting on any part of a machine depend not only on the energy supplied and the useful work done, but also upon the frictional resistances and the inertia of the moving parts. As a rule the frictional resistances are small and may often be neglected. The effect of the inertia of the moving parts depends upon the magnitudes of the linear and angular accelerations of the parts and upon their form, size, and weight, or material. The linear and angular accelerations are known from the analysis of the kinematic scheme. The form, size, and material for each part must, therefore, be assigned before the inertia forces may be calculated, and the magnitude and direction of the resulting forces acting on each part determined. Hence, for high-speed machinery, the proportions for each part must be assigned before the data become available for determining the resulting forces acting on each part and before the degree of unbalance of the machine can be determined. Where, as in some slow-speed machinery, the inertia forces may be neglected, it is not necessary to have the form, size, and material of the parts to approximate the magnitude and direction of the forces acting on each part. This information may be obtained from the velocities of the parts and the energy supplied or useful work done.

(c) *Proportioning the Various Parts.* The size and form of each part are fixed by the function or functions of the part and the magnitude and direction of the forces acting upon it. With this information available, the proportioning of the parts becomes a rational application of the laws and methods of mechanics and machine design. Each part must be sufficiently strong to avoid failure, and sufficiently rigid to avoid distortions that would interfere with its functioning properly.

(d) *Specification and Drawing.* These are absolutely essential to a satisfactory and successful design, and just as essential to the complex problems of organizing and directing the building of the machine.

It is evident from what has been said above that kinematics has a very important place in the design of mechanisms and machines. Not only is a knowledge of kinematics essential in determining the correct geometric form of the parts to transmit and to transform the motions properly, but it is also essential to the study and application of Kinetics.

A moving part or body may be considered as made up of particles, or material points. Since the study of the motion of bodies is largely

a study of the motion of their particles, the fundamental kinematic relations between distance, time, velocity, and acceleration will first be developed for a particle, or a material point.

3. Kinds of Quantities. Two kinds of quantities, *scalar* and *vector*, are dealt with in kinematics. Scalar quantities, as, for example, time, distance, volume, etc., possess magnitude only. Vector quantities, as, for example, displacement, velocity, acceleration, etc., possess direction as well as magnitude.

4. Vectors. A vector is a straight line of definite length and direction. The point where a vector begins is called its *origin*, and the point where it ends is called its *terminus*. An arrow-head is used to indicate which end is the origin and which the terminus. This arrow-head is usually placed at the terminus and should point away from the origin. The length of a vector represents to scale the magnitude of the quantity being considered, and the arrow-head indicates the direction to be associated with the magnitude. If a vector representing the magnitude and direction of the velocity or acceleration of a point is drawn from the point as an origin, it is called a *localized vector*, and is called a *free vector* if drawn anywhere else in the plane of motion. Likewise, a vector representing the magnitude and direction of a force is a localized vector if drawn from the point of application of the force and is a free vector if drawn anywhere else in the plane of the force.

Many problems can be solved more easily and quickly by the use of vectors than by algebraic means. By a proper choice of scale the degree of accuracy of such graphical solutions may be made almost anything desired. Furthermore, the graphical method has the advantage of showing a number of quantities in their true relations and proportions, thus appealing to the mind through the eye much more effectively than would merely the numerical values.

5. Addition and Subtraction of Vectors. When dealing with vector quantities the symbol $+\!\!\!+$ is commonly used to denote addition, and the symbol $\rightarrow\!\!\!\!-$ to denote subtraction. The sum of vectors A and B is expressed thus; $A +\!\!\!+ B$: and the subtraction of vector B from A thus: $A \rightarrow\!\!\!\!- B$.

Two vectors, A and B, are shown in Fig. 1. The addition of these vectors is shown in Fig. 2. This is performed by placing the origin of B at the terminus of A while retaining the magnitude and direction of each. The vector $A +\!\!\!+ B$ will than have its origin coincident with that of vector A, and its terminus coincident with that of vector B. If the origin of vector A, had been placed at the terminus of vector B, the vector sum, $B +\!\!\!+ A$ would have been the same as $A +\!\!\!+ B$ but the lines of construction would be differently placed.

Fig. 3 shows the construction for the vector difference, $A \rightarrow B$. As in algebraic problems, this difference $A \rightarrow B$ is equal to the vector sum, $A \leftrightarrow (-B)$. From the terminus of A draw a vector $(-B)$ of the same magnitude but opposite in direction to $(+B)$. The vector difference, or resultant, $A \rightarrow B$, must have its origin coincident with that of A and its terminus coincident with that of $(-B)$. The determination of the vector difference $B \rightarrow A$ is illustrated in Fig. 4. In this case the vector $(-A)$ is added to vector B. The procedure is similar to that followed for Fig. 3. It is to be noted that the vector differences $A \rightarrow B$ and $B \rightarrow A$ are the same in magnitude but opposite in direction, which relation a little consideration will show is correct.
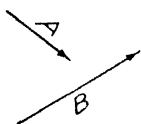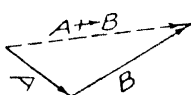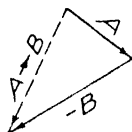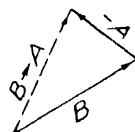
FIG. 1   FIG. 2   FIG. 3   FIG. 4

6. Composition and Resolution of Vectors. The sum of any number of vectors is called their *resultant*, and the vectors are called *components* of that resultant. The process of finding the resultant of any number of vectors is called *composition of vectors*, and the converse process of breaking up a vector into a number of components is called *resolution of vectors*. It should be noted that a group of related vectors can have but one resultant, but that a given vector may have any number of sets of components. The various components may or may not be confined to one plane.

Vectors may be resolved or composed by the use of vector triangles, as in Art. 5, or by the use of parallelograms. Thus, in Fig. 5, vector
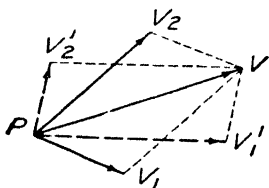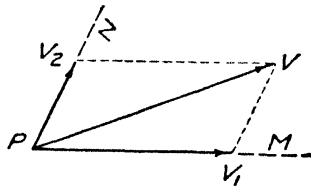
FIG. 5   FIG. 6

PV may be resolved into components PV_1 and PV_2, or into any other components PV'_1 and PV'_2, or vector PV may be regarded as the resultant obtained by composing the pair of vectors PV_1 and PV_2 or the pair PV'_1 and PV'_2.

There are four general cases for the resolution of a given vector into

two components: (a) When the direction of each of the components is known; (b) when the magnitude of each of the components is known; (c) when the magnitude and direction of one component are known; (d) when the magnitude of one component and the direction of the other are known.

(a) *To resolve a given vector PV, Fig. 6, into two components in the known directions PM and PN*: Through the terminus V of the given vector draw lines parallel respectively to PN and PM. The intersections V_1 and V_2 of these lines with PM and PN determine the required vector components PV_1 and PV_2.

(b) *To resolve a given vector PV, Fig. 7, into two components known in magnitude but not in direction*: Let M and N be the known magni-
tudes of the components. With
radius M draw arcs M_1 and M_2
from P and V as centers, and then
with radius N draw arcs N_1 and
N_2, using the same centers. Arcs
N_1 and M_2 intersect at V_1 and
V'_1, and arcs N_2 and M_1 intersect
at V_2 and V'_2. Joining these
intersections with P and V forms
two parallelograms, $PV_1\,VV_2$ and

FIG. 7

$PV'_1\,VV'_2$, each having PV as a diagonal, with sides PV_1 and PV_2, also PV'_1 and PV'_2, respectively, equal to the required components. Hence there are two solutions satisfying the condition that vector PV be resolved into two components equal in magnitude to M and N.

(c) *To resolve a given vector PV, Fig. 8, into two components when the magnitude and direction of one of the components are known*: Let the
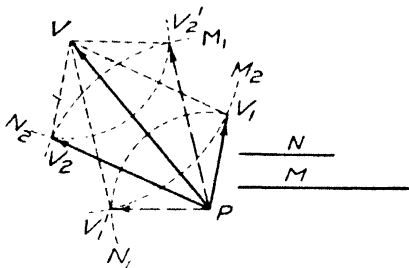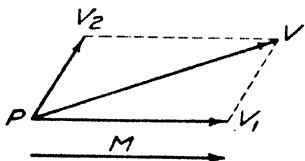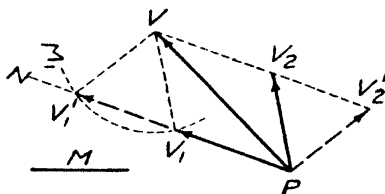
FIG. 8   FIG. 9

vector M represent the magnitude and direction of one of the components. From the origin P draw PV_1 equal and parallel to M, and join the termini V_1 and V. The intersection V_2 of a line through P parallel to V_1V with a line through V parallel to PV_1 determines the desired vector PV_2 in magnitude and direction.

(d) To resolve a given vector PV, Fig. 9, *into two components when the magnitude of one and the direction of the other are known*: Let M represent the magnitude of one component and PN the direction of the other. With V as a center and M as a radius draw the arc M_1 intersecting the line PN at V_1 and V'_1. Lines through P parallel, respectively, to VV_1 and VV'_1 will intersect a line through V parallel to PV_1 at V_2 and V'_2, thus determining two pairs of vectors, PV_1 and PV_2 and PV'_1 and PV'_2, that fully satisfy the conditions imposed.

7. Path and Motion of a Point. The path of a moving point is the locus of the successive positions of the point. The position of a moving point at any instant may be specified by stating the rectangular coordinates or the polar coordinates of the point. The path may be a line of any form whatever located in a plane or in space. If the path of a moving point is a straight line, the point is said to have *rectilinear* motion, and is said to have *curvilinear* motion if the path of the moving point is a curved line. A moving point is said to have *uniform* motion if it describes equal distances in equal intervals of time, however small, and to have *non-uniform*, or *variable*, motion if it describes unequal distances in equal intervals of time.

8. Linear Displacement. The change of position of a moving point is called its *displacement*. Thus, in Fig. 10, the change of position, or linear displacement, of the point P as it moves along its path M from B to C is the vector Δs drawn from B to C. This displacement may be expressed as the vector sum of its x and y components, Δx and Δy, or, as the vector difference of its radius vectors ρ_2 and ρ_1 to the two positions of the moving point. Thus,

$$\Delta s = \Delta x \mathbin{+\!\!\!+} \Delta y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Delta s = \rho_2 \rightarrow \rho_1 \quad \text{or} \quad \rho_2 = \rho_1 \mathbin{+\!\!\!+} \Delta s \ . \ . \ . \ . \ . \ (2)$$

It is to be noted that any displacement has both magnitude and direction and is, therefore, a vector quantity, and can be combined or resolved like other vector quantities. Any convenient unit of length, such as the inch, foot, mile, etc., may be used to express the magnitude of a displacement. It is also to be noted that one vector equation is sufficient to express both the magnitude and direction of a displacement, while two algebraic equations are required, one for the magnitude and the other for the direction. Thus,

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\tan \phi = \frac{\Delta y}{\Delta x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If in Fig. 10 the displacement is decreased indefinitely, C, in the limit, will coincide with B, and the chord Δs will coincide with the tangent to the path at B. Hence, the direction of motion of a particle at any point in its path is tangent to the path at that point.

9. Angular Displacement. · The change in the angle made by the radius vector of the moving point with some fixed reference line is called the *angular displacement* of the point. Thus, in Fig. 10, the angular displacement due to the point moving from B to C along its path is,

$$\Delta\theta = \theta_1 - \theta_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$
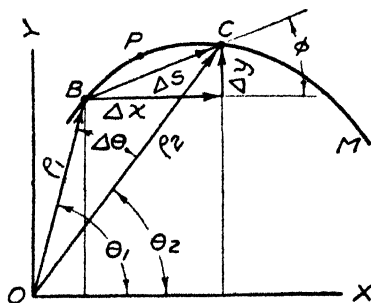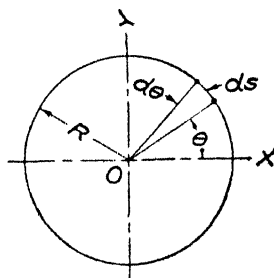
FIG. 10   FIG. 11

The magnitude of an angular displacement may be expressed in any convenient unit of angular measure, such as the radian, degree, revolution, etc.

10. Relation between Linear and Angular Displacement. In Fig. 10 the linear displacement Δs is equal to the vector sum of $\rho_1\Delta\theta$ and $\Delta\rho$, where $\Delta\rho = \rho_2 - \rho_1$; or

$$\Delta s = \rho_1\Delta\theta \dplus \Delta\rho$$

In the limit,

$$ds = \rho d\theta \dplus d\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When ρ represents the radius of a circle or the radius of curvature of the path of the particle at the instantaneous position of the point, then $\rho_2 - \rho_1 = d\rho$ becomes zero, and

$$ds = \rho d\theta$$

Hence, if the path of a moving point is a circle of radius R, as in Fig. 11, or, if R is the radius of curvature of the path at the instantaneous position of the point, then the relation between the linear and angular displacements of the point for an indefinitely small displacement is

$$ds = Rd\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

11. Linear Speed and Velocity. The linear speed of a moving point is the time rate of change of distance, while the linear velocity of a moving point is the time rate at which the particle is changing its position, or, more briefly, the time rate of change of linear displacement. Distance possesses magnitude only and is, therefore, a scalar quantity, while displacement possesses both magnitude and direction and is, therefore, a vector quantity. Hence speed is a scalar quantity and velocity a vector quantity. Speed represents the magnitude of the velocity. To definitely define the velocity of a point at any instant it is necessary to know its direction as well as its magnitude, or speed, at the instant.

If a moving point traverses a distance Δs in an interval of time Δt, the average speed for the interval is

$$V = \frac{\Delta s}{\Delta t}$$

If the point has *uniform* motion, the speed is constant from instant to instant and is given by this expression. The velocity, while constant in magnitude from instant to instant, may or may not be constant in direction. If the point has uniform rectilinear motion, its velocity at any instant becomes known as soon as its path is located. If, however, the point has uniform curvilinear motion, the velocity, while remaining constant in magnitude, changes its direction from instant to instant. In accordance with Art. 8, the direction of motion of the particle at any point in its path is tangent to the path at the point.

If the motion of the point is variable, the above expression will not give the speed of the point at each instant during the interval, Δt. It can give only the average speed for the interval. Thus, if a train, moving at a variable rate, travels 440 feet in 10 seconds, its average speed during the 10-second interval is 44 feet per second, or 30 miles per hour. Its speed at any instant during the 10 seconds may be greater or less than 44 feet per second. If the speed of the train is measured for a smaller interval than 10 seconds, say 2 seconds, the average speed for the two seconds will undoubtedly be nearer the speed *at any instant* during the 2 seconds than the average speed for the 10-second interval. If the distance traveled in a still smaller interval, including the specified instant, can be measured, a still closer approximation to the speed at the instant is obtained. Hence, the speed at any instant is the average speed for an indefinitely small period of time including the specified instant. Expressed mathematically, the speed, or magnitude of the velocity, at any instant, is

$$V = \frac{\text{Limit}}{\Delta t = 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (8)$$

Speed, or the magnitude of the velocity, may be expressed in any convenient units of distance per unit of time, such as feet per second, feet per minute, miles per hour, etc.

12. Angular Velocity. The angular velocity of a moving point is the time rate of angular displacement of its radius vector. For uniform angular motion, equal angular displacements in equal intervals of time, the angular velocity ω is the ratio of any angular displacement $\Delta\theta$ to the time interval Δt required for the displacement. Thus,

$$\omega = \frac{\Delta\theta}{\Delta t}$$

For variable angular motion this expression would give only the average angular velocity for the interval. The angular velocity at any instant is the average angular velocity for an indefinitely small interval of time including the specified instant. Expressed mathematically, the angular velocity at any instant of a variable angular motion is,

$$\omega = \underset{\Delta t = 0}{\text{Limit}} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9)$$

Angular velocity may be expressed in any convenient units of angular displacement per unit of time, such as radians per second, degrees per second, revolutions per minute or per second, etc. It is usual to consider the angular velocity as positive if the rotation is counter-clockwise, and negative if clockwise.

13. Relation between Linear and Angular Velocities. The magnitude of the linear velocity V and the angular velocity ω at any instant for a moving point, were found in Arts. 11 and 12 to be

$$V = \frac{ds}{dt} \quad \text{and} \quad \omega = \frac{d\theta}{dt}$$

It was shown in Art. 10 that if the point is moving in a circular path of radius R, or is moving in a path whose radius of curvature at the instantaneous position of the point is R, then the indefinitely small displacement of the point is

$$ds = Rd\theta$$

Substituting this value of ds in the equation for V and then substituting ωdt for $d\theta$, gives

$$V = \frac{Rd\theta}{dt} = R\omega \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (10)$$

The relation just expressed is found very useful in kinematics. If a point has a uniform motion in a circular path of radius R feet at the rate

of N revolutions per second, it has an angular velocity of $2\pi N$ radians, or N revolutions, or $360N$ degrees per second, and a speed at any instant of $2\pi RN$ feet per second. Since velocity is a directed speed, the velocity of the point at any instant will be tangent to the circle at the instantaneous position of the point.

14. Components of Velocity. For a point moving in a circular path of radius R, or moving in a path whose radius of curvature at the instantaneous position of the point is R, the relation between the angular velocity and the magnitude of the linear velocity has just been shown to be

$$V = R\omega$$

If the center of curvature of the path of a moving point is not taken as the origin of the radius vector of the point at the instant, then the term $\rho d\theta$ is one component only of the linear velocity. The linear velocity at the instant would be the vector sum of the *transverse* and *radial* components of velocity, V^t and V^r, where

$$V^t = \frac{\rho d\theta}{dt} \quad \text{and} \quad V^r = \frac{d\rho}{dt} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In a similar manner the velocity of a point at any instant may be considered as the vector sum of its *axial* components, V^x and V^y, where

$$V^x = \frac{dx}{dt} \quad \text{and} \quad V^y = \frac{dy}{dt} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

15. Linear Acceleration. The time rate at which the linear velocity of a moving point is changing at any instant is called its *linear acceleration*. The linear velocity of a moving point possesses both direction and magnitude, or speed. A change in the velocity of a moving point means, therefore, a change in the speed or a change in the direction of motion or a simultaneous change in both speed and direction of motion. Hence, the linear acceleration of a moving point at any instant may be the time rate of change of the velocity due to a change in speed or due to a change in the direction of motion or due to a simultaneous change in both speed and direction of motion. Where, as in variable rectilinear motion, there is a change in speed but not in the direction of motion, the linear acceleration is due to a change in speed only. In curvilinear motion the linear acceleration is due either to a change in the direction of motion or to a change in both the speed and the direction of motion. Where, as in uniform curvilinear motion, only the direction of motion changes, the acceleration is due to the change in the direction of

motion. In variable curvilinear motion both the speed and direction of motion change, and the acceleration is the vector sum of the component accelerations due to the change in speed and in direction of motion.

If the velocity of a moving point is increasing, its acceleration is positive; and it is negative if the velocity is decreasing.

The magnitude of an acceleration may be expressed in any convenient units of magnitude of velocity per unit of time, such as feet per second per second (ft./sec.²), miles per hour per second (mi./hr./sec.), etc.

16. Accelerated Rectilinear Motion. For rectilinear motion, as stated above, the acceleration of a moving point is due entirely to the change in the magnitude of the velocity. If the motion of the point is *uniformly* accelerated, the time rate of change of the velocity is constant, and the acceleration of the point at any instant is

$$A = \frac{\Delta V}{\Delta t} = \frac{V - V_0}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

or

$$V = V_0 + At \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where V_0 and V are the respective velocities at the beginning and at the end of any interval of time Δt, or simply t.

The linear displacement s during the interval t is the product of the average velocity and the time interval; or,

$$s = \tfrac{1}{2}(V_0 + V)t = \tfrac{1}{2}(V_0 + V_0 + At)t = V_0t + \tfrac{1}{2}At^2 \quad . \quad (15)$$

Substituting the value of t from (14) in (15) gives,

$$V^2 = V_0^2 + 2As$$

Hence, for *uniformly* accelerated rectilinear motion,

$$V = V_0 + At \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$V^2 = V_0^2 + 2As \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$s = V_0t + \tfrac{1}{2}At^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$s = \tfrac{1}{2}(V_0 + V)t \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

For *non-uniformly* accelerated rectilinear motion equation (13) gives only the average acceleration for the interval and not the acceleration at any instant. The acceleration at any instant would be the average

acceleration for an indefinitely small interval including the specified instant; or, mathematically,

$$A = \frac{\text{Limit}}{\Delta t = 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt} \quad \ldots \ldots \quad (20)$$

But

$$V = \frac{ds}{dt} \quad \ldots \ldots \ldots \ldots \quad (21)$$

Hence,

$$A = \frac{dV}{dt} = \frac{d^2s}{dt^2} \quad \ldots \ldots \ldots \ldots \quad (22)$$

Substituting the value of dt from (21) in (20) gives,

$$Ads = VdV \quad \ldots \ldots \ldots \ldots \quad (23)$$

The above equations express the fundamental relations between displacement, velocity, acceleration, and time for *non-uniformly* accelerated motion of a point in a straight line path. These equations also apply to a component of the motion of a point, provided s, V, and A, as applying to the displacement, velocity, and acceleration of the component, are used in the equations.

17. Simple Harmonic Motion. Simple harmonic motion is a special case of non-uniformly accelerated rectilinear motion of such engineering importance as to deserve special notice. A point is said to have simple harmonic motion if it moves in a straight line with an acceleration proportional to its distance from a point, or origin, and is directed towards the origin. If a point moves in a circular path at constant speed, it can be shown that the projection of the point on a diameter of the circle will have simple harmonic motion.
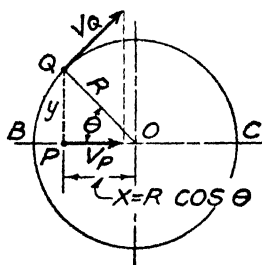
FIG. 12

In Fig. 12, let Q be a point moving in a circular path of radius R at a constant speed V_Q or at a constant angular velocity $\omega = \dfrac{V_Q}{R}$. If Q moves from B to Q in an interval of time t, then,

$$\vartheta = \omega t$$

and

$$x = R \cos \theta = R \cos \omega t \qquad y = R \sin \omega t$$

Hence, P, the projection of Q, will have a velocity at the instant of,

$$V_P = \frac{dx}{dt} = \frac{d(R \cos \omega t)}{dt} = - \omega R \sin \omega t = - \omega y \quad . \quad . \quad . \quad (24)$$

and the acceleration of P at the instant will be,

$$A_P = \frac{d^2 x}{dt^2} = \frac{d(- \omega R \sin \omega t)}{dt} = - \omega^2 R \cos \omega t = - \omega^2 x \quad . \quad (25)$$

The angular velocity, ω, is a constant, and the minus sign shows A_P to be directed towards O. Therefore, the point P has simple harmonic motion, since it moves in the line BOC with an acceleration proportional to its distance from the origin O and is directed towards O.

For simple harmonic motion, the *amplitude* is defined as one-half the distance traversed by the point, which, for the above example, gives an amplitude equal to the radius R of the circle. The *period* is defined as the time required for one complete oscillation, and the *frequency* as the number of complete oscillations per unit of time. Hence, for the above example, the period T is,

$$T = \frac{2\pi}{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and the frequency N is,

$$N = \frac{1}{T} = \frac{\omega}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

A weight, if suspended by a spring and set in motion, will vibrate up and down with simple harmonic motion. The higher the ratio of the connecting-rod length to the crank length, the more nearly does the motion of the cross-head of an engine approach simple harmonic motion. The motion of a pendulum approximates closely a simple harmonic motion if the arc through which the pendulum swings is small.

18. Tangential and Normal Acceleration. For a point having variable curvilinear motion, the velocity changes in magnitude and in direction. A point having such a motion in a circular path of radius R is illustrated in Fig. 13. As the point moves from B to C in an interval of time Δt, the velocity changes from V_1 to V_2. The total change in velocity, ΔV, is the vector difference of V_1 and V_2, as shown in Fig. 14. It is also equal to the vector sum of two velocity components, $\Delta V'$ and $\Delta V''$, respectively parallel and perpendicular to V_1. Thus,

$$\Delta V = V_2 \to V_1 = \Delta V' \nrightarrow \Delta V''$$

Expressed algebraically,

$$\Delta V = \sqrt{(\Delta V^t)^2 + (\Delta V^n)^2}$$

By definition the average acceleration for the interval is,

$$A = \frac{\Delta V}{\Delta t} = \sqrt{\left(\frac{\Delta V^t}{\Delta t}\right)^2 + \left(\frac{\Delta V^n}{\Delta t}\right)^2}$$
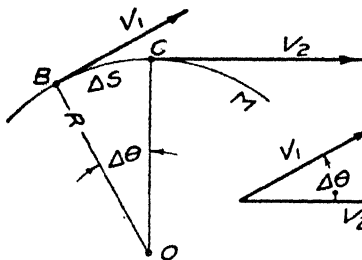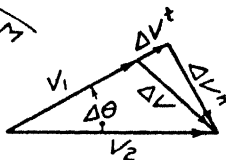
FIG. 13   FIG. 14   FIG. 15

The acceleration at the instant is the average acceleration for an indefinitely small interval of time including the instant; or,

$$A = \underset{\Delta t = 0}{\text{Limit}} \frac{\Delta V}{\Delta t} = \sqrt{\left(\frac{dV^t}{dt}\right)^2 + \left(\frac{dV^n}{dt}\right)^2}$$

As Δt approaches zero, $\dfrac{\Delta V^t}{\Delta t}$ approaches the time rate of change of the magnitude of the velocity, $\dfrac{dV}{dt}$, in the direction of V_1, or tangent to the path at the instantaneous position of the point. This component, A^t, of the acceleration is called the *tangential acceleration*. Hence,

$$A^t = \frac{dV}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The acceleration component, $\dfrac{dV^n}{dt}$, is called the *normal acceleration*, A^n, and acts at right angles to the tangential acceleration along the radius, R, toward the center of curvature, O. From Fig. 14,

$$\Delta V^n = V_2 \sin \Delta\theta$$

or,

$$\frac{\Delta V^n}{\Delta t} = V_2 \frac{\sin \Delta\theta}{\Delta t}$$

As Δt approaches zero, V_2 approaches V_1, or V, the velocity of the point at the instant; $\sin \Delta\theta$ approaches $\Delta\theta$; and $\dfrac{\Delta\theta}{\Delta t}$ approaches $\dfrac{d\theta}{dt}$. Hence, in the limit,

$$A^n = \frac{\text{Limit}}{\Delta t=0} \frac{\Delta V^n}{\Delta t} = V\frac{d\theta}{dt}$$

But $\dfrac{d\theta}{dt}$ is the angular velocity, ω, and $V = R\omega$; therefore,

$$A^n = V\omega = \omega^2 R = \frac{V^2}{R} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (29)$$

Hence, the acceleration of the point at the instant is the vector sum of A^t and A^n as shown in Fig. 15, or, expressed algebraically, is,

$$A = \sqrt{(A^t)^2 + (A^n)^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (30)$$

The direction of the acceleration relative to R is,

$$\tan \phi = \frac{A^t}{A^n} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (31)$$

The expressions above for A^t and A^n apply to any curvilinear motion of a point where R is the radius of curvature of the path at the instantaneous position of the point. As will appear later, most of the graphical methods for determining accelerations are based on these components.

19. Components of Acceleration. It is sometimes convenient in plane motion to refer the motion of a point to X and Y coordinate axes, and to use the axial components of acceleration, A^x and A^y, instead of the components A^t and A^n. Thus,

$$A^x = \frac{dV^x}{dt} = \frac{d^2x}{dt^2} \quad \text{and} \quad A^y = \frac{dV^y}{dt} = \frac{d^2y}{dt^2}$$

Hence, the acceleration of a point at any instant is the vector sum of the components A^x and A^y, or, expressed algebraically, is,

$$A = \sqrt{(A^x)^2 + (A^y)^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (32)$$

The direction of the acceleration with reference to the Y axis is,

$$\tan \phi = \frac{A^x}{A^y} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (33)$$

20. Angular Acceleration. The angular acceleration of a moving point is the time rate of change of the angular velocity of the radius vector of the point. For a point having a *uniformly* accelerated motion in a circular path, the time rate of change of the angular velocity is constant, and the angular acceleration of the point at any instant is,

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

or

$$\omega = \omega_0 + \alpha t \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Where ω_0 and ω are the respective angular velocities at the beginning and at the end of any interval of time Δt, or simply t.

The angular displacement θ during the interval t is the product of the average angular velocity and the time interval; or

$$\theta = \tfrac{1}{2}(\omega_0 + \omega)t = \tfrac{1}{2}(\omega_0 + \omega_0 + \alpha t)t = \omega_0 t + \tfrac{1}{2}\alpha t^2 \quad . \quad (36)$$

Substituting the value of t from (35) in (36) gives

$$\omega^2 = \omega_0{}^2 + 2\alpha\theta$$

Hence, for *uniformly* accelerated motion of a point in a circular path,

$$\omega = \omega_0 + \alpha t \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\omega^2 = \omega_0{}^2 + 2\alpha\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\theta = \tfrac{1}{2}(\omega_0 + \omega)t \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

For *non-uniformly* accelerated motion of a point in a circular path, equation (34) gives only the average angular acceleration for the interval and not the angular acceleration at any instant. The angular acceleration at any instant would be the average angular acceleration for an indefinitely small interval including the specified instant; or, mathematically,

$$\alpha = \frac{\text{Limit}}{\Delta t = 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \cdot \quad . \quad . \quad . \quad . \quad . \quad (41)$$

But

$$\omega = \frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Hence,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Substituting the value of dt from (42) in (41) gives

$$\alpha d\theta = \omega d\omega \qquad \dots \dots \dots \quad (44)$$

The above equations express the fundamental relations between angular displacement, velocity, acceleration, and time for *non-uniformly* accelerated motion of a point in a circular path. These equations also apply to any angular motion of a point whose radius vector at the instant is drawn from the center of curvature of the path.

Angular acceleration may be expressed in any convenient units of angular velocity per unit of time, such as radians per second per second (rad./sec.²), degrees per second per second (deg./sec.²), revolutions per minute per second (rev./min./sec.), etc.

21. Relation between Linear and Angular Accelerations. It was shown in Art. 13 for a point moving in a circular path of radius R, or for a point moving in a path whose radius of curvature at the instantaneous position of the point is R, that

$$V = R\omega$$

Differentiating this expression with respect to time, gives

$$\frac{dV}{dt} = R\frac{d\omega}{dt} \qquad .$$

But, by definition, $\dfrac{dV}{dt} = A^t$, and $\dfrac{d\omega}{dt} = \alpha$; hence,

$$A^t = R\alpha \qquad \dots \dots \dots \dots \quad (45)$$

From Art. 18,

$$A^n = V\omega = \omega^2 R = \frac{V^2}{R} \qquad \dots \dots \quad (46)$$

Hence,

$$A = \sqrt{(A^t)^2 + (A^n)^2} = R\sqrt{\alpha^2 + \omega^4} \qquad \dots \dots \quad (47)$$

It is thus evident that the magnitude of the linear acceleration of a point moving in a circular path of radius R, or in a path whose radius of curvature at the instantaneous position of the point is R, is proportional, like the linear velocity of the point, to the rotational radius, R, of the point.

22. Relative Motion. In the preceding discussion the motion of a particle, or material point, is defined with reference to a point or a set of axes assumed as at absolute rest, or *fixed*. By *absolute* motion of a point or body is meant the change of position of the point or body with

reference to a body at absolute rest.   Since nowhere in the universe is there a body known to be at absolute rest, it is impossible to know that the motion of any point or body is absolute.   Hence, all motion is relative.   It is customary, however, in mechanics and kinematics to regard the earth as at rest, and to speak of any motion relative to the earth as absolute motion.   Hence, the *absolute* motion of a point or body is the motion of the point or body relative to the earth.   The *relative* motion of a point or body is the motion of the point or body defined with respect to another body.

The terms absolute and relative may be illustrated by considering an express and a freight train running in the same direction on straight parallel tracks, say the express train at 50 miles per hour and the freight train at 20 miles per hour.   Hence, the absolute linear velocities of the two trains are 50 and 20 miles per hour in the same direction.   The linear velocity of the express train relative to the freight train is 30 miles per hour; that is, to an observer riding on the freight train the express train seems to be advancing at a rate of 30 miles per hour, because, to the observer, the freight train itself appears to be fixed, and in looking at the express train only the *difference* between the velocities of the express train and the freight train is apparent.   On the other hand, to an observer riding on the express train, the freight train appears to be backing at a velocity of 30 miles per hour; that is, the linear velocity of the freight train relative to the express train is 30 miles per hour backwards.   This last velocity is negative, because to the reference body, the express train in this instance, the body under observation is moving in a direction opposed to that which it has relative to the earth. Thus it is possible to gain a more complete understanding of the motion of a body by referring its motion to different bodies.

As will be shown, the motion of a rigid body can be defined by the motion of one or more points of the body.   An understanding, therefore, of the relative motion of points is of great importance in the study of the relative motion of bodies.

In Fig. 16 let the two points B and C move parallel to the fixed plane O.   The velocity of B relative to O at the instant is represented by the localized vector V_{BO}, and the velocity of C relative to O by the localized vector V_{CO}.   It is to be noted that V_{CO} is read, the velocity of C relative to O.   In determining the *relative* velocity V_{BC}, velocity of B relative to C, use will be made of the important principle that the addition of equal velocities to each particle does not disturb or change the relative velocities of the two particles.   Thus, if C and B are each given a velocity equal and opposite to V_{CO}, the point C ceases to move relative to O; B acquires a new velocity component, $-V_{CO}$, relative to O; and

the relative velocity V_{BC} becomes the absolute velocity of B, since C is fixed, relative to O. This is shown by the diagram of free vectors in Fig. 17. This shows V_{BC} as the vector sum of V_{BO} and $-V_{CO}$ or as the vector difference of V_{BO} and V_{CO}. Hence,

$$V_{BC} = V_{BO} \rightarrow V_{CO} \quad \ldots \ldots \ldots \quad (48)$$

The dotted lines of Fig. 17 show that

$$V_{BO} = V_{CO} \leftrightarrow V_{BC} \quad \ldots \ldots \ldots \quad (49)$$
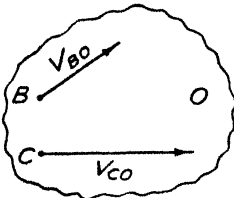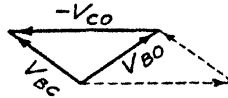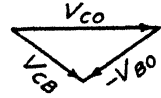

FIG. 16   FIG. 17   FIG. 18

If the relative velocity V_{CB} is desired, B and C would each be given a velocity $-V_{BO}$. Constructing the corresponding diagram of free vectors shown in Fig. 18, gives

$$V_{CB} = V_{CO} \rightarrow V_{BO} \quad \ldots \ldots \ldots \quad (50)$$

or

$$V_{CO} = V_{BO} \leftrightarrow V_{CB} \quad \ldots \ldots \ldots \quad (51)$$

It is to be noted that V_{CB} is equal in magnitude but opposite in direction to that of V_{BC}.

These same relations hold true whether dealing with displacements, velocities, or accelerations. In proving the relation for displacements, each point would be given a displacement equal and opposite to the displacement of one of the points. In dealing with accelerations each of the points would be given a velocity and an acceleration equal and opposite to the velocity and acceleration of one of the points.

There can be no relative translation of the particles of a rigid body, since any change of relative position of the particles would mean a change of form, which is contrary to the conception of a rigid body. Bodies which have no relative motion can, therefore, be joined together without affecting any motion they may have relative to other bodies. This is true, since, if there is no relative motion between them, there is no relative displacement of the particles, and they are as one body. It can, therefore, be stated that *bodies which have no motion relative to each other have the same motion relative to any other body*. It is, however,

not generally true that bodies which have the same motion relative to another body have no motion relative to each other. The front and rear wheels of an automobile on a straight road are an example. Each wheel has the same motion relative to the frame and relative to the ground, but they are not without relative motion, since to rigidly fasten them together would arrest all motion.

In the study of machines we are interested, as a rule, either in the motions of the parts of a machine relative to the frame, or in the motion of the frame, or machine as a whole, relative to the earth. For these reasons it is generally found most convenient to treat the frame of a machine as the stationary, or *fixed*, member, and to relate the motions of the parts to the frame. Where the frame of a machine has no motion relative to the earth, this procedure is equivalent to referring the motions of the various parts to the earth, since the motions of the parts relative to the earth and to the frame would be the same. This, however, would not be the case for such machines as automobiles, locomotives, and marine engines, where the frames have motion relative to the earth.

It should now be evident that a machine part can have as many relative motions as there are objects with which to compare its change of position. Thus, for example, a pair of locomotive driving wheels rotate on their axes relative to the frame, they roll relative to the rails or to the earth, they rotate about the axes of their pins relative to the attached side rods, and they have still different motions relative to the pistons, connecting-rods, and other parts.

The study of a mechanism is at times found to be much simplified by referring the motion of one member to some other member in motion rather than to the frame. While such related motions will be discussed, the word *motion* when used without qualification is to be understood throughout this book to refer to a change of position relative to the frame.

A clear conception of what is meant by relative motion will be found essential to an understanding of kinematics.

23. Free and Constrained Motion. A free body is a body having no material connection with other bodies. By *free motion* is meant the motion of a free body. A projectile in flight is an example of a body having free motion. The forces determining its path do not act through material connections to a fixed body. A body whose path with reference to some other body is determined by forces acting through material connections to the reference body is said to have *constrained motion*. Thus, in a steam engine, Fig. 19, every point of the crank-pin or crank-web is constrained by the crank-shaft bearings to move in a circular path in a fixed plane, the connecting-rod is constrained to move in a

definite path which it repeats for each revolution of the crank, and the piston is constrained by the walls of the cylinder and by its connection with the crank to move backward and forward in a straight line. The parts of the engine just mentioned have *constrained* motion since they
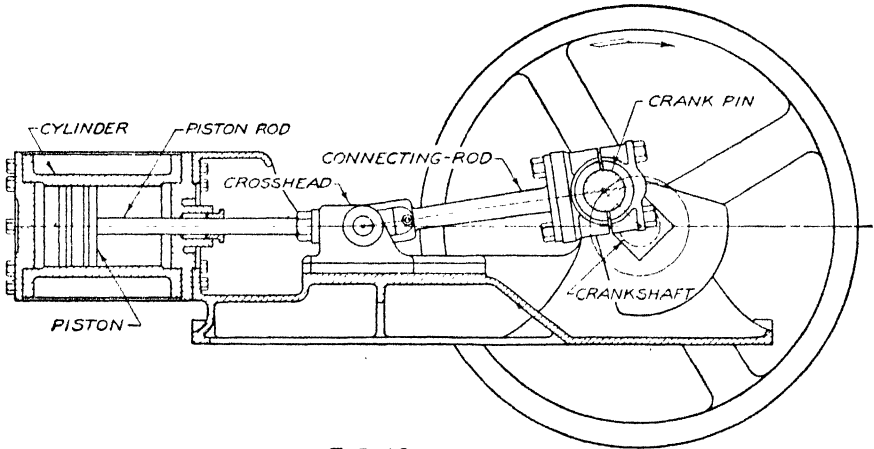
FIG. 19

move in fixed paths that are fully determined by the material connections to the reference body, the frame of the engine. There are cases where the constrained motion is not fully determined by the material connections. Thus the path of travel of the platen of a planing machine, Fig. 20, is determined by the action of gravity in conjunction with the
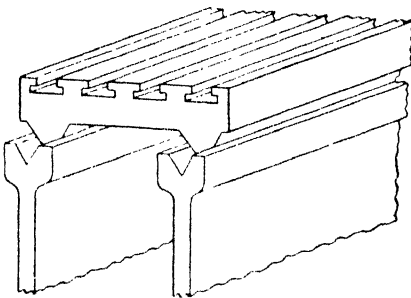
FIG. 20
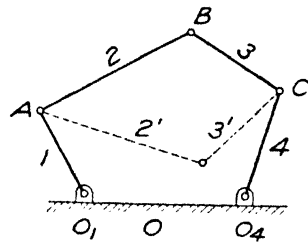
FIG. 21

material connections or contacts. This is likewise true of some cam followers, while in others constrainment is made complete by the use of some auxiliary device such as a spring.

For the combination of members shown in Fig. 21, the motion is neither free nor constrained but unconstrained so far as concerns the

plane motion of links *1*, *2*, *3* and *4*, relative to the fixed member *O*. For *1* and *4* in the positions shown, *2* and *3* may be in the positions shown or in the positions *2'* and *3'*. Clearly the motions of *1*, *2*, *3*, and *4* relative to *O* will depend on the friction at the joints, the action of gravity, and the direction and point of application of other forces that may be applied. Since, with one member fixed, each point in the other members is not constrained to move in a fixed path, the combination is not a mechanism, and the motion is *unconstrained*.

24. Cycle, Period, and Phase of Motion. The parts of a machine have completed a *motion cycle*, when, starting from some simultaneous set of relative positions, they have passed through all the possible positions they could assume and have returned to their original relative positions. The time required for a motion cycle is called a *period*. The simultaneous relative positions occupied by the parts of a mechanism or machine at any instant during a motion cycle, constitutes a *phase*.

In the ordinary steam engine, Fig. 19, the motion cycle corresponds to one revolution of the crank-shaft, whatever the time occupied by the revolution. In the common type of gas engine, each energy cycle requires two revolutions of the crank-shaft or four strokes of the piston; a suction stroke, a compression stroke, a working stroke, and an exhaust stroke. Hence, for each motion cycle of the valve gear parts, the principal parts complete two motion cycles. For the engine as a whole, the moving parts return to their initial relative positions only after the completion of two revolutions of the crank-shaft or four strokes of the piston. Gas engines of this type are, therefore, referred to as *four-stroke cycle* engines.

25. Continuous, Intermittent, and Reciprocating Motion. The motion of a body is *continuous* if during each successive cycle it neither stops nor reverses. The motion of a body is *intermittent* if during each cycle it stops moving for a finite interval of time and then starts moving again. A body has a *purely reciprocating* motion if during each cycle its motion is reversed without dwell. Thus it is possible for a part to have a purely reciprocating motion or an intermittent reciprocating motion. The crank-shaft of an automobile has a continuous motion, the valve tappet rods an intermittent reciprocating motion, and the piston a purely reciprocating motion.

26. Principal Classes of Motions. Of the motions available for machinery, a very large proportion are included in three classes comparatively simple in nature; *plane motion, helical motion,* and *spherical motion.*

27. Plane Motion. A rigid body has *plane motion* when all its points move parallel to a fixed plane. Any point in a body having

plane motion may trace any path in the plane of motion, and every other point on a line through the first point perpendicular to the plane of motion will trace paths of identically the same form. Thus, in Figs. 22 and 23, if the sections of the bodies M and N, shown shaded, lie in the plane of motion of point P, the successive positions of a line through P perpendicular to the sections will be parallel, and every point such as Q or S in this perpendicular will have a path identical with that of P. This property of plane motion greatly simplifies the treatment of the plane motion of bodies, since the motion of a point or group of points in a section in the plane of motion represents the motion of all corresponding points in parallel sections. Hence, the motion of all the points in a body having plain motion may be represented by the motion of a single plane figure. The motions, for example, of the points P, Q,
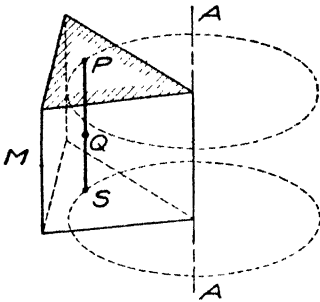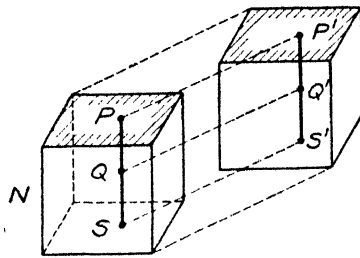
FIG. 22   FIG. 23

and S in Fig. 22 or 23 are in identical paths, and the motion of any one of these points may be taken to represent that of the other points.

Plane motion is the simplest and by far the most common class of motion. It is either a *rotation*, a *translation*, or a motion that can be reduced to a *combination of rotation* and *translation*.

28. Rotation. If each point of a rigid body having plane motion remains at a constant distance from a fixed straight line which is perpendicular to the plane of motion, the body has a motion of *rotation*. In Fig. 22 is represented a body M having a motion of rotation about the fixed straight line, or *axis of rotation*, AA. The rotation may be continuous, intermittent, or reciprocating. The shaded section lies in one of the parallel planes of motion. Any point in this section, such as P, traces a circular path in the plane of motion; and every other point, such as Q, in a line through P perpendicular to the plane of motion will trace a path of identically the same form. Hence, every point in a rotating body has the same angular displacement, angular velocity, and angular acceleration, and to define the motion of

one point defines the motion of every other point in the body.   Gears,
pulleys, cranks, and levers rotating about
fixed axes are examples of bodies having
plane motion of rotation.
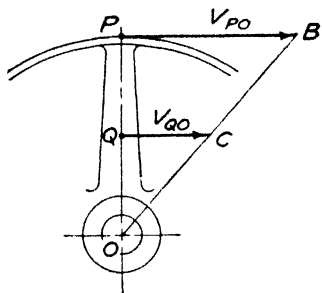
FIG. 24

Fig. 24 represents a portion of a
pulley having plane motion of rotation
about the fixed axis O. In accordance
with Art. 13, the direction of motion
of P at the instant is represented by a
line in the plane of motion tangent
to the path at P, and, therefore, per-
pendicular to the rotational radius OP.
The magnitude of the velocity of P at
the instant relative to the fixed body O is

$$V_{PO} = R\omega = (OP)\omega$$

Likewise for any other point in OP, such as Q, the magnitude of the
velocity is

$$V_{QO} = (OQ)\omega$$

Hence,

$$\frac{V_{QO}}{V_{PO}} = \frac{(OQ)\omega}{(OP)\omega} = \frac{OQ}{OP}$$

which indicates that OQC and OPB are similar triangles, and that the
vector termini C and B lie on the same straight line.   It is to be noted
that this triangular relation is true whether or not the points considered
are on the same rotational radius.   It is also to be noted that the tangent
of the angle BOP is proportional to the angular velocity of the body.

Line OB is call a *gauge line* since it measures, or gauges, the lengths
of all vectors which are proportional to their distance from a pole.

29. Translation. When a rigid body so moves that the position of
each straight line of the body is parallel to all its other positions, the
body has a motion of *translation*.   It follows that in any translation of
a rigid body each point of the body has the same motion, that is, the
same displacement, velocity, and acceleration at any instant.

If the translation of a body is such that all points of the body are
moving in identical paths parallel to a fixed plane, it is called *plane trans-
lation*.   Plane translation may be either *rectilinear* or *curvilinear*.   It is
rectilinear if all points of the body move in parallel straight-line paths,
and curvilinear if the paths of the points are identical curves parallel to
a fixed plane.

Rectilinear translation is illustrated in Fig. 23, where, obviously,
the points of line PS and of all other lines of the body traverse parallel

straight-line paths. The carriage of a lathe, the platen of a planer, Fig. 20, and the piston and cross-head of an engine, Fig. 19, are examples of bodies having rectilinear translation. The side rods of a locomotive and the cars of a Ferris wheel are examples of bodies having plane curvilinear translation.

Rectilinear translation may be regarded as a special case of rotation in which the distance from the points of the body to the axis of rotation is infinite. When the word translation is used without qualification it is to be understood to mean rectilinear translation.

It has been shown that the plane motion of a body is completely defined by the motion of any section of the body parallel to the plane of motion, that is, by the change of position of a plane figure. Since two points suffice to locate a figure in a plane, the plane motion of a body is determined by the motion of any two points in a plane of motion or by the motion of the line joining the points.

30. Rotation and Translation. Any plane motion of a rigid body may be reduced to an equivalent rotation and translation. Thus the motion of the connecting-rod of a steam engine is a combination of rotation and translation. Suppose, in the plane motion of the body shown in Fig. 25, the body changes its position from B to B''. As before pointed out, the plane motion of a body may be represented by the motion of any two points in the plane of motion or by the motion of a line joining these points. Hence, the change of position of body B may be looked upon as a

FIG.25

translation of the line CD to the position $C''' D'$ and then as a rotation from this position to the position $C'' D''$, or the line may be looked upon as first being rotated and then translated. Since the example illustrates a perfectly general case, the same method of reasoning applies to all cases of plane motion, no matter how great or how small the motion may be.
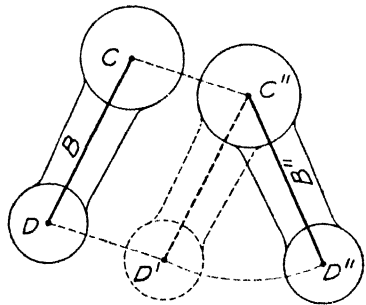
31. Helical Motion. A rigid body has helical motion when each point of the body has a motion of rotation about a fixed axis combined with a simultaneous translation parallel to the axis. If the ratio between the angular and translational components is constant, the body is said to have a regular helical motion. The motion of a nut on a screw, Fig. 26, is an example of regular helical motion. Practically all of the helical motions used in machines are of this character. The *lead* of a

helix is the translation along the axis per complete turn, or rotation. In a regular helical motion the lead is constant.

32. Spherical Motion. A rigid body has spherical motion when each point of the body has a motion about a fixed point while remaining at a constant distance from it. Hence each point moves in the surface of a sphere having the fixed point as the center. A ball-and-socket joint permits spherical motion. The balls of a fly-ball, or Watt, governor, Fig. 27, have spherical motion when the speed of the engine is varying. The balls *3* and *4* are pivoted to the rotating shaft *1* at *O*; *2* is free to slide up and down on *1* and is connected to the valve gear mechanism of the engine. When the engine is running at a constant
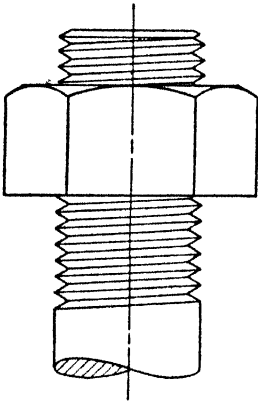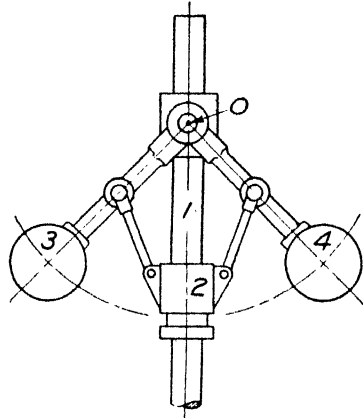
FIG. 26   FIG. 27

speed, *3* and *4* are held in fixed positions relative to *1* by the centrifugal force and the action of gravity, and any point in *3* or *4* is thus constrained to move in a circular path in a plane perpendicular to *1*. If, because of a change of load on the engine, the speed of the engine is increased or decreased, the centrifugal force will be increased or decreased, and *3* and *4* will swing about the pivot *O* as they rotate, thus causing any point in *3* or *4* to travel in the surface of a sphere having a radius equal to the distance of the point from *O*. Hence, points in *3* and *4* would have spherical motion.

33. Relation between Plane, Helical, and Spherical Motions. If, in a helical motion, the translational component, or lead, is reduced to zero, the motion is reduced to a plane motion of rotation. If the angular component is reduced to zero or the lead is increased until it is of infinite length, the motion is reduced to rectilinear translation. Thus the two

limits of a helical motion are plane motions, and either plane rotation or plane translation may be treated as a special case of helical motion.

If, in a spherical motion, the distance from the moving body to the fixed point is increased to infinity, the surfaces of the spheres in which the points of the body move are reduced to planes, and the motion is reduced to plane motion. Hence, plane motion may be treated as a special case of spherical motion.

CHAPTER II

TRANSMISSION OF MOTION

34. Pairs. The parts of a machine may make point or line contact, as in a ball or roller bearing, as between meshing gear teeth, between friction wheels, or between a cam and its follower; or the parts may have surface contact, as between a journal and its bearing, between a slide and its guide, between a screw and its nut, or between a piston and its cylinder.

The connections which permit relative motion between the parts of a machine are called *pairs*. Where surface contact is maintained the connection is called a *lower pair*, and is called a *higher pair* if point or line contact is maintained. The surfaces or forms which work together in a pair are called the *elements* of the pair. Thus the cylindrical surface of the crank-shaft of an engine and the inside surfaces of the supporting bearings are pairing elements. The surfaces of a cross-head and its guide, of a screw and its nut, and of a cam and its follower are other examples of pairing elements. A pair that permits of only relative rotation between the parts connected is called a *turning pair*, and one that permits of only relative sliding is called a *sliding pair*. A screw and its nut is called a *screw pair*, and a ball-and-socket joint a *spherical pair*. A pair which completely constrains the relative motion of the parts connected is called a *closed pair*, and a pair which does not completely constrain the relative motion of the parts connected or in contact is called an *unclosed pair*. All lower pairs that are complete are closed pairs. In general, higher pairs are unclosed pairs.

Point or line contact usually involves greater wear than surface contact, and is generally to be avoided if possible. The contact between the teeth of gears and that between most cams and their followers is necessarily line contact. There are many cases, however, where it is perfectly feasible to introduce modifications which will distribute the contact over a surface without affecting the related motions. By so doing the intensity of pressure, and consequently the wear, are reduced. Where line contact cannot be eliminated, it is often possible to improve the contact by getting a nearer approach to surface contact. While the three forms of the same mechanism shown in Figs. 28, 29, and 30

transmit the same motion from driver to follower, the quality of the contact for each form is different. The contact between the driver and follower in Fig. 29 has been improved over that in Fig. 28 by increasing the radius of curvature of the driver contact with the follower, thus giving a nearer approach to surface contact. In Fig. 30 the nature of

FIG. 28   FIG. 29   FIG. 30
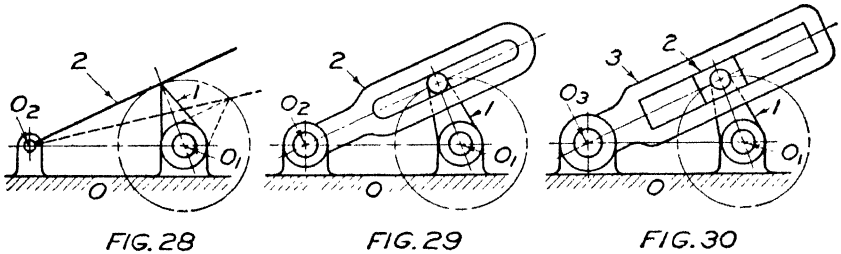
the contact has been still further improved over that shown in Fig. 28 by fitting the driver pin with a sliding block, thus changing the contact from line to surface contact.

Surface contact is only possible between plane surfaces, surfaces of revolution, and helicoidal surfaces. Hence, line contact cannot be eliminated unless it can be arranged to use such surfaces in contact. In gear teeth, for example, it is impossible to avoid line contact.

It is often found desirable, as in cams, to introduce an intermediate piece. Thus, in Fig. 31, considerable advantage is gained by introducing the roller 2 as an intermediate piece between the cam 1 and the rod, or follower, 3. The cam could act directly on the end of the rod, 3, but this would result in unnecessary friction and wear, and the operation would not be smooth, especially if the form of the cam departed much from
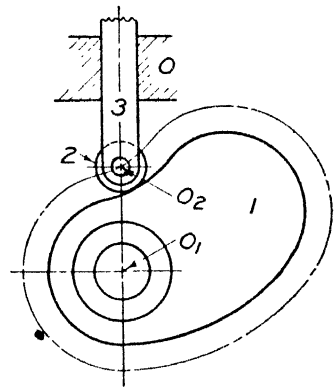
FIG. 31

a surface of revolution having its axis of rotation somewhere near its geometric axis. By introducing the roller, much smoother action is obtained. Rolling between the roller and cam surface is substituted for direct sliding between the cam and rod, and what sliding exists, is transferred to the pin carrying the roller where surface contact and a much lower speed of rubbing obtain.

It is interesting to note in connection with the preceding discussion

that it is sometimes an advantage to employ point or line contact where surface contact could be employed. Ball and roller bearings are examples. In ball or roller bearings point or line contact exists, while the ordinary bearing has surface contact between the journal and bearing shell. The more unfavorable contact in the ball or roller bearing is accepted since it is more than offset by the reduction in the friction loss due to the substitution of rolling for sliding friction.

35. Kinematic Links. A *kinematic link* is a rigid body provided with two or more elements of pairs which permit of its being connected to at least two other bodies. A body with two elements of pairs is called a *binary* link; one with three elements is called a *ternary* link; and one with four elements is called a *quaternary* link.

One link of the engine shown in Fig. 19 is made up of the parts of the cross-head, piston-rod, and piston. The parts of the crank and crank-shaft form a link, the parts of the connecting-rod another, and the parts of the frame still another. Neglecting the valve gear, the engine is seen to be composed of four binary links. The cross-head, piston-rod, and piston form a binary link since they provide an element of a sliding pair and an element of a turning pair. The sliding element pairs with a sliding element of the engine frame, and the turning element pairs with a turning element of the connecting-rod. The engine frame, neglecting the valve gear, is a binary link since it provides an element of a sliding pair and an element of a turning pair. The sliding element pairs with the sliding element of the cross-head, piston-rod, and piston; and the turning element pairs with the turning element of the crank-shaft. The crank and connecting-rod are binary links since each contains two elements of turning pairs. One turning element of the crank and shaft pairs with the turning element of the frame, and the other with a turning element of the connecting-rod. One turning element of the connecting-rod pairs with a turning element of the crank, and the other with a turning element of the cross-head. To simplify matters, these links are not, in kinematics, represented by the actual parts of the engine, but are represented diagrammatically. This is possible since the motions of the parts of a machine depend entirely on the relative positions of the pairs and not at all on the size, shape, and weight of the parts. Thus, the sliding element of the cross-head, piston-rod, and piston is diagrammatically represented in Fig. 32 by the outlines of a rectangular block, and the turning element by a small circle at the center of the block. The sliding element of the frame is represented by a heavy line joined to a small circle representing the turning element of the frame. The crank and shaft are represented by a heavy line joining two small circles representing the two turning elements of

the crank and shaft. The connecting-rod is similarly represented. The fact that the engine frame is the fixed, or reference, link may be indicated by shading, or edging, as in Fig. 38. A ternary link is represented as shown in Fig. 33, and a quaternary link as shown in Fig. 34.

Each link of the engine mechanism contains two elements of pairs, each element belonging to a lower pair. The cam, 1, in Fig. 31, illustrates a kinematic link containing an element of a higher pair and an element of a lower pair. The surface of the cam making line contact with the roller constitutes the element of the higher pair, and the surface of the shaft which turns the cam constitutes the element of the lower pair.
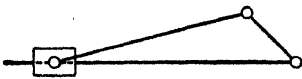
FIG. 32   FIG. 33   FIG. 34

36. Kinematic Chains. Links that are provided with suitable pairing elements may be connected by joining the elements. A combination of links that are thus connected, without leaving any elements unpaired, is called a *chain*. If the combination is such as to admit of no relative motion of the links, it is called a *structure*, or *locked chain*. Thus the three-link chain shown in Fig. 35 is a structure, since it does not admit of any relative change of position of the pairs and, therefore, does not admit of any relative motion of the links. A combination
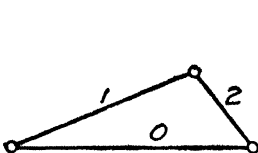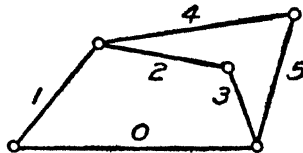
FIG. 35   FIG. 36

which admits of definite relative motion of the pairs or links such that, with one of the links fixed, each point in every other link is constrained to move in a fixed path, is called a *kinematic chain*. The chain illustrated in Fig. 32 is such a chain. A combination in which, with one of the links fixed, each point in every other link is not constrained to move in a definite path, is called an *unconstrained chain*. Such a chain is illustrated in Fig. 21.

Whether or not any ordinary chain is a structure, a kinematic chain, or an unconstrained chain, can usually be told by inspection. It be-

comes, however, increasingly difficult, as the number and kinds of links in a chain are increased, to tell whether the chain is locked, constrained, or unconstrained. A *criterion of constraint** applying to chains having plane motion has been derived in algebraic form. This criterion is

$$J + \tfrac{1}{2}H = \tfrac{3}{2}N - 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

In this equation J represents the number of binary joints in the chain, H the number of unclosed pairs, and N the number of links in the chain. The term *joint* is restricted to apply to the connection of two links by a closed pair. If three links are joined at the same connection, the joint is called a *ternary joint;* and if four links are joined at the same connection, the joint is called a *quaternary joint.* An ordinary joint of two elements of a closed pair is called a *binary joint.* Evidently a ternary joint is equivalent to two binary joints, since one of the three links joined could carry the pin for the other two links. Similarly, a quaternary joint is equivalent to three binary joints.

FIG. 37

The above equation indicates that a chain is locked if the right-hand member comes out less than the left-hand member. If the two members come out equal it indicates that the chain is constrained; and it indicates that the chain is unconstrained if the right-hand member exceeds the left-hand member.

A six-link chain having 3 binary joints and 2 ternary joints is represented in Fig. 36. The number of joints is, therefore, equivalent to 7 binary joints. Hence, $J = 7$ and $N = 6$ and $H = 0$, since there are no unclosed pairs. Substituting in equation (52) gives

$$7 + 0 = \tfrac{3}{2}(6) - 2 = 7$$

The chain is, therefore, constrained.

An 8-link chain having 10 binary joints is shown in Fig. 37. Hence $J = 10$ and $N = 8$ and $H = 0$, or

$$10 + 0 = \tfrac{3}{2}(8) - 2 = 10$$

indicating that the chain is constrained.

* See "Kinematics of Machinery," by A. W. Klein, or "Kinematics and Kinetics of Machinery," by J. A. Dent and A. C. Harper.

The chain illustrated by Fig. 21 has 5 links and 5 binary joints.
Hence $N = 5$ and $J = 5$ and $H = 0$, or

$$5 + 0 = \tfrac{3}{2}(5) - 2 = 5\tfrac{1}{2}$$

indicating that the chain is unconstrained.

In certain cases equation (52) will indicate that a chain is unconstrained when it is actually constrained. This is true of the simple chain suggested by Fig. 31, which has 4 links, 3 binary joints, and one unclosed pair. However, link *2* is an intermediate piece which could be rigidly attached to *3* without affecting the relative motion of *1* and *3*. Hence for the equivalent chain $N = 3$ and $J = 2$ and $H = 1$, or

$$2 + \tfrac{1}{2}(1) = \tfrac{3}{2}(3) - 2 = 2\tfrac{1}{2}$$

showing that the chain is constrained.

While the criterion of constraint as stated in equation (52) cannot always be applied with certainty, it will, in general, be found very useful in connection with chains whose constraint is not obvious.

The simplest complete kinematic chain is made up of four links. Kinematic chains having a greater number of links are called *compound* kinematic chains. There are, however, kinematic chains which have less than four links. Most cams acting on direct contact followers have only three links. By using a bar with the ends bent and so formed as to work in slots a mechanism equivalent to that shown in Fig. 70 is obtained which has only two links. Such apparent exceptions to the basic four-link chain are classed as *incomplete* four-link chains. They will be found to have links which vary in length, or to have two or more links which, kinematically, are of infinite length.

37. A Mechanism. A mechanism is a kinematic chain with one of its links fixed. Hence, from a kinematic, or constrained, chain having n links, n mechanisms may be formed by taking in turn each link of the chain as the fixed, or reference, link. The various mechanisms that can thus be obtained from a given kinematic chain are said to be inversions of the chain.

38. Inversion. As will be evident, many different and useful mechanisms may be obtained by the *inversion* of various kinematic chains.

Figs. 38, 39, 40, and 41 illustrate the various inversions of the slider crank chain shown in Fig. 32. By making the link *O* of Fig. 32 the fixed link, or frame, we obtain the ordinary steam-engine mechanism shown in Fig. 38. Fig. 39, the inversion obtained by fixing the crank, or link *1*, illustrates the mechanism of the Gnome type of engine with rotating cylinders. This inversion is also used for the quick return mechan-

ism of the Whitworth and other types of metal shapers. Fig. 40, the inversion obtained by fixing the connecting-rod, or link *2*, illustrates the mechanism of the oscillating cylinder engine, once used as a type of marine engine, and now found, because of its simplicity, on toy steam engines. It is also used as a shaper mechanism. Fig. 41, the inversion obtained by fixing link *3*, the slide, or cross-head, of Fig. 32, has been used in several varieties of pumps.
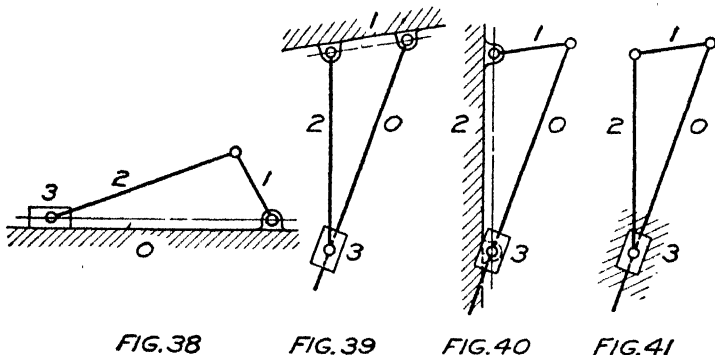

FIG. 38   FIG. 39   FIG. 40   FIG. 41

39. A Machine. A machine is a mechanism or a combination of mechanisms constructed not only for the purpose of transmitting and transforming motion, but also for the purpose of transmitting forces and doing useful work.

While a machine contains one or more mechanisms, a mechanism or a group of mechanisms is not necessarily a machine. Many models, which are mechanisms or combinations of mechanisms, are not machines since they are not built to do useful work but merely to illustrate transformations of motions. The same can be said of instruments, watches, governors, etc., since the energy they transmit is only that required to overcome their own friction.

It must be evident that the same mechanism may be used in many different machines, and that many different mechanisms may be used in a given machine to accomplish the same purpose. The working, or moving, parts of a machine may be composed of levers, arms, links, screws, nuts, rods, shafts, cams, wheels with teeth or with grooved rims, or other rigid members, also flexible members such as belts, ropes, chains, and springs, and occasionally a body of confined fluid such as water, oil, air, etc. Many combinations and modifications of these are used, resulting in an innumerable variety of forms, which, kinematically, reduce to a few fundamental classes of mechanisms or combinations of the same. It might here be pointed out that it is not always a simple

step to successfully apply to a machine an apparently suitable mechanism.

The frames which support the working parts and determine their motions are almost as varied in form as the moving parts themselves. Kinematically, the frame reduces to a single member for each kinematic chain involved. It is usually regarded as the fixed member. However, so far as relative motion is concerned, it matters not which member of a kinematic chain is taken as the fixed member.

As pointed out in Art. 23, the complete constrainment of the parts of a mechanism may or may not be due entirely to the material connections. The platen of a planing machine was cited as an example. A crane where the hoisting block and load are raised and lowered by means of a chain or cable, is an example where the motion of the block and load is constrained entirely by the action of gravity. The slightest horizontal force will sway the crane hook and, therefore, change its path from the vertical in which it normally moves. This does not, however, affect the useful operation of the crane. For all practical purposes the action is better than if the hook were rigidly constrained. In fact, a certain degree of freedom is sometimes desirable.

40. Methods of Transmitting Motion. Since the purpose of a mechanism is to transmit or to transform motion, or both, a study of the laws governing the methods by which motion may be transmitted is of prime importance. Although the variety of forms of mechanisms is almost unlimited, there are only a few classes or kinds of mechanisms. As will be evident in the following articles, this greatly simplifies the study of the laws governing the transmission of motion by mechanisms.
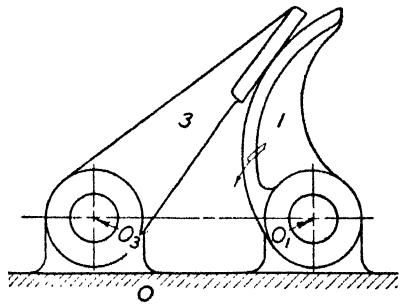
FIG. 42

The nature of the transmission is dependent upon the form of connection between the two members, the driver and the follower, as well as upon the general type of mechanism. Figs. 42, 43, and 44 illustrate some of the possible ways of transmitting motion. In each of the figures the shaft of *1* has a given angular motion, oscillation. Because of the nature of the connection between them, *1* transmits angular motion to *3*. Within a limited range each of the three ways of transmitting motion could give the same angular velocity change between the shaft of *1* and the shaft of *3*.

In Fig. 42, motion is imparted to *1* by its shaft, to which it is rigidly attached. The cam by *direct contact* with its follower *3*, transmits motion to *3* and hence to its shaft. The motion which *1* gives to *3* depends upon the shape of each and the distance between their shaft centers O_1 and O_3. The mechanism of Fig. 42 could be so designed as to transmit the same angular motion from the driver to the driven shaft as the mechanism shown in Fig. 43 or 44.

In Fig. 43, the motion of member *1* is transmitted to member *3* by the *intermediate rigid link 2*. In this mechanism the lengths of the links and angular relations can be changed to produce considerable variation in the related motions of the driver and driven shafts.
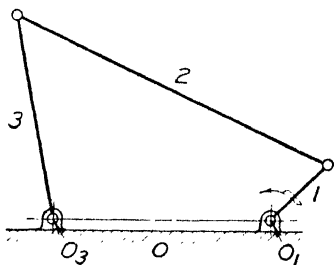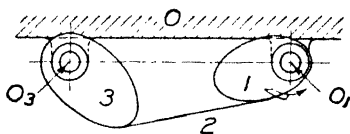

FIG. 43   FIG. 44

In Fig. 44, the motion of member *1* is transmitted to member *3* by the intermediate member *2*, a flexible band. In this mechanism also the related motions between the driver and driven shafts may be greatly varied by changing the size and shape of the band lobes and the distance between their centers of rotation. This is transmission of motion by a *flexible connector*.

Motion transmitted from one body to another without any *material connection* between them, constitutes a fourth method of transmitting motion. This method is illustrated by the action of gravity and by the action on bodies of a magnetic or electric field.

The various ways of transmitting motion from one body to another give rise to the following classification:

(*a*) Transmission of motion by direct contact, as in toothed wheels, friction wheels, cams without rollers, etc.

(*b*) Transmission of motion through intermediate rigid connectors, such as links, rollers, shoes, etc.

(*c*) Transmission of motion through intermediate flexible connectors, such as belts, ropes, chains, springs, fluids, etc.

(*d*) Transmission of motion without material connection, as in electrical machinery.

41. Driver and Follower. That part of a mechanism which causes motion is called the *driver*, and that part to which motion is transmitted is called the *follower*. In each of Figs. 42, 43, and 44, illustrating the three principal modes of transmitting motion, member *1* is the driver, rotating in a counter-clockwise direction, and member *3* is the follower. Each of these three mechanisms is taken as a representative of its class, illustrating transmission of motion by direct contact, by a rigid connector, and by a flexible connector. The following analysis of a representative of each class will cover all but a few special cases of each class, and the fundamental principles deduced will cover all but a few special cases without reference to class.

42. Line of Transmission. In a mechanism the velocities of the points of connection between the driver and the follower may be resolved into components by which the driver causes the follower to move, and into components which apply only to the relative motions of the driver and follower, as the sliding of one on the other. The *line of transmission* is that line along which motion of the driver must act in order to cause motion of the follower. If the direction of motion at the point of connection of the driver is such that it has no component along the line of transmission, it will not drive its follower; and, if the possible direction of motion at the point of connection of the follower is such that it has no component along the line of transmission, it will not be driven.

Brief study will disclose that in a mechanism transmitting motion by direct contact, the line of transmission is the common normal to the two surfaces in contact. In a mechanism transmitting motion by a link, the line of transmission is along the link, while in a flexibly connected mechanism it is along the line of the wrapping connector.

43. Angular Velocity Ratio. The three mechanisms shown in Figs. 42, 43, and 44 illustrate the three common methods of transmitting motion. The velocity relations of these three mechanisms are shown in Figs. 45, 46, and 47. In each of these figures corresponding points and lines are represented by the same letters.

In Fig. 45, the contact point P as a point in the driver 1 rotates about the center O_1. Its velocity vector PM_1 is, therefore, perpendicular to the rotational radius O_1P. PT and NN' are lines through P, respectively tangent and normal to the contact surfaces, and PT_1 and PS are, respectively, the tangent and normal components of the velocity vector PM_1. The contact point P as a point in 3 rotates about the center O_3. The normal component of its velocity must be equal to PS, since, if it were larger, 3 would leave contact with 1, and, if it were smaller, 1 and 3 would be mutually deformed. This would be contrary

to the assumption that *1* and *3* are rigid bodies. The direction of motion of P as a point in *3* is perpendicular to its rotational radius O_3P. Having its direction of motion and the normal component of its velocity, the velocity PM_3 of P as a point in *3*, likewise its tangential component PT_3, can be found by composition and resolution. Since the linear velocities and the rotational radii of the point P as a point in both *1*
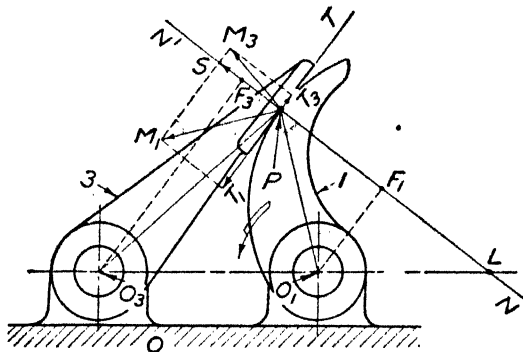

FIG. 45

and *3* are known, the angular velocities ω_{10} and ω_{30} of *1* and *3* relative to O are

$$\omega_{10} = \frac{V}{R} = \frac{PM_1}{O_1P} \quad \text{and} \quad \omega_{30} = \frac{PM_3}{O_3P}$$

Hence,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{PM_1}{O_1P} \times \frac{O_3P}{PM_3} \quad \dots \dots \dots \quad (53)$$

Line O_1F_1, perpendicular to the line of transmission NN', forms a triangle F_1O_1P similar to triangle SPM_1, and line O_3F_3, perpendicular to NN', forms a triangle F_3O_3P similar to triangle SPM_3. Hence,

$$\frac{PM_1}{O_1P} = \frac{PS}{O_1F_1} \quad \text{and} \quad \frac{PM_3}{O_3P} = \frac{PS}{O_3F_3} \quad \dots \quad (54) \text{ and } (55)$$

Dividing (54) by (55) gives

$$\frac{PM_1}{O_1P} \times \frac{O_3P}{PM_3} = \frac{PS}{O_1F_1} \times \frac{O_3F_3}{PS} = \frac{O_3F_3}{O_1F_1} \quad \dots \quad (56)$$

Hence, by substituting (56) in (53),

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_3F_3}{O_1F_1} \quad \dots \dots \dots \quad (57)$$

But triangles $F_1 O_1 L$ and $F_3 O_3 L$ are similar, and

$$\frac{O_3 F_3}{O_1 F_1} = \frac{O_3 L}{O_1 L} \quad \cdots \cdots \cdots \cdots \quad (58)$$

Hence, from (57) and (58),

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_3 L}{O_1 L} \quad \cdots \cdots \cdots \cdots \quad (59)$$
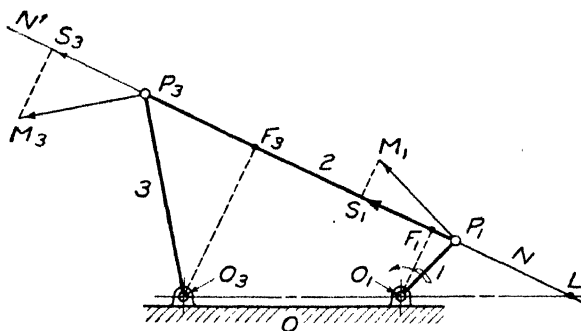

FIG. 46

It will be evident by inspection that similar treatment applied to the mechanisms shown in Figs. 46 and 47 will yield the same results as expressed in equations (57) and (59). Hence, the relations expressed by these equations for the three common methods of transmitting motation may be stated as follows: *The angular velocities of driver and follower are*
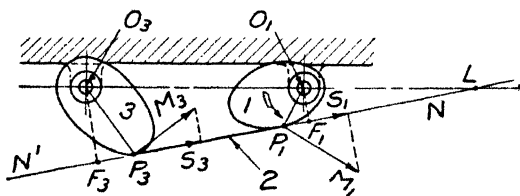
FIG. 47

inversely as the perpendiculars let fall from their fixed centers upon the line of transmission; or inversely as the segments into which the line of transmission cuts the line of centers.

It is possible to further clarify the angular velocity relations just stated by considering the points F_1 and F_3, in Fig. 45, as respective points in *1* and *3*, which could be so extended as to actually include F_1 and F_3. The linear velocities of F_1 and P as points in *1* are as their rotational radii, or, from similar triangles, as PS is to PM_1. Hence, the total

velocity of F_1 is PS equal to the normal component of PM_1. Since the normal component of the velocity of P as a point in 3 must be equal to PS, the total velocity of F_3 as a point in 3 is, by the same reasoning, equal to PS, and, therefore, equal to the total velocity of F_1. Since the linear velocities of F_1 and F_3 as points in 1 and 3 are equal, their angular velocities are inversely proportional to their rotational radii, O_1F_1 and O_3F_3, which are the perpendiculars from the fixed centers to the line of transmission. In a similar manner, the relation is also evident for Figs. 46 and 47.

44. Constant Angular Velocity Ratio. It has just been shown, for the three common methods of transmitting motion, that

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_3L}{O_1L}$$

If the angular velocity ratio is to be constant, L must be a fixed point on the line of centers. *Hence, for constant angular velocity ratio, the line of transmission must always cut the line of centers at a fixed point.*

This requirement is fulfilled by many pairs of curves which may be used as the outlines for the acting faces of direct-contact members. As will appear later, it is a requirement of fundamental importance in treating the theory of gear tooth outlines.

Constant angular velocity ratio is secured in link transmission when the driving and driven arms are equal and the length of the connecting link is equal to the distance between the fixed centers, as in the side rods connecting locomotive driving wheels.

Constant angular velocity ratio is secured in the case of wrapping connectors when the driver and follower are circular cylinders rotating about their geometric axes, as do belted wheels.

45. Directional Relation. In Figs. 45, 46, and 47, the rotation of the follower is in the same direction as the driver. It is to be noted that for each one of these three common methods of transmitting motion, the centers O_1 and O_3 lie on the same side of the line of transmission NN'. In Fig. 48 is illustrated a direct-contact mechanism in which the direction of rotation of the follower is evidently the opposite to that of the driver. It is also evident that the centers of rotation O_1 and O_3 are on

FIG. 48

opposite sides of the line of transmission NN'. This can easily be shown to be true for the two other methods of transmitting motion.

Hence, for any of the three common methods of transmitting motion, the directions of rotation of driver and follower are the same if the centers of rotation lie on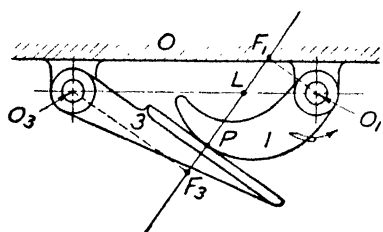 the same side of the line of transmission, and the directions of rotation are opposite if these centers lie on opposite sides of the line of transmission.

46. Nature of Rolling and Sliding. The action between two members in direct contact may be pure rolling, pure sliding, or a combination of rolling and sliding.   Two members acting by direct contact are shown in Fig. 49.   For the phase shown they are in contact at P.   Let R and S be any two points that meet and become coincident contact points as the action continues.   For pure rolling action between 1 and 2 the arcs PR and PS must be equal.   If for any increment of motion the corresponding arcs of action are not equal, the rolling must be accompanied by some sliding.   For pure rolling action no single point on either member comes in contact with two successive points of the other.   Where a point on one body, within the limits of its path, comes
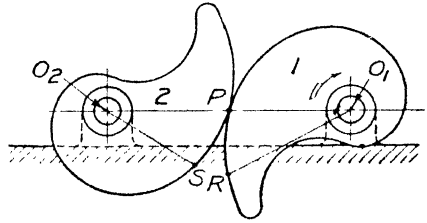

FIG. 49

in contact with all the successive points on the acting surface of another body, the action is pure sliding.   The action between a cross-head and its guide is an example of pure sliding   The action between gear teeth and between many cams and followers is mixed rolling and sliding.

47. Velocity of Sliding in a Direct-Contact Mechanism. Since, as was shown in Art. 43, two rigid bodies in direct contact must have the same component of velocity along the common normal through the point of contact, and, hence, have no relative velocity along this common normal, or line of transmission, all motion of the contact points relative to each other must be along the common tangent.   The tangential components may have any values, either in the same or in opposite directions.   If, as is usually the case, the velocities of the contact points relative to the frame of the mechanism are known or can be determined, the relative sliding between the driver and the follower may be found by subtracting the tangential velocity components of the contact points. *Hence, in direct-contact mechanisms, the velocity of sliding is the vector difference of the tangential components of the velocities of the points of contact.*   Since these components lie in the same line, the algebraic difference is the same as the vector difference.

Fig. 50 illustrates the graphical determination of the velocity of sliding between two direct-contact members 1 and 3 in the phase shown.

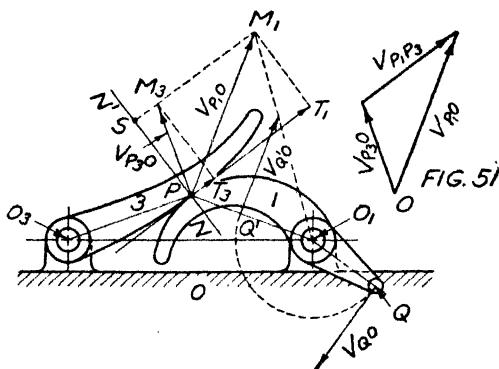Let *1* drive *3*, and suppose *1* to be driven by another member acting at Q giving Q a known velocity V_{Qo} relative to the frame O. Q' is a position of Q revolved so that it lies on the radius of P. The vector PM_1, representing the velocity of P as a point in *1*, is perpendicular to radius O_1P, and its magnitude is determined by the dotted gauge line drawn through the extremity of the vector $V_{Q'o}$, since the velocities of points in a rotating body are proportional to their rotational radii. The vector PM_1 is now resolved into its components PS along the common normal NN', and PT_1 along the common tangent at the point of contact.

FIG. 51

FIG. 50

The contact point P as a point in *3* rotates about the center O_3. The normal component of its velocity must be equal to PS, since, if larger, *3* would leave contact with *1*, and, if smaller, *1* and *3* would be mutually deformed. The direction of motion of P as a point in *3* is perpendicular to its rotational radius O_3P. The next step is to compose a resultant velocity PM_3, knowing its direction and the magnitude and direction of its normal component PS. The line M_1S, parallel to the common tangent which determined the magnitude of the normal component PS, will also determine by its intersection, M_3, the magnitude of the resultant vector PM_3. This velocity vector PM_3 will have also the tangential component PT_3 along the common tangent. The velocity of sliding is the vector difference of the tangential components. Hence

$$PT_1 \rightarrow PT_3 = T_3T_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

or, since PT_1 and PT_3 are in the same line, T_3T_1 may be obtained algebraically;

$$PT_1 - (+PT_3) = T_3T_1 \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Since velocity vectors PM_1 and PM_3 must have the same normal component PS, it is to be noted that vector subtraction of PM_1 and PM_3 must be the same as T_3T_1; that is, velocity vector M_3M_1 equals T_3T_1, the velocity of sliding of the driver *1* with respect to the follower *3*. Hence, in a direct-contact mechanism, if the velocities of the contact point as a point in the driver and as a point in the follower are known,

the velocity of sliding is equal to the vector difference of these velocities, as shown in Fig. 51.

It is also evident that if in some phase, for the mechanism shown in Fig. 50, PT_3 and PT_1, were to come out equal and in the same direction, PM_3 and PM_1 would coincide, and at the instant there would be pure rolling and no sliding between *1* and *3*.

48. Condition of Pure Rolling. As was pointed out immediately above, if one member is to transmit motion to another without relative sliding, that is, with pure rolling, the velocity of the point of contact of one member must coincide with the velocity of the point of contact of the other member. In Figs. 52 and 53, P, as a point in *1*, moves at right angles to O_1P; and, as a point in *3*, it moves at right angles to O_3P. Hence, when PM_1 and PM_3 coincide, radii O_1P and O_3P are perpendicu-
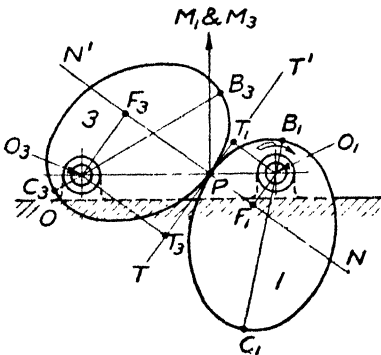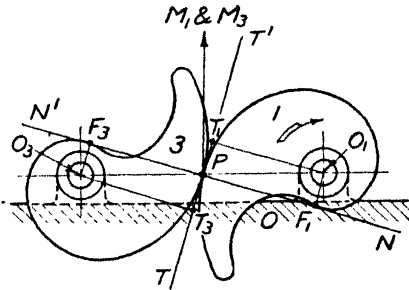
FIG. 52   FIG. 53

lar to the same line at the same point, and must, therefore, lie in the same straight line. This means that P must lie on the line of centers. *Hence, the condition of pure rolling is that the point of contact shall always lie on the line of centers.* Conversely, if the point of contact does not lie on the line of centers, there will be relative sliding between the two members in contact. There are many sets of curves which may be employed to thus transmit motion by direct contact with pure rolling action, among which may be mentioned: tangent circles or circular arcs rotating about their centers; pairs of equal ellipses, Fig. 52, each rotating about one of its foci with a distance between the fixed centers equal to the common major axis; and pairs of similar logarithmic spirals, Fig. 53, rotating about their foci.

For two direct-contact members it was shown in Art. 43 that the angular velocity ratio of the members is inversely as the perpendiculars let fall from the fixed centers to the common normal, or inversely as the

segments into which the line of centers is divided by the common normal, or line of transmission. Since for pure rolling, the intersection of the line of transmission with the line of centers is the point of contact, it follows that *in pure rolling action the angular velocity ratio is inversely as the contact radii.* Hence, in Figs. 52 and 53,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_3 P}{O_1 P} = \frac{O_3 F_3}{O_1 F_1} . \quad . \quad . \quad . \quad . \quad (62)$$

If in Figs. 52 and 53 the perpendiculars $O_1 T_1$ and $O_3 T_3$ be dropped from the fixed centers O_1 and O_3, upon the common tangent TT', two similar triangles, $PO_1 T_1$ and $PO_3 T_3$, are formed. Hence,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_3 P}{O_1 P} = \frac{O_3 T_3}{O_1 T_1} . \quad . \quad . \quad . \quad . \quad (63)$$

For direct-contact members having *pure rolling* action, it may therefore
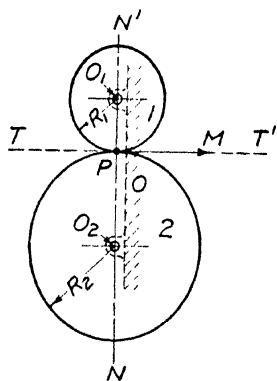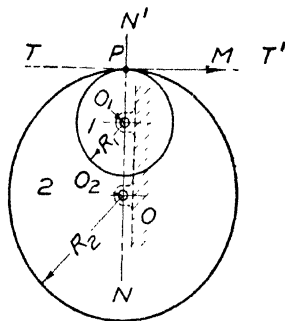
FIG. 54   FIG. 55

be stated that *the angular velocity ratio is inversely as the contact radii or inversely as the perpendiculars let fall from the fixed centers to the common tangent or to the common normal.*

It is to be noted that the circular forms shown in Figs. 54 and 55 fulfil the geometric requirements for pure rolling action. It is also to be noted that the perpendiculars from the fixed centers to the common tangent coincide with the contact radii, or, more simply stated, the common tangent is normal to the line of centers. This means that for pure rolling action the friction of contact must be sufficient to drive without slipping.

49. Constant Angular Velocity Ratio Combined with Pure Rolling. As stated in the preceding articles, there are many pairs of curves which will satisfy either the condition of constant angular velocity ratio or the condition of pure rolling action. Circles and circular arcs rotating about their geometric axes are, however, the only class of curves which can have pure rolling action combined with constant angular velocity ratio. For constant angular velocity ratio, the common normal must cut or pass through a fixed point in the line of centers; and for pure rolling, the contact point through which the normal passes must lie on the line of centers. If both of these requirements are to be met, the contact point must be a fixed point in the line of centers. Hence, the contact radii cannot vary in length, and the curves must be circles or circular arcs.

50. Positive Driving. In order to have pure rolling action between the right cylinders shown in Figs. 54 and 55, the friction of contact must be sufficient to assure driving without slipping. In the absence of friction the motion of one cylinder would not cause any motion of the other, resulting in relative sliding without any rolling action. With friction, the motion of one cylinder would cause motion of the other, but there would be no positive assurance that a certain amount of slipping or sliding did not take place. If there is to be pure rolling action between the cylinders, one must positively drive the other. To secure this the cylinders must be supplied with teeth whose outlines will be such that the line of transmission will always cut the line of centers in a fixed point, the requirement deduced in Art. 44.

For such direct-contact mechanisms as are shown in Figs. 48 to 53, inclusive, one member will positively drive the other so long as the contact radius of the driver is increasing as action proceeds. This condition for positive driving is not, however, true for all drivers and followers, as shown by the example illustrated in Fig. 56. Here the contact radius of the driver remains practically constant, decreasing or increasing only slightly as action proceeds. It is evident, however, that
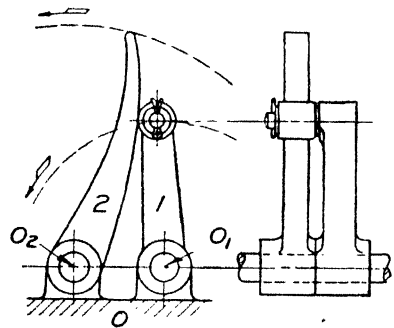

FIG. 56

1 will positively drive *2*, the follower, throughout each revolution. This is likewise true of the driver and follower shown in Fig. 30. For the mechanism shown in Fig. 56, either member may be the driver for either clockwise or counter-clockwise rotation. For the mechanism shown in Fig. 29, while *1* may be the driver for either clockwise or

counter-clockwise rotation, member *2* will not act so satisfactorily as the driver. A better statement of one of the conditions for positive driving applying to all drivers and followers is that the motion of the point of contact of the driver must have a component directed towards the follower. Thus, in Fig. 52, the motion of the point of contact of *1* the driver, has, for the phase shown, a normal component directed toward the follower. This continues to be true until C_1 and C_3 become coincident contact points, at which instant positive driving would cease since the motion of the point of contact of the driver would be directed along the common tangent and, therefore, would have no normal component. Hence, for the rolling ellipses of Fig. 52, *1* may positively drive *3* or *3* may positively drive *1* for one-half revolution only.

It has been brought out immediately above that it is only the normal component of the motion of the contact point of the driver that can possibly cause any motion of the follower. Positive driving ceases when this component becomes zero, or when it is directed away from the follower, or, if directed toward the follower, when it passes through the fixed center of the follower. When the normal component of the motion for the contact point of the driver is zero, it means that it is directed along the common tangent, and that the common normal, or line of transmission, passes through the fixed center of the driver. When the normal component is directed through the fixed center of the follower, it means that the pressure of the driver on the follower can exert no turning moment on the follower.

The criterion for positive driving can, therefore, be stated as follows: *For positive driving the common normal, or line of transmission, shall not pass through the fixed center of either the driver or the follower, and the normal component of the motion of the contact point of the driver must be directed toward the follower.*

CHAPTER III

ANALYSIS OF PLANE MOTION

51. Plane Motion. This chapter will deal with the analysis of the plane motion of bodies and with the methods of determining the velocities and tangential accelerations of any point in a mechanism whose parts move parallel to a fixed plane. In this chapter the following deductions of Arts. 13, 27, 28, and 29 will be applied:

(a) Any plane motion of a body may be fully represented by the motion of a single plane figure parallel to the plane of motion. Hence, in dealing with the plane motion of a body having appreciable depth, the body may be fully represented by a plane figure outlined on the surface of a sheet of paper.

(b) The plane motion of a body is determined by the motion of any two of its points not in the same perpendicular to the plane of motion, or by the motion of the line joining the two points.

(c) Rectilinear translation may be regarded as a special case of plane rotation in which the rotational radii of the points of the body are infinitely long and hence are parallel.

(d) For a body having a motion of plane rotation, the velocity or direction of motion of any point in the body is perpendicular to the rotational radius of the point at the instant. The direction of motion is usually represented by a vector.

(e) The linear speeds of any two points of a body having a motion of plane rotation are proportional to the respective rotational radii of the points.

52. Instant Center. At any instant the *velocity* relations of a body having plane motion are the same as they would be if the body were rotating, *at the instant*, about a certain axis.

In Fig. 57, let O be a rigid body considered as fixed, and let 1 be a rigid body having plane motion relative to O. The instantaneous motion of 1 relative to O is represented by the velocities, V_{PO} and V_{RO}, of the two respective points, P and R. Since these two points are in a rigid body their instantaneous velocities must always be such that their components, V and V', along the line joining the points, shall be equal and in the same direction. From P and R draw perpendiculars to V_{PO} and V_{RO} intersecting at O_{10}. Draw $O_{10}M$ perpendicular to the line PR.

The triangles PNQ and $O_{10}MP$ are similar, as are also the triangles $RN'S$ and $O_{10}MR$, having their respective sides perpendicular to each other. From these pairs of similar triangles,

$$\frac{V_{PO}}{V} = \frac{O_{10}P}{O_{10}M} \quad \text{and} \quad \frac{V_{RO}}{V'} = \frac{O_{10}R}{O_{10}M}$$

whence, since $V = V'$,

$$\frac{V_{PO}}{V_{RO}} = \frac{O_{10}P}{O_{10}R} \qquad \cdots \cdots \cdots \quad (64)$$

Since the velocities of P and R are at right angles to $O_{10}P$ and $O_{10}R$, and are proportional, respectively, to these distances, the rigid body 1 appears at the instant to be rotating about the point O_{10}. Therefore,

FIG. 57

O_{10} is the instantaneous center of the velocities of the points in the rigid body 1 relative to the fixed body O. The point O_{10} has no motion relative to 1 or O, and may be considered as a point common to both bodies at the instant. It should be noted that the above relation applies only to the relative motion of the two bodies, 1 and O, and that one of the bodies was considered as fixed for the sole purpose of clarifying the relations and of making the velocities definite. Both 1 and O might have motion relative to other bodies, and, therefore, have other instant centers relative to those bodies. It should be noted also that the instant center O_{10} applies only to the instantaneous velocities and not to the accelerations. Throughout this text the term *instant center* will be used in this customary sense. It is to be understood to be an abridgment of the longer expression, instantaneous center of velocity, and does not refer or apply to the center of acceleration, which will be dealt with in the next chapter.

The body 1 really rotates relative to O about an *instant axis*. Since the axis of rotation of a body is always perpendicular to the plane of rotation and pierces every section of the body parallel to the plane of motion at its center of rotation, the point O_{10} is the projection of the instant axis on the working plane, in this case the plane of the paper. This conclusion is in accordance with the deductions stated in **Art. 51.**

For greater simplicity in dealing with plane motion, it is customary to refer to instant centers rather than to the corresponding instant axes which they represent.

For plane motion the *instant center* of two bodies is the point about which one body may be considered to rotate at any instant relative to the other body. It is the point at which they have no relative velocity, and, hence, at which they have the same velocity. It is, for the instant, the only point common to both bodies. The use of instant centers will be found to be of great assistance in the analysis of complex motions and in the determination of the relations of the linear and angular velocities.

53. Permanent and Fixed Centers. For two bodies having plane motion, the instant center may be a fixed center, a permanent center, or a center that is neither fixed nor permanent. When one body is materially connected to another body by a turning pair, the instant center of one body relative to the other is either a fixed or a permanent center. Thus the crank of an engine is connected to the connecting-rod by means of a pin. The geometric center of the pin and its bearing is the *permanent* instant center of the two bodies. While a permanent center, it is not a fixed center, since it moves relative to the frame, the fixed member of the mechanism. The geometric center of the crank-shaft and its bearings is the instant center of the crank and frame. This center is not only the permanent center of these two members, but is also their *fixed* center since it has no motion relative to the frame. The instant center of the connecting-rod relative to the frame is neither a fixed nor a permanent center.

54. Notation. The instant center of the plane relative motion of two bodies *1* and *2* will be indicated by the symbol O_{12}. This symbol when used indicates the instant center of the plane motion of body *1* relative to body *2*, and also of body *2* relative to body *1*. Hence, whether *2* or *1* is regarded as the fixed body, the symbol for their instant center may be written O_{12} or O_{21}.

It is to be noted that the instant center O_{12} is a double point, since it is a point in *2* as well as a point in *1*. The velocity of any point in *1* relative to *2* is perpendicular to the radius from O_{12} to the point, and the velocity of any point in *2* relative to *1* is perpendicular to the radius from O_{12} to the point in *2*.

Throughout this book the links of mechanisms will be designated by numbers, and, in general, the frame, or fixed member, will be marked *O*, zero. While Fig. 57 does not represent a mechanism, it is to be noted that the bodies in this figure have been numbered in accordance with this scheme, the body regarded as fixed being marked *O*.

55. Location of Instant Centers. It is quite apparent from the discussion in Art. 52 that the location of the instant center of the plane motion of one rigid body relative to another depends only on the directions of motion of two points in one body relative to the other. Thus, in Fig. 58, let PQ and RS rep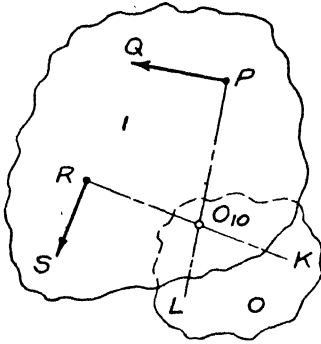resent the respective directions of motion at the instant 'of points P and R of body 1, relative to body O. Since PQ represents the direction of motion of point P, the point may be considered at the instant as rotating about any point in a line through P perpendicular to PQ. Likewise, point R at the instant may be considered as rotating about any point in a line through R perpendicular to RS. Where these two lines, PL and RK, intersect at O_{10} is the only point about which both points, and therefore, body 1, may be considered to rotate relative to body O at the instant. Hence, O_{10} is the instant center of body 1

FIG. 58

relative to body O. While, as brought out in Art. 52, the velocities of the respective points must be related in magnitude, only the directions of motion of the points are necessary to locate the instant center unless the vectors are parallel.

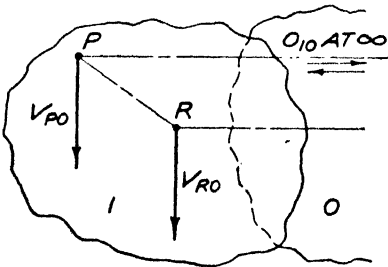If the velocities of any two points in a rigid body are both equal
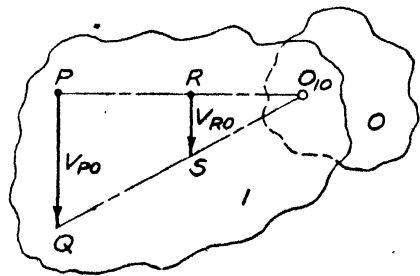

FIG. 59   FIG. 60

and parallel, the body has a motion of rectilinear translation. This is shown in Fig. 59. Since V_{PO} and V_{RO} are both equal and parallel, lines drawn through points P and R perpendicular to these velocity vectors would be parallel, and would, therefore, intersect at infinity. Hence, all rotational radii are of infinite length, and the body has a motion of rectilinear translation.

It is to be noted that the instantaneous velocities of two points in a rigid body cannot be parallel and *unequal* unless the line joining the points is perpendicular to the velocity vectors.   Brief reflection will show that this limitation is necessary to satisfy the assumption of rigidity. P and R in Fig. 60 are two such points in 1 having motion relative to the fixed body O.   Since the linear speeds of points in a rigid body are proportional to their distances from the center of rotation, the location of the instant center O_{10} can be found by the construction of two right triangles based on this proportionality. V_{PO} and V_{RO} are the velocities of the two points of body 1 relative to O.   Through the extremities of these vectors draw the line QS to intersect the line PR at O_{10}.   In the similar right triangles, $O_{10}RS$ and $O_{10}PQ$, thus formed,

$$\frac{V_{RO}}{V_{PO}} = \frac{O_{10}R}{O_{10}P} \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (65)$$
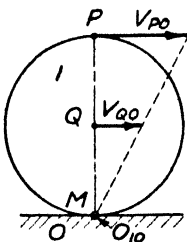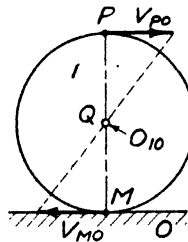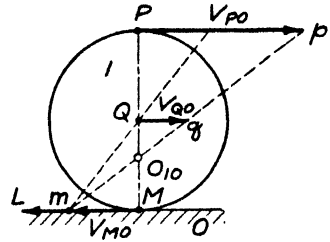

FIG. 61   FIG. 62   FIG. 63

This is merely a statement of the above proportionality, and shows that body 1 is at the instant rotating about O_{10} relative to body O considered as fixed.

Figs. 61, 62, and 63 illustrate three interesting practical examples of instant center location from the known velocities of certain points.   Let 1 be one of the driving wheels of a locomotive which runs on the horizontal track O.   In Fig. 61 it is assumed that the driving wheel 1 has a motion of pure rolling on the track O.   Since pure rolling means an absence of slip, the point of contact O_{10} between the wheel and the track is the only point of 1 that has no motion relative to O.   It is, therefore, the instant center about which each moves relative to the other.   Hence, if the wheel center Q is moving to the right with a velocity V_{QO}, the highest point P of the wheel, having an instant radius twice that of Q, will move with twice the velocity of Q.

In Fig. 62 it is assumed that the load of the train is so great that the locomotive driving wheels turn without starting the train from rest.

Hence the driving wheel *1* slips or slides but does not roll upon the track. Since there is no rolling of the wheel *1* on the track *O*, the wheel center *Q* can have no motion relative to the track, and is, therefore, the instant center O_{10} about which each moves relative to the other. Points *P* and *M*, having equal rotational radii, will have equal peripheral velocities, which velocity is also the velocity of sliding of *1* on *O*.

In Fig. 63 it is assumed that the load of the train is such that in starting the locomotive driving wheels both roll and slide. Suppose that when the peripheral velocity of the driving wheels relative to the locomotive frame is *ML*, the velocity of the train, and, therefore, of the wheel center *Q*, is observed to be V_{QO}. If there were no movement of *Q* the velocity of sliding between *1* and *O* would be *ML*. Since, however, the forward velocity of *Q* is V_{QO}, the velocity of sliding of *1* on *O* is the
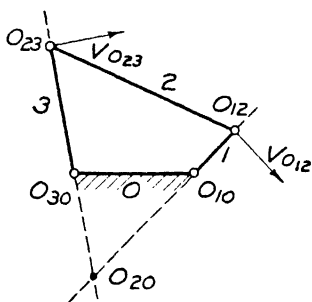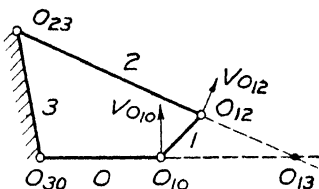

FIG. 64   FIG. 65

vector sum of *ML* and V_{QO}, or equal to V_{MO}. Hence we have the velocities of the two points *Q* and *M* in *1* relative to the track *O*, the velocities being parallel and unequal and normal to the line joining the points. This enables us to determine the instant center O_{10} and the velocity V_{PO} of point *P* relative to *O* by drawing the gauge line *mqp*. It is to be noted that the velocity of *P* relative to *O* is equal to the vector difference of twice the velocity of the train and the velocity of sliding. Thus,

$$V_{PO} = 2V_{QO} \rightarrow V_{MO}$$

56. Instant Centers of Four-Link Chains. Two common types of four-link chains are illustrated in Figs. 64, 65, 66, and 67. The relative motions of the adjacent links of mechanisms are usually quite simple, being, as a rule, rotation, oscillation, or rectilinear translation.

In Fig. 64 link *O* has been taken as the fixed, or reference, member. Evidently *1* and *3* rotate relative to *O* about the respective instant centers O_{10} and O_{30}. Since *O* is fixed O_{10} and O_{30} are also fixed centers.

The motion of 1 relative to 2 and of 2 relative to 3 are about the respective instant centers O_{12} and O_{23}. It is to be noted that these last two centers are permanent but not fixed centers. The remaining instant centers O_{20} and O_{13} are neither fixed nor permanent centers.

With link O fixed, as in Fig. 64, the instant center O_{20} for the motion of 2 relative to O can be found. Since 1 rotates about the fixed center O_{10}, the direction of motion of all points in the line $O_{12}O_{10}$ is perpendicular to $O_{12}O_{10}$. Hence, at the instant, the point O_{12}, which is common to both 1 and 2, may be considered as rotating about some point in the line $O_{12}O_{10}$ or its extension. For the same reason, the point O_{23}, as a point common to both 2 and 3, may be considered as rotating at the instant about some point in the line $O_{23}O_{30}$. The intersection O_{20} of lines $O_{12}O_{10}$ and $O_{23}O_{30}$ extended, is the only point common to both lines, and is, therefore, the only point about which points O_{12} and O_{23} can rotate at the instant. Since points O_{12} and O_{23} are points in 2 as well as respective points in 1 and 3, O_{20} is the instant center of 2 with respect to O.

If link 3 is fixed, as in Fig. 65, the same reasoning as applied in the preceding paragraph will locate the instant center O_{13} at the intersection of the lines $O_{23}O_{12}$ and $O_{30}O_{10}$ extended.

Fig. 66 illustrates the slider crank chain. As in Fig. 64, the fixed center O_{10} and the permanent centers O_{12} and O_{23} are determined by inspection. The location of the fixed center O_{30} is not so evident. Since 3 has a motion of rectilinear translation, the velocities of all points in 3 are equal and parallel. Hence, the rotational radius of each point is of infinite length, and the instant center O_{30} for any point such as O_{23}, common to both 2 and 3, is located at infinity on a line through O_{23} perpendicular to the path of 3. Center O_{23} as a point in 3 may be considered, at the instant, as rotating about some point in the line $O_{23}O_{30}$; and O_{12} as a point in 1 may be considered, at the instant, as rotating about some point in the line $O_{10}O_{12}$. Hence, the intersection O_{20} of $O_{10}O_{12}$ and $O_{23}O_{30}$ extended, is the only point about which both points O_{23} and O_{12} can rotate at the instant. Since these points are also points in 2 and since the motion of two points in a member determines the motion of the member, O_{20} is the instant center of 2 relative to O.

If, as in Fig. 67, member 3 is considered as fixed, member O would have a motion of rectilinear translation, and the instant center of all points in O would be at infinity. Hence, O_{10} as a point in O is moving relative to 3 about a center O_{30} located at infinity on a line through O_{10} perpendicular to the path of O. At the instant O_{10} may be considered to rotate about some point in the line $O_{10}O_{30}$, and O_{12} may be considered to rotate at the instant about some point in the line $O_{23}O_{12}$.

Hence, the intersection O_{13} of lines $O_{10}O_{30}$ and $O_{23}O_{12}$ is the only point about which points O_{10} and O_{12} can rotate at the instant. Since these points are also points in 1, and since the motion of two points in a member determines the motion of the member, O_{13} is the instant center of 1 relative to 3.

The instant center O_{13} may also be located by the use of Fig. 66, where O is the fixed member, by imagining 3 extended to include O_{10} as a coincident point at the instant. As a point in 3 it may be considered at the instant to rotate about some point in a line through O_{10} parallel to $O_{23}O_{30}$, and O_{23} may be considered at the instant to rotate about some point in line $O_{23}O_{12}$. The intersection of these two lines is obviously the instant center O_{13}. It must not be forgotten that O_{30} is the instant center of O with respect to 3 as well as of 3 with respect to O.
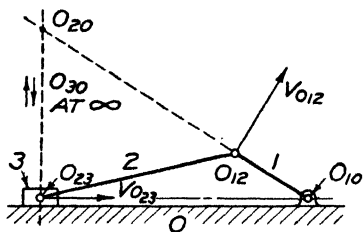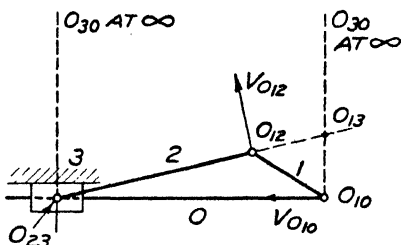

FIG. 66   FIG. 67

57. Number of Instant Centers. Kinematic chains composed of four links were considered in Art. 56. Since an instant center governs the relative motion of two links, six instant centers, corresponding to the possible number of pairs for four links, were found for each four-link chain. The number of instant centers N for any kinematic chain composed of n links is the number of possible combinations of the links into pairs, that is, the number of combinations of n objects taken two at a time. Hence,

$$N = \frac{n(n-1)}{2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (66)$$

A method of locating the instant centers of kinematic chains from the possible motions of the links was demonstrated in Art. 56. It is obvious that this method, which was found to be quite complicated for four-link chains, would be much more complicated for kinematic chains of six, seven, eight, or more links. The following articles describe a much simpler and more practical method of locating instant centers.

58. Kennedy's Theorem. The application of this important theorem is very helpful in locating the instant centers of any kinematic chain and will be found to save much time. The statement of the theorem as given by Professor Kennedy is: " *If any three bodies O, 1, and 3 have plane motion, their virtual (instant) centers O_{10}, O_{30}, and O_{13}, are three points upon one straight line.*" This theorem applies to any three bodies having plane motion. With two of the centers known, it can be shown that no point lying outside of the line through the known centers can be the required center. The third center must, therefore, lie on a line through the other two points as stated in the theorem.

In Fig. 68 let O, 1, and 3 be any three bodies moving parallel to a fixed plane. Suppose, at a given instant, that 1 moves relative to O about the center O_{10}, and that 3 moves relative to O about the center O_{30}. Center O_{10} is a point common to 1 and O, and center O_{30} is a point
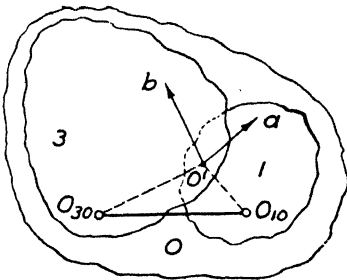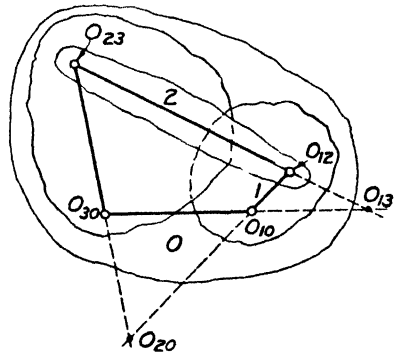

FIG. 68   FIG. 69

common to 3 and O. Assume any point off the line through O_{10} and O_{30}, such as O', to be the instant center of the relative motion between 1 and 3, thus making O' a point common to 1 and 3. All points in 1 must rotate relative to O about O_{10} and all points in 3 must rotate relative to O about O_{30}. Center O' as a point in 1 must move relative to O in a direction $O'a$ perpendicular to the instant radius $O_{10}O'$, while as a point in 3 it must move relative to O in a direction $O'b$ perpendicular to the instant radius $O_{30}O'$. Since, at a given instant, a point can have but one direction of motion relative to a given body, the perpendiculars $O'a$ and $O'b$ should coincide. This is possible only when $O_{10}O'$ and $O_{30}O'$ are in one straight line, that is, in a line through the given centers O_{10} and O_{30}. It is thus evident that O' cannot be the center required, and that O_{13}, the center required, must be on a line through the centers O_{10} and O_{30}. Just where on this line O_{13} is located is not indicated by the

theorem. This is to be expected, since the relative motion of *1* and *3* is not definitely constrained. Hence, to definitely locate O_{13}, the members *1* and *3* must be definitely constrained by being connected by a fourth member.

In Fig. 69 members *1* and *3* have been given definite, relative constrained motion by the introduction of a fourth member *2*. Member *2* is connected to *1* at O_{12} and to *3* at O_{23}. According to the theorem O_{13} is not only on the line of centers of the bodies *O*, *1*, and *3*, but is also on the line of centers of the bodies *1*, *2*, and *3*. Hence, O_{13} is at the intersection of a line through O_{30} and O_{10} with a line through O_{23} and O_{12}. It is to be noted that the system of notation makes it evident by inspection what third center is to be found on a line passing through any two known centers. Thus, a line passing through O_{12} and O_{10} must contain the center O_{20}, *2* and *0* being the numbers not common to the two known centers. This same center, O_{20}, is likewise on a line through O_{23} and O_{30}, which fact definitely locates O_{20}.

59. Application of Kennedy's Theorem. Fig. 70 will be used to demonstrate the method of procedure in locating instant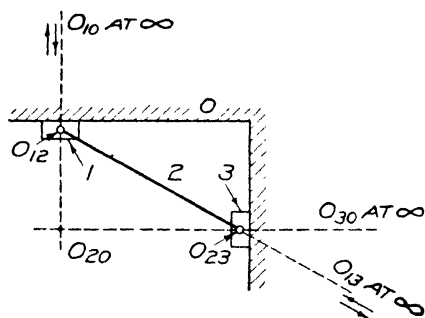 centers by means of Kennedy's theorem. The figure illustrates a unique but not uncommon four-link chain. Link *1* sliding horizontally on *O* is connected by link *2* to link *3* which slides vertically on *O*. Link *2* is pivoted to *1* and *3* at the respective permanent centers O_{12} and O_{23}. Since *1* and *3* have a motion of rectilinear translation relative to *O*, their instant centers, O_{10} and O_{30}, are at infinity. Instant centers O_{10} for point O_{12} in *1* is at infinity on a line through O_{12} perpendicular to the path of *1*, and instant center O_{30} for point O_{23} in *3* is at infinity on a line through O_{23} perpendicular to the path of *3*. For the three links *1*, *2*, and *O*, Kennedy's theorem states that their three instant centers O_{10}, O_{12}, and O_{20} lie on one straight line. The known centers O_{10} and O_{12} locate this line which contains the unknown center O_{20}. Similarly for links *2*, *3*, and *O*, the instant center O_{20} is on the line through the centers O_{30} and O_{23}. Since O_{20} is on both lines, their intersection definitely determines the location of the instant center O_{20}.

FIG. 70

The three centers O_{12}, O_{23}, and O_{13}, for links *1*, *2*, and *3*, lie on a line through the known centers O_{12} and O_{23}. Likewise, the instant center

O_{13} lies on a line through the centers O_{10} and O_{30}. Since O_{10} and O_{30} are at infinity, a line through them would intersect a line through O_{12} and O_{23} at infinity. Hence, the instant center O_{13} is at infinity on a line through O_{12} and O_{23}.

The number of links of a chain which are of infinite length is indicated by the instant centers which are located at infinity. Since each instant center is a point common to two links, an instant center at infinity means that there are two links of infinite length. It is, therefore, impossible to have a single link of infinite length. Hence a four-link chain may have four finite links, two finite and two infinite links, one finite and three infinite links, or four infinite links. The four-link chain of Fig. 70 may be taken as an illustration. This chain is called the double slider-crank chain, and is found in the Scotch yoke, in the Oldham coupling, and in instruments for drawing ellipses. It has already been shown that the instant centers O_{10}, O_{30}, and O_{13} are located at infinity, and that the other instant centers are in finite locations. Hence, links O, 1, and 3 must be infinitely long to include their instant centers, and link 2 must be of finite length. The chain is, therefore, composed of one finite and three infinite links.

60. Instant Centers of Incomplete Four-Link Chains. Fig. 71 illustrates an incomplete four-link chain made up of a frame O, a cam 1 and a flat-faced follower 2. The instant centers O_{10} and O_{20} of the
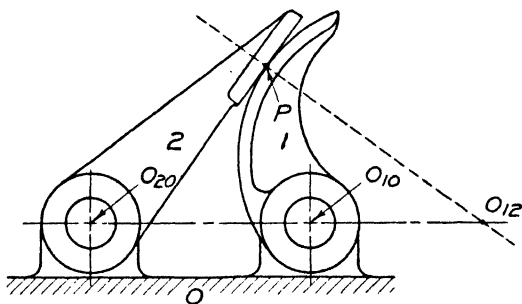

FIG. 71

respective motions of 1 and 2 relative to O are fixed centers, and their locations are obvious. The only remaining center is O_{12}, which, by Kennedy's theorem, must be somewhere on the line through O_{10} and O_{20}. But, there being only three links, the theorem does not yield any other line to definitely determine the location of the center. Information may be obtained, however, from the motions of the various members.

The instant center O_{12} has to do with the relative motion of 1 and 2. It was demonstrated in Art. 43 that P as a point in 1 can have no motion relative to 2 along the common normal through P, and that the only motion that P as a point in 1 can have relative to 2 is along the common tangent through P. Hence, it is known that P as a point in 1 can have a direction of motion relative to 2 only along the common tangent, or

perpendicular to the common normal. The instant center O_{12} of the motion of *1* relative to *2* must, therefore, be on the common normal and hence at the intersection of this normal with the line through O_{20} and O_{10}. This, or some similar method of reasoning, must be used in the analysis of incomplete four-link chains.

61. Effect of Inversion on Instant Centers. It was explained in Art. 37 that a mechanism is a kinematic chain with one of its links fixed. As many mechanisms can therefore be obtained from a given kinematic chain as there are links in the chain. It was explained also that this process of obtaining different mechanisms from the same kinematic chain is known as inversion. Thus Figs. 38, 39, 40, and 41 illustrate mechanisms that are inversions of the same kinematic chain, the slider-crank chain. It is to be noted that the determination of the location of the instant centers is unaffected by the inversion, or, in other words, is unaffected by the particular link of the chain that is fixed to obtain the mechanism. Hence, the instant centers of a kinematic chain in some phase can be located, and all of the possible mechanisms that can be obtained by inversion can be studied without changing or adding any line to the original figure.

As the subject is pursued it will be discovered that there are only a few elementary kinematic chains, and that the apparent multiplicity of mechanisms is due to changes in these chains by inversion or by changing the proportions of the links or by the addition of other links or chains.

62. Instant Centers of Compound Kinematic Chains. The use of Kennedy's theorem in determining the instant centers of a simple four-link chain was explained in Art. 59. While the same general method is applicable to compound chains, it is found necessary in addition to analyze the chain and to keep a record of the known and unknown centers.

Almost any compound chain may be considered from the standpoint of having in its composition at least one simple chain to which have been added extra links or even extra kinematic chains. Therefore, a compound chain should be analyzed and the instant centers of the four-link chains located before the added links are considered. In Figs. 73 and 74 are illustrated two forms of charts which are considered to be the most useful of the many schemes for the purpose of recording the analysis of compound chains.

The Whitworth shaper mechanism, illustrated in Fig. 72, has been chosen for analysis. In this mechanism, as crank *1* rotates, the block *2*, to which it is pivoted, slides on *3* and causes *3* to rotate. The rotation of *3*, through the connecting-rod *4*, causes the block *5* to reciprocate. In an actual shaper *5* is the reciprocating ram which carries the cutting

tool. The length of the stroke of the ram is changed by shortening or lengthening that part of 3 which constitutes the crank member of the common slider-crank mechanism made up of links $3, 4, 5$, and O, illustrated in Fig. 38. The links $1, 2, 3$, and O constitute the fixed-crank inversion of the slider-crank chain illustrated in Fig. 39. Let the four-link chain made up of $3, 4, 5$, and O be designated as simple chain a,
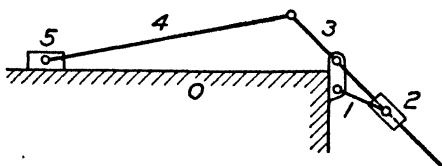

FIG. 72

	1	2	3	4	5
0	b	b	ab	a	a
1		b	b		
2			b		
3				a	a
4					a

FIG. 73

and the four-link chain made up of $1, 2, 3$, and O be designated as simple chain b.

The compound linkage illustrated in Fig. 72 is composed of 6 links. From equation (66) it is known that it has $\dfrac{6(6-1)}{2}$, or 15 instant centers. These centers, some of which are known, are represented by the unshaded squares of the chart shown in Fig. 73. Chain a has 6 instant

0	1	4
0	b	a
2	b	
3	b	a
5		a

0	1	5
0	b	a
2	b	
3	b	a
4		a

0	2	4
0	b	a
1	b	
3	b	a
5		a

0	2	5
0	b	a
1	b	
3	b	a
4		a

FIG. 74

centers. The letter a has been entered in the proper squares of the chart to represent these centers. Likewise chain b alone has 6 instant centers. The letter b has been entered in the squares of the chart to represent these centers. It is to be noted that since links 3 and O are common to chains a and b, the instant center O_{30} is common to each, and is so shown in the chart by the letters a and b falling in the same square. Hence, the location of 11 of the 15 instant centers of the compound linkage is obvious. The location of the 4 unknown centers O_{14}, O_{15}, O_{24}, and O_{25} of the compound linkage, indicated by the blank squares of Fig. 73, are yet to be determined. From Kennedy's theorem

it is known that instant center O_{14} is on a line through O_{10} and O_{40}, or through O_{12} and O_{24}, or through O_{13} and O_{34}, or through O_{15} and O_{45}. These facts are represented by the first chart to the left in Fig. 74, and similar charts are shown for the remaining unknown centers O_{15}, O_{24}, and O_{25}. By recording the known centers in these charts, it is at once apparent whether there is sufficient information at hand to determine the location of the unknown instant centers. The first chart to the left in Fig. 74 shows that we have instant centers O_{10} and O_{40}, also O_{13} and O_{34}. The intersection of the two lines through these pairs of centers determines the location of instant center O_{14}. Having O_{14} adds one more center to two of the remaining charts in case it is needed to determine one or all of the remaining centers.

Locating the instant centers of compound kinematic chains consists largely in adhering to a consistent notation and of applying a well-devised scheme for recording the centers found by Kennedy's theorem. The compound chain should be analyzed for what simple chains it contains, and then the centers for each simple chain should be located and recorded. The information thus gained enables the remaining centers of the compound linkage to be located. When all the centers have been located for a six-link chain, each center will be found to be located at the intersection of four lines, the lines indicated by the individual charts constructed for each center to be located. The location of a center on so many lines makes checking convenient, although some of these lines may coincide or may be unknown because their locating centers are not on the paper.

63. Centrodes and Axodes. With one link of a kinematic chain fixed, all the instant centers except the fixed centers will trace paths on the fixed link or an extension of it. The paths so traced are called *fixed centrodes*. The path traced by an instant center on a moving link, or an extension of it, is called a *moving centrode*. The surfaces generated by the instant axes are called *axodes*. Hence, an axode is a surface perpendicular to the plane of motion, and a centrode is the intersection of an axode with the plane of motion or with a parallel plane.

The kinematic chain shown in Fig. 75 with link O fixed, may be used as an illustration. Instant centers O_{10} and O_{30} are fixed centers, and O_{12} and O_{23} are permanent centers. The permanent center O_{12} will trace a fixed centrode in the form of a circular arc with O_{10} as a center and the length of link *1* as a radius. The permanent center O_{23} will trace a similar path on the fixed link. Hence, so far as the constrainment of link *2* is concerned, two slots in the form of circular arcs could be cut in the fixed member in lieu of links *1* and *3*. For the phase shown, link *2* at the instant is rotating about the center O_{20}. Since this instant

center is neither fixed nor permanent, its location changes with the relative motion of 2 and O. As 1 is rotated clockwise until it coincides with O extended, this center traces the path $O_{20}O_{30}$ on the fixed member, and traces the path $O_{20}O_{10}$ as link 3 is rotated counter-clockwise until it coincides with O extended. Hence, $O_{10}O_{20}O_{30}$ is a portion of the *fixed centrode* for the motion of 2 relative to O. Since O_{20} is a point common to 2 and O, it likewise traces a path on link 2 as motion progresses. This path is the *moving centrode* for the motion of 2 relative to O. Now O_{20} is the center about which O rotates at the instant relative to 2 as well as being the center about which 2 rotates relative to O. The fixed centrode traced by O_{20} with 2 as the fixed link is, therefore, the same as the moving centrode traced on 2 by O_{20} with O as the fixed link.

With 2 fixed, O_{20} traces the curve $O_{20}O_{23}$ on link 2 as 1 is rotated clockwise until it coincides with 2, and traces the curve $O_{20}O_{12}$ as 3 is rotated counter-clockwise until it coincides with 2. Hence, $O_{12}O_{20}O_{23}$ is a ·portion of the moving centrode for the motion of 2 relative to O. It is to be noted that the moving centrode M is tangent to the fixed centrode F at the instant center O_{20} for the phase considered. No slipping can occur at O_{20}, since it is a point common to 2 and O and is instantaneously at rest relative
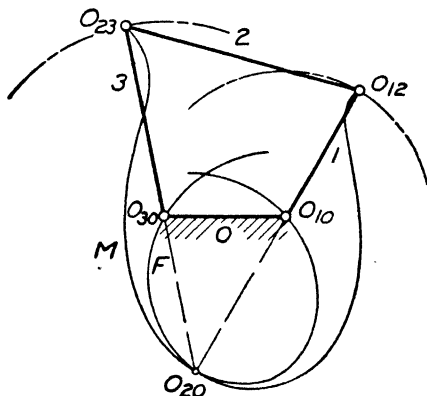

FIG. 75

to both. It follows, therefore, that there is no slipping between these centrodes as motion continues, and that the pure rolling of M on F in the absence of links 1 and 3 would result in identically the same motion of 2 relative to O as given by the original four-link chain. Using F as the moving and M as the fixed centrode would give the same motion of O relative to 2 as given by the original four-link chain with link 2 fixed.

To make a practical mechanism by using only the two links M and F to secure the given motion of 2 relative to O would require that the centrodes M and F be fitted with suitably mating teeth in order that they could be rolled together without slipping. As in the present example, this is often possible for only a portion of the motion of one link relative to another as given by the original linkage. Even where the rolling centrodes are of such form as to be readily supplied with

teeth, it is not always easy or possible to secure complete constrainment. If, in the present example, M and F were used as pitch curves for teeth, the most obvious way of keeping them in mesh would be to introduce links *1* and *3*. While theoretically the motion of any link relative to some other link regarded as fixed may be shown to be equivalent to a *pair of rolling centrodes*, the practical application of this fact may be attended with unsurmountable difficulties.

For the mechanism shown in Fig. 76 the instant center O_{20} as a point in O is always at the intersection of two lines through O_{12} and O_{23} at right angles to the guide slots in O. Since these slots are at right angles, the distance from C to O_{20} is always equal to $O_{12}O_{23}$. Hence, the *fixed centrode*, F, for the motion of *2* relative to O, is a circle having its center at C and a radius equal to $O_{12}O_{23}$. The *moving centrode* for the motion of *2* relative to O may be found
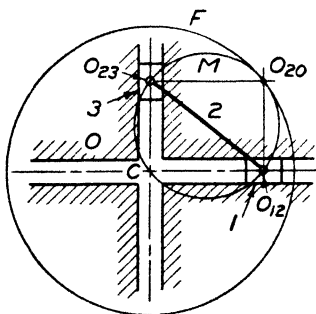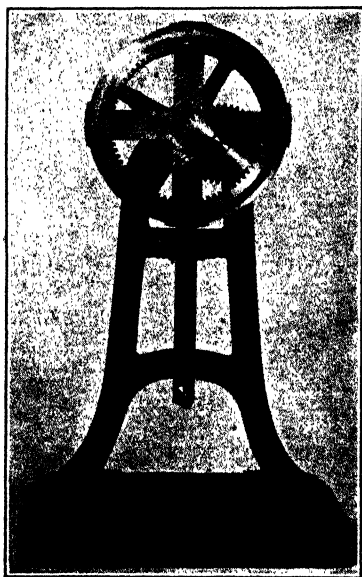

FIG. 76

FIG. 77.

by making *2* the fixed link. With *2* fixed, it is evident that the instant center O_{20} is always the apex of a right-angled triangle formed on $O_{12}O_{23}$ as an hypotenuse. Hence, the locus of O_{20}, or the moving centrode, M, is a circle on $O_{12}O_{23}$ as a diameter. These centrodes may be used as the pitch circles of mating gears. If the moving centrode gear is carried on a crank having its shaft center at C and a radius equal to one-half of $O_{12}O_{23}$, any point on the pitch circle of the moving gear will travel in a straight line as illustrated by the straight-line motion shown in Fig. 77.

In Fig. 78 is shown a right cylinder *2* rolling on a fixed track O. For pure rolling action, the instant center O_{20} is always at the point of

contact between the wheel and the track. Hence, the track is the fixed centrode, and the circumference of the wheel is the moving centrode. When supplied with teeth, this combination becomes the familiar rack and pinion.

Actual mechanisms were used above to show that the motion of one body relative to another is equivalent to the rolling of a pair of centrodes. It can be shown that if the motion of one body relative to another is known, the rolling centrodes to produce the desired motion can be determined without having in mind an actual mechanism for the purpose. In Fig. 79 the desired plane motion of one body relative to another body O, regarded as fixed, is represented by the motion of two points B and C of the body. The initial position of the points B
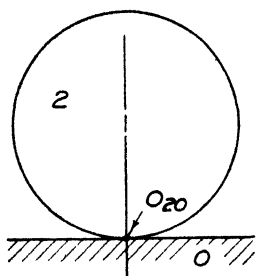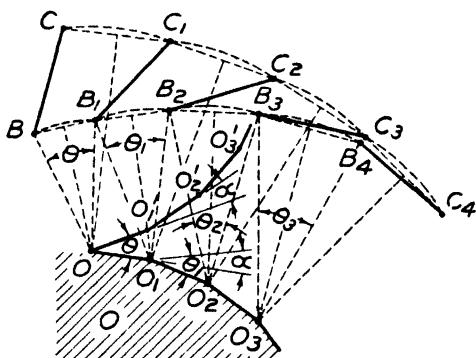

FIG. 78   FIG. 79

and C of the moving body is represented by the line BC, and the successive positions of the body by the lines B_1C_1, B_2C_2, etc. Obviously, the intersection of the perpendiculars drawn from the mid-points of the lines BB_1 and CC_1 locates the center O about which the moving body may be considered to rotate relative to the fixed body O as the moving body is displaced from BC to B_1C_1. During this displacement the moving body has rotated through an angle θ. In a similar manner the centers O_1, O_2, etc., and the angles θ_1, θ_2, etc., may be found. The centers O, O_1, O_2, etc., are points on the fixed centrode. Corresponding points on the moving centrode may now be found. Point O'_1, corresponding to O_1, is found by drawing line OO'_1 equal to and at an angle θ with line OO_1. When O'_1 is in contact with O_1, line $O'_1O'_2$ must make an angle θ_1 with O_1O_2. That this may be true, angle α must be determined. Obviously angle α is the angle made by the extension of OO_1 with a line drawn from O_1 at an angle θ_1 with O_1O_2. Point O'_2 is then determined by making $O'_1O'_2$ equal to O_1O_2 and at an angle α with the

extension of OO'_1. *Other points on the moving centrode, such as* O'_3, *are found in a similar manner.* *Practical rolling centrodes may be* obtained by drawing smooth curves through each series of points found. The greater the number of points obtained between the initial and final positions the more nearly will these curves approach the true centrodes.

64. Linear Velocities by the Use of Instant Centers. In determining the linear velocity of any point in a mechanism by the use of instant centers, the following principles are applied:

(*a*) If one rigid body is moving relative to another, the direction of motion of any point in the first body relative to the second is perpendicular to the radius of the point from the instant center of motion, and the magnitude of the velocity is proportional to the length of the instant radius of the point.
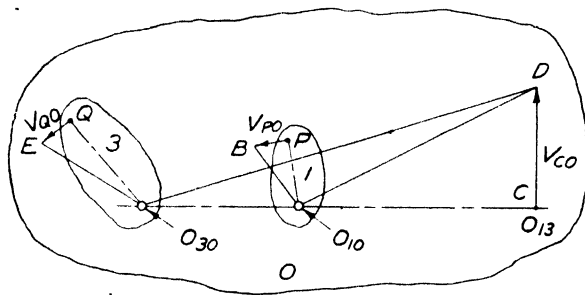

FIG. 80

(*b*) The instant center of the relative motion of two bodies is the point at which they have no relative motion and hence at which they have the same motion relative to a third body.

The general method of determining the linear velocities of points in a mechanism by the use of instant centers will first be developed before applying the method to an actual mechanism. In Figs. 80 and 81, the three rigid bodies *O*, *1*, and *3* have relative plane motion. To gain in simplicity, the necessary connection between *1* and *3* to give them definite relative constrained motion is omitted. The instant centers O_{10} and O_{30} govern the motions of all of the respective points of *1* and *3* relative to the reference member *O*. These centers will be called *pivot points*. By Kennedy's theorem the instant center O_{13} is on the line through O_{10} and O_{30}, its exact location depending upon the nature of the connection between *1* and *3*. The assumed location of O_{13} is shown in Figs. 80 and 81.

If, in Fig. 80, the velocity V_{PO} of a point *P* in *1* is known, the corresponding velocity V_{QO} of any point such as *Q* in *3* can be found by

going step by step through the mechanism from point P in 1 to point Q in 3. Since, however, the instant center O_{13} is a point in both 1 and 3, it may be used as a *transfer point*, and the velocity of Q may be found by transferring directly from 1 to 3. The velocities of P and O_{13}, or C, are proportional to their distances from their center of rotation, O_{10}. Hence,

$$\frac{V_{CO}}{V_{PO}} = \frac{O_{10}C}{O_{10}P}$$

Evidently the velocity of O_{13}, or C, as a point in 1 may be determined graphically by employing two similar right triangles with their vertices at the pivot point O_{10}. Thus V_{CO} has been determined in Fig. 80 by constructing the right triangle $O_{10}CD$ similar to the known right triangle
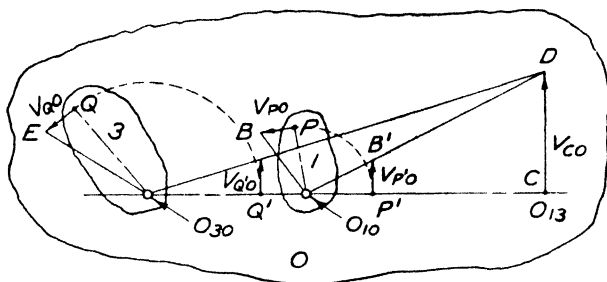

FIG. 81

$O_{10}PB$. Since O_{13} is a point in 3 as well as in 1 and since the speeds of all the points in 3 are proportional to the distances of the points from the center of rotation O_{30}, it follows that

$$\frac{V_{QO}}{V_{CO}} = \frac{O_{30}Q}{O_{30}C}$$

The velocity of Q may be determined by employing, as before, two similar right triangles. Thus, since V_{CO} is known, the required velocity V_{QO} is found by constructing the triangle $O_{30}QE$ similar to the known right triangle $O_{30}CD$.

Since any point in 1 at the same distance from O_{10} as P will have the same speed as P, and since any point in 3 at the same distance from O_{30} as Q will have the same speed as Q, the graphical construction for determining V_{QO} may be considerably simplified over that shown in Fig. 80 by employing the construction shown in Fig. 81. Point P and its known velocity vector are first rotated about O_{10} into the position $P'B'$. The intersection of a gauge line through O_{10} and B' with a line through

O_{13} parallel to $P'B'$, determines the velocity vector V_{co}. Having V_{co} the velocity of Q is readily determined. Point Q is first revolved to Q'. Where a gauge line through O_{30} and D intersects a line through Q' parallel to CD determines $V_{Q'o}$, which may be revolved to its true position at Q.

65. Application of the Instant Center Method of Determining Velocities. For the four-linked mechanism in Fig. 82, it is required to find the linear velocity of R relative to the fixed member O having given the linear velocity of P relative to the fixed member. The known velocity is that of a point in 1, and, as this velocity is with reference to O, the pivot point of the known velocity is O_{10}. The unknown
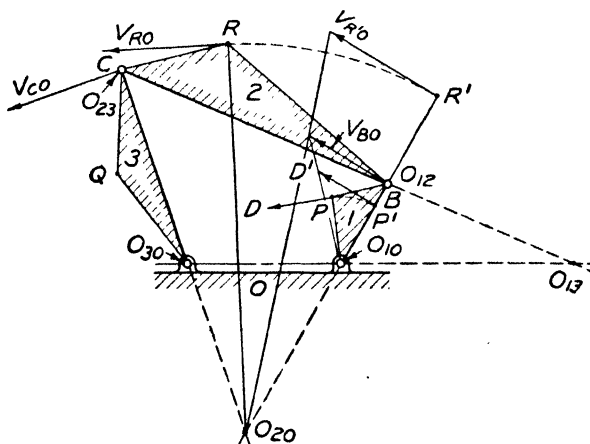

FIG. 82

, velocity is that of R, a point in 2, and, as this velocity is with reference to O, the pivot point of the unknown velocity is O_{20}. Since the velocities are being transferred from 1 to 2, the direct transfer point is O_{12}. The solution is concerned with O, 1, and 2 only, and the natural base line is the line on which are found the three centers O_{10}, O_{20}, and O_{12}.

Rotate P and its vector about its pivot O_{10} to the position P' on the base line, and draw the gauge line $O_{10}D'$. The intersection of this gauge line with a perpendicular to the base line through O_{12} determines the velocity vector V_{BO}. Since O_{12} is a point in 2 as well as in 1, the velocity of one point in 2 is known. By drawing a gauge line from the pivot point O_{20} through the terminus of vector V_{BO} the velocity of any other point of 2 may be found. Revolve R about its pivot point O_{20} to the position R' in the base line. The intersection of the gauge line from O_{20} with a perpendicular to the base line through R' determines

the length of the velocity vector $V_{R'O}$. This vector when rotated back to R represents the desired velocity, V_{RO}.

The method of determining the velocity of Q by the use of the transfer point O_{13} has already been covered in Art. 64. In finding above the velocity of R, a gauge line was drawn which makes it possible to find the velocity of any other point in 2 in the same manner as the velocity of R was found. Hence, the velocity of O_{23}, or C, a point common to both 2 and 3, can be found. By drawing a gauge line from the pivot O_{30} to the terminus of vector V_{CO} the velocity of Q may be determined.

Occasionally the velocity relations of some member as 2 in Fig. 83 are wanted relative to a member O, and the pivot point O_{20} is so far removed as not to be available for use. In such a situation, the construction shown in Fig. 83 will be found convenient. Suppose that the velocity of B relative to O is known and· that the velocity of C relative to O is desired. On $O_{10}B$ extended lay off BM equal to the length of vector V_{BO}, and draw MN parallel to BC, cutting $O_{30}C$ extended at N. The intercept CN will be equal in length to the desired vector V_{CO}, which vector in its true
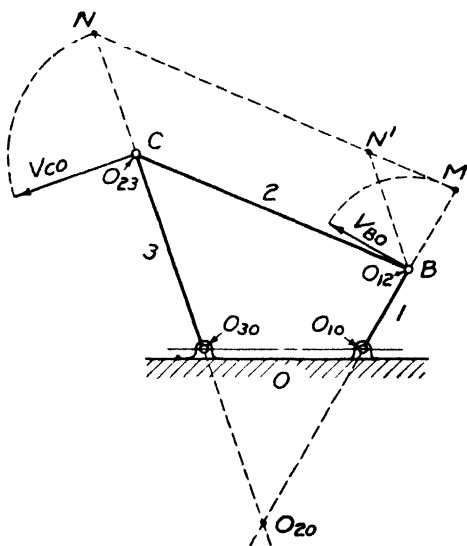

FIG. 83

position is perpendicular to its rotational radius $O_{30}C$ at C. The proof of this construction rests upon the fact that triangles BCO_{20} and MNO_{20} are similar. By drawing BN' parallel to CN, we may write

$$\frac{BM}{BN'} = \frac{O_{20}B}{O_{20}C}$$

But the velocities V_{BO} and V_{CO} are respectively proportional to their rotational radii, or

$$\frac{V_{BO}}{V_{CO}} = \frac{O_{20}B}{O_{20}C}$$

Hence,

$$\frac{V_{BO}}{V_{CO}} = \frac{BM}{BN'}$$

Since *BM* was made equal in length to vector V_{BO}, it follows that $BN' = CN$ must be equal in length to the desired velocity vector V_{CO}. The velocity of *R* in Fig. 82 could have been found in a similar manner.

Fig. 84 shows how the *parallel line construction* outlined above may be applied to determine the velocity of the cross-head, or point *C*, and the velocities of points *P* and *Q* on the connecting-rod of a slider-crank mechanism, when the velocity, V_{BO}, of the crank-pin is given. Pivot point O_{20}, being at the intersection of a line through O_{10} and O_{12} with a line through O_{23} normal to $O_{23}O_{10}$, is not available for use. As shown, *Bb* is made equal to the length of the known vector V_{BO}, and line *bc* is drawn parallel to *BC*. In accordance with the proof above, *Cc* represents the length of the desired vector V_{CO} to the same scale that V_{BO} represents the known velocity of *B*. Hence, vector V_{CO} is equal in length to *Cc*, and is perpendicular to *Cc* at *C*. The point *q* in line *cb* is located from the proportionality

FIG. 84

$$\frac{cq}{cb} = \frac{CQ}{CB}$$

Obviously a line through *Q* and *q* will pass through the pivot point O_{20}, and *Qq* will represent the length of the desired vector V_{QO} which is perpendicular to *Qq* at *Q*. Lines through *b* and *c* parallel, respectively, to *BP* and *CP* will intersect at *p*, and the line joining *P* and *p* will represent the length of the desired vector V_{PO} which is perpendicular to *Pp* at *P*.

Even where the required instant centers are available, the so-called parallel line construction demonstrated above in connection with Figs. 83 and 84 will frequently be found more convenient to apply in determining the velocities of points in a mechanism than the triangular construction demonstrated in connection with Figs. 81 and 82.

The instant center method of finding the magnitude and direction of the velocity of any point in a mechanism will, in general, be found more convenient to apply and more certain and accurate in its results than the determination of velocities by the method of resolution.

66. Linear Velocities by Resolution. Fig. 85 shows a construction for determining, by the method of resolution, the linear velocity V_{DO} of the cross-head, link 5, for a given velocity V_{BO} of point B common to links 1, 2, and 3. That component of the velocity of B which is perpendicular to the direction of motion of 2 relative to 3, imparts to 3 a motion of rotation about O_{30}. This component, V''_{BO}, is the velocity of point B as a point in 3.

The velocity of C as a point in 3 relative to O is perpendicular to $O_{30}C$. Hence, the length of the velocity vector V_{CO} is determined by the intersection of a gauge line from the terminus of V'_{BO} through O_{30} with a line
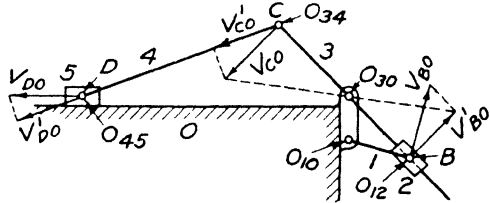
FIG. 85

from C perpendicular to $O_{30}C$. The component of V_{CO} along CD is V'_{CO}. Since link 4 is rigid, the velocity component, V'_{DO}, of D relative to O is equal to V'_{CO}. The intersection of a perpendicular through the terminus of V'_{DO} with a line through D, in the known direction of its motion, determines the desired velocity vector V_{DO}.

In velocity determinations it may happen that the point whose velocity is desired is not a permanent center conveniently located as in the previous example. Thus in Fig. 86 let it be required to find the velocity of P in link 2, having given the velocity V_{BO} of point B. The
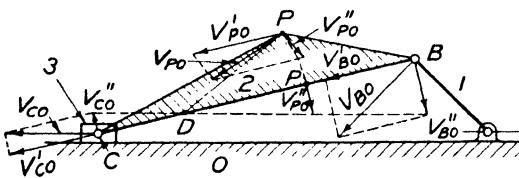
FIG. 86

construction shown treats this velocity as composed of a translational and a rotational component. Link 2 may be regarded at the instant as being translated parallel to BC while being simultaneously rotated about some point in BC. Accordingly V'_{BO} and V''_{BO} become the ·translational and rotational components of B as a point in link 2. Since link 2 is rigid, the respective translational components, V'_{CO} and V'_{PO}, of C and P must be equal and parallel to V'_{BO}. The direction of motion of C being known, V_{CO} and the rotational component V''_{CO} are readily found as shown. The intersection D of line BC with a gauge line drawn through the termini of V''_{CO} and V''_{BO}, is the center about which 2 is rotating. Hence, the rotational com-

ponent of P is normal to DP and must have a *length in accordance* with the proportionality

$$\frac{V'''_{PO}}{V''_{BO}} = \frac{DP}{DB}$$

The desired velocity V_{PO} of P relative to O is the resultant of the translational and rotational components, V'_{PO} and V''_{PO}.

In the determination of the motion of a point by resolution, its velocity is regarded as the resultant of two components. Each of these components must be determined in magnitude and direction. The instant center method of determining velocities rests upon the fact that each point of a body may be considered as rotating about the instant center of the body. From this it follows that the direction of motion of any point is perpendicular to its instant radius, and the magnitude of its velocity is proportional to the length of the instant radius. The magnitude and direction of the velocity of a point are, therefore, determined with more assurance and certainty by the method of instant centers than by the method of resolution. Because of this and also because of the danger of confusing the resolution of forces with the resolution of velocities, the instant center method is to be preferred for general use to the method of resolution.

67. Angular Velocity Theorem. In many kinematic problems it is necessary to determine the ratio of the angular velocities of two members of a mechanism with respect to a third member which may be either stationary or in motion. For such determinations the following theorem is valuable for its simplicity and generality.

Let O, 1, and 3, in Fig. 87 represent any three bodies having plane motion. It will be assumed that O is the reference body. The instant
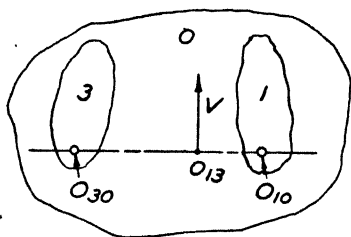
FIG. 87

centers O_{10} and O_{30} are the *pivot* points about which bodies 1 and 3 rotate relative to the reference body. In accordance with Kennedy's theorem, the instant center O_{13}, a point common to 1 and 3, must be on the same straight line as O_{10} and O_{30}. In the absence of any definite connection between 1 and 3, the location of this instant center, or *transfer point*, has been arbitrarily assumed. Since, as before stated, the transfer point O_{13} is a point common to 1 and 3, it has, as a point in 3, the same linear velocity V relative to O that it has as a point in 1. Also, since O_{13} must be in the same straight line as O_{10}

and O_{30}, its linear velocity V is perpendicular to this line. Considering the transfer point, O_{13}, as a point in 1, the angular velocity of 1 relative to O must be

$$\omega_{10} = \frac{V}{O_{10}O_{13}} \quad \ldots \quad \ldots \quad \ldots \quad (67)$$

Considering O_{13} as a point in 3, the angular velocity of 3 relative to O must be

$$\omega_{30} = \frac{V}{O_{30}O_{13}} \quad \ldots \quad \ldots \quad \ldots \quad (68)$$

Dividing (67) by (68) gives

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_{30}O_{13}}{O_{10}O_{13}} \quad \ldots \quad \ldots \quad \ldots \quad (69)$$

Equation (69) is the angular velocity theorem expressed algebraically. The theorem may be stated as follows: *The angular velocities of any two bodies with respect to a third body are inversely as the distances of the pivot points of the two bodies from their transfer point.*
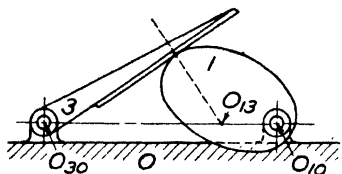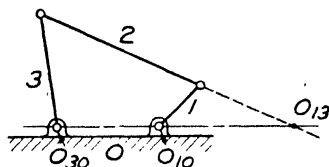
FIG. 88   FIG. 89

In the two mechanisms illustrated in Figs. 88 and 89 the angular velocities under scrutiny are those of the moving members 1 and 3 each referred to the fixed member O. In each case the pivot points are the fixed centers O_{10} and O_{30}, while the transfer point for 1 and 3 must be O_{13}. In each case the angular velocity ratio is

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_{30}O_{13}}{O_{10}O_{13}}$$

Figs. 88 and 89 are typical mechanisms quite similar to those of Figs. 45 and 46. It is to be noted that the point where the line of transmission, Arts. 42 and 43, cuts the line of centers is the transfer point in these simple mechanisms; that the angular velocity theorem just stated is in agreement with the statements of Art. 43; and that the directional relation of Art. 45 may be stated in the following terms: If the two pivot points lie on opposite sides of the transfer point, as in

Fig. 88, the angular motions of the corresponding members are opposite in direction, while, if the pivot points lie on the same side of the transfer point, as in Fig. 89, the angular motions are the same in direction.

68. Graphical Determination of Angular Velocities. Since angular velocities are inversely proportional to certain distances or lengths, they are readily determined graphically by the use of homologous lines of similar triangles. The construction, however, differs from that given for linear velocities in Art. 65 in that inverse ratios are used and the distances involved have a different significance.

Fig. 90 shows a construction for determining the angular velocity ω_{30} of *3* relative to *O*, having given the angular velocity ω_{10} of *1* relative to *O*. This is the cam mechanism of Fig. 88. The angular velocity ω_{10} may be represented by a vector drawn in any convenient direction
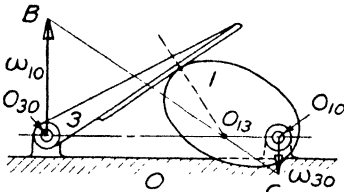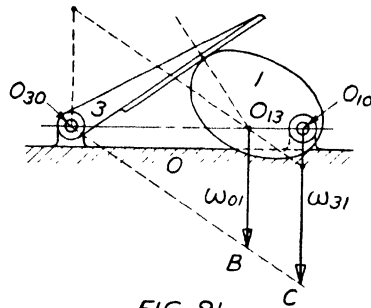
FIG. 90   FIG. 91

from the pivot point O_{30} of *3*. A gauge line through the terminus of vector ω_{10} and the transfer point O_{13} will intersect a line through the pivot point O_{10} parallel to ω_{10} at *C*. From the two similar triangles thus formed with the line of centers,

$$\frac{\omega_{30}}{\omega_{10}} = \frac{O_{10}O_{13}}{O_{30}O_{13}}$$

Since this is in accordance with the angular velocity theorem, equation (69), line $O_{10}C$ represents the angular velocity ω_{30} to the same scale that $O_{30}B$ represents ω_{10}.

In order to indicate direction and to distinguish between linear and angular velocity vectors, a special kind of arrow-head may be used for angular velocity vectors, an arrow-head of the kind used for electric current vectors. By Art. 67, the pivot points O_{10} and O_{30} being on opposite sides of the transfer point O_{13}, the directions of the angular motion of *1* and *3* are opposite. This is verified in Fig. 90 by the two vectors ω_{10} and ω_{30} extending in opposite directions from the base line.

It is easily possible by the angular velocity theorem to change from one member to another as the reference member. For the mechanism shown in Fig. 90, the angular velocity ω_{10} of 1 relative to O was given. If 1 is taken as the reference member, the known angular velocity becomes ω_{01}, equal and opposite in direction to. ω_{10}, that is, $\omega_{01} = -\omega_{10}$. If the angular velocity ω_{31} of 3 with respect to 1 is desired, an application of the angular velocity theorem, equation (69), would give

$$\frac{\omega_{31}}{\omega_{01}} = \frac{O_{01}O_{03}}{O_{31}O_{03}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Since instant centers O_{10}, O_{30}, and O_{13} mean the same as O_{01} O_{03}, and O_{31}, it is not necessary in Fig. 91 or in equation (70) to write the instant centers differently than in Fig. 90. Hence, equation (70) can be written

$$\frac{\omega_{31}}{\omega_{01}} = \frac{O_{10}O_{30}}{O_{13}O_{30}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

It should, however, not be overlooked that ω_{01} and ω_{31} are opposite in direction though equal in magnitude to ω_{10} and ω_{13}.

The graphical determination of ω_{31} from the proportionality of (71) is represented in Fig. 91. Centers O_{10} and O_{13} are the pivot points, and center O_{30} the transfer point. Since ω_{01} is. opposite in direction to ω_{10}, it is drawn from the pivot point O_{13} downward from the line of centers. The intersection of a line from the pivot point O_{10} parallel to ω_{01} with a gauge line through the transfer point O_{30} and the terminus of vector ω_{01} forms two similar triangles with the line of centers in accordance with the proportionality stated in (71), and determines the length of the desired vector ω_{31}.

The angular velocity ω_{31}, Fig. 91, can also be obtained by the addition or subtraction of vectors. Angular motions are either clockwise or counter-clockwise, and angular velocities may, therefore, be represented by parallel vectors in the same or in opposite directions, as the case may be. Hence, relative angular velocities can be obtained by the addition or subtraction of vectors. Vector ω_{31} is, therefore, either the vector sum of ω_{30} and ω_{01} or the vector difference of ω_{30} and ω_{10}.

$$\omega_{31} = \omega_{30} \nrightarrow \omega_{01} \qquad \text{or} \quad \omega_{31} = \omega_{30} + \omega_{01} \quad . \quad . \quad . \quad (72)$$

and

$$\omega_{31} = \omega_{30} \rightarrow \omega_{10} \qquad \text{or} \quad \omega_{31} = \omega_{30} - \omega_{10} \quad . \quad . \quad . \quad (73)$$

These additions and subtractions are made clear in Figs. 90 and 91 by drawing a line through O_{13} parallel to $O_{30}BC$ in Fig. 91. In such vector

additions it is evident that the directions of rotation must be known and utilized.

In the graphical method outlined above for determining angular velocities by the use of instant centers, the question of directions is, however, automatically taken care of. This method is very simple and direct, and will be found to have many applications in the study of different mechanisms.

69. Graphs. A graph is a curve drawn with respect to a pole point or with respect to rectangular coordinate axes where the coordinates of any point on the curve represent the simultaneous values of the magnitudes of the variables employed. The numerical magnitudes of such quantities as displacement, velocity, acceleration, force, volume, cost, etc., may be plotted against such quantities as time, space, etc. The first four quantities named involve both magnitude and direction, and are, therefore, vector quantities. The magnitude of velocity is called speed. Hence, where magnitudes of velocity are plotted, the resulting curves are frequently called speed graphs. There are, however, no corresponding terms for the magnitudes of displacement, acceleration, and force. Where such magnitudes are plotted, the curves are called displacement-time graphs, acceleration-time or acceleration-space graphs, etc. In such graphs it is understood that it is the magnitudes of the vector quantities that are plotted. In the articles to follow the same understanding will be assumed with respect to velocity. Graphs resulting from plotting magnitudes of velocity will, therefore, be referred to as velocity graphs instead of speed graphs.

70. Velocity Graphs. In the four-link mechanism shown in Fig. 92, the rotation of the driving arm, or crank, *1*, imparts an oscillatory motion to arm *3*. As crank *1* rotates counter-clockwise from position *Bo* to position *B8*, arm *3* moves from its extreme right position at *Eo* to its extreme left position at *E8*, and returns to position *Eo* as crank *1* completes its rotation from *B8* to *Bo*. In constructing the velocity graph *od8* for point *D* on its curved path as a base, the path, for convenience, was first divided into eight equal parts, and the corresponding positions of the driving point *C* in its circular path were found. The figure shows the construction for determining one point on the velocity graph of *D*. Length *Cc*, laid off on *BC* extended, represents the known speed of point *C* for the phase shown. Then, in accordance with Art. 65, length *Dd* on *ED* extended, determined by a line through *c* parallel to *CD*, represents the magnitude of the velocity of *D* to the same scale that *Cc* represents the magnitude of the velocity of *C*. By finding other points in a similar manner, the velocity graph *od8* was drawn. Since, for each phase, the length representing the magnitude of the

velocity of D was laid off on an extension of the radial position of link 3, the curve $od8$ is called a *radial* velocity-space graph. For a constant angular velocity of the driving crank, point C would move at a uniform velocity, and its radial velocity graph would be a circle concentric with the path of C as shown. It is to be noted, however, that the construction for determining the velocity graph of D remains the same whether the known velocity of C is constant or variable.

FIG. 92

FIG. 93
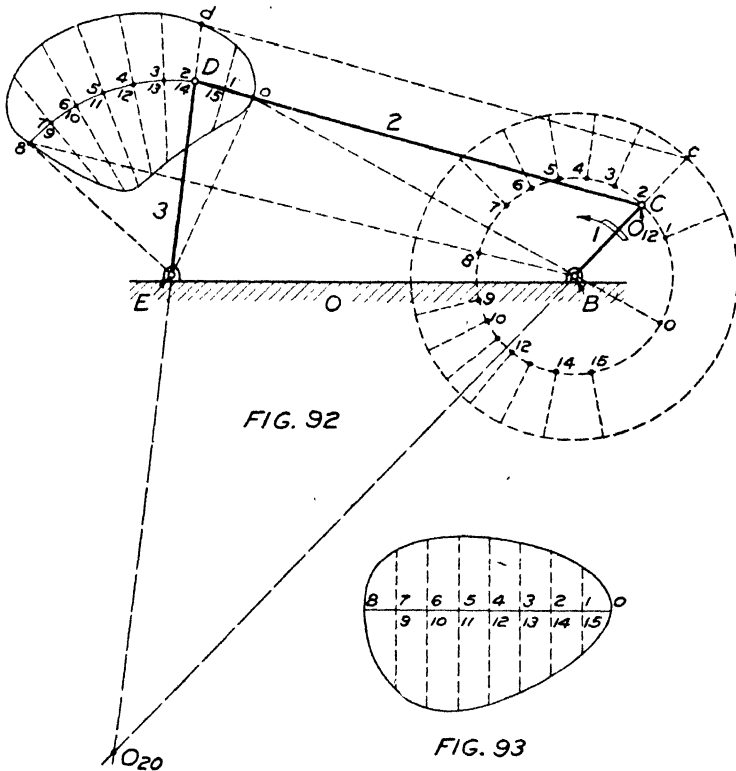
A velocity-space graph with rectangular coordinates may be constructed from the radial velocity-space graph just found by rectifying the path of D and erecting at the various points parallel ordinates corresponding in length to those already determined. This derived velocity-space graph is shown in Fig. 93.

Fig. 94 illustrates the mechanism of the ordinary reciprocating engine. Because this mechanism is employed so frequently, its study is of considerable value. In Fig. 94 instant center O_{12} is a point common to links 1 and 2, and instant center O_{13} is a point common to links 1

and *3;* hence, at the instant, points O_{12} and O_{13} are common to link *1.* As points in link *1,* they are rotating at the instant about O_{10} relative to $O.$ Hence,

$$\frac{V_{O_{13}}}{V_{O_{12}}} = \frac{O_{10}O_{13}}{O_{10}O_{12}}$$

or, since O_{13} has been designated B and O_{12} has been designated $P,$

$$\frac{V_{BO}}{V_{PO}} = \frac{O_{10}B}{O_{10}P}$$
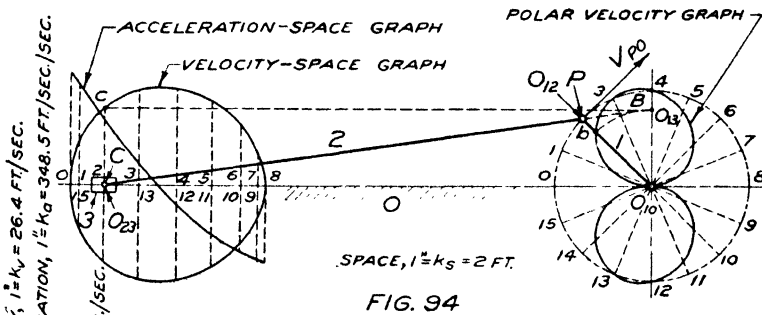
FIG. 94
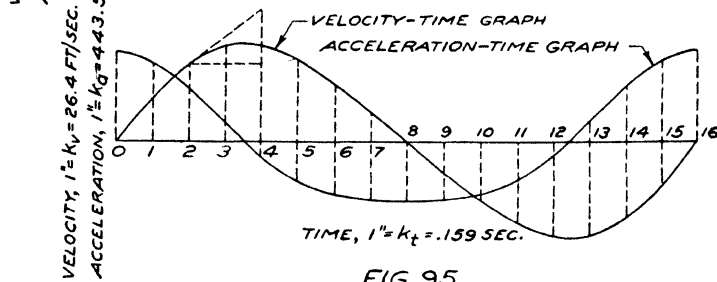
FIG. 95

If the crank is assumed to rotate at a uniform rate, then the linear velocity V_{PO} will be constant and may be represented by the fixed length $O_{10}P;$ in which event, $O_{10}B$ would represent the linear velocity V_{BO} to the same scale that $O_{10}P$ represents the constant linear velocity $V_{PO}.$ Since O_{13} is a point in *3* as well as in *1,* V_{BO} is equal to the linear velocity V_{CO} of the cross-head.

The theory outlined immediately above was applied in Fig. 94 to determine graphically the velocity of the cross-head for each position of the crank.   The polar velocity graphs and the velocity-space graphs for the cross-head for one revolution of the crank are shown in Fig. 94.   For each crank position such as $O_{10}P,$ the magnitude of the cross-head velocity V_{CO} was laid off from O_{10} on the crank to determine a point b on

the polar velocity graph for the cross-head. Also, for each position of the crank, the magnitude of the velocity V_{CO} of the cross-head was laid off on a line through the corresponding cross-head position and perpendicular to its path to determine a point c on the velocity-space graph for the cross-head. Since the crank is rotating at a uniform rate, equal crank spaces represent equal time intervals. Hence the magnitudes of the cross-head velocities may be used as ordinates and time as abscissae, resulting in the velocity-time graph shown in Fig. 95. In the velocity-space and in the velocity-time graphs, the magnitude of the velocity of the cross-head when traveling to the right was laid off upwards, and laid off downwards when traveling to the left.

To make these velocity graphs of any use in a quantitative way, it is necessary to determine the scales to which they were drawn. For the mechanism shown in Fig. 94, the crank was assumed to rotate in a clockwise direction at a uniform rate at 126 revolutions per minute. The stroke is 24 inches, and the connecting-rod length is 60 inches. The space scale of Fig. 94 is $\frac{1}{2}$ in. = 12 in. or 1 in. = k_s = 2 ft. The crank is, therefore, represented by a length of $\frac{1}{2}$ in. The velocity or speed scale was fixed when the length of the crank was taken to represent the constant linear velocity of the crank-pin P, which is

$$V_{PO} = \pi D(R.P.S.) = \pi(\tfrac{24}{12})\tfrac{126}{60} = 13.2 \text{ ft./sec.}$$

Hence the velocity or speed scale of Figs. 94 and 95 is

$$1 \text{ in.} = k_v = \frac{V_{PO}}{O_{10}P} = \frac{13.2}{0.50} = 26.4 \text{ ft./sec.}$$

In Fig. 95, the time for one revolution, or 60/126 of a second, is represented by a line 3 in. long. Hence, the time scale is

$$1 \text{ in.} = k_t = \tfrac{1}{3}(\tfrac{60}{126}) = 0.159 \text{ sec.}$$

In any distance-time graph, the speed, or magnitude of the velocity, for any point on the curve, is proportional to the slope of the tangent to the curve at that point. This statement corresponds to equation (8) Art. 11, and to the definition of the speed of a point as the time rate of change of distance. In Fig. 96 let OPQ be any space-time curve, and let P be any point on the curve for which BP is the ordinate and PE the tangent. Draw PD of some convenient length parallel to the axis of abscissae, and erect DE parallel to the axis of ordinates. In the slope triangle PDE, PD represents an infinitesimal increment of time dt, and DE the corresponding increment of distance ds. The distance, or space, scale is such that one inch represents k_s feet, and the time scale is

such that one inch represents k_t seconds. In Fig. 96 the increment of distance at the instant is $ds = (DE)k_s$, and the increment of time is $dt = (PD)k_t$. Hence, the speed, or magnitude of the velocity, at the instant, is

$$V = \frac{ds}{dt} = \frac{(DE)k_s}{(PD)k_t} \qquad \cdots \cdots \cdots \quad (74)$$

If the same length of line is used for PD for each point on the distance-time graph, then DE would represent the speed, or magnitude of the velocity, for each point, to a scale of

$$1 \text{ in.} = k_v = \frac{k_s}{(PD)k_t} \qquad \cdots \cdots \cdots \quad (75)$$

71. Acceleration Graphs. Owing to the desire or necessity for more rapid transportation or for more power per unit weight or for more production per capita, machinery speeds have been increasing, especially in the past twenty-five years. This is particularly true of engines and other machines utilizing the slider-crank mechanism. As speeds have increased, the unbalanced inertia forces and moments in machines have become more serious. Hence, a study of the accelerations which accompany inertia forces and moments, and of the problem of balancing such forces and moments, has become of increasing importance.
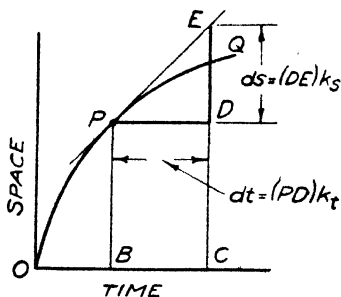
FIG. 96

It is possible to derive acceleration graphs from velocity graphs. It is important to note that such methods give only the accelerations which are due to the change in the magnitude of the velocity. They are applicable, therefore, only to total accelerations for rectilinear motion, or to tangential accelerations for curvilinear motion. Since an engine cross-head has rectilinear motion, its tangential acceleration A' is its total acceleration, and can, therefore, be found from its velocity graphs.

It will now be shown that in any velocity-space graph the subnormal to the curve at any point is proportional to the corresponding tangential acceleration. Let OPQ, Fig. 97, be any velocity-space curve, and P be any point on the curve. Lines BP, PE, and PC, are, respectively, the ordinate, tangent, and normal to the curve at the point. Let PD be drawn parallel to the axis of abscissae and of some convenient length, and let DE be erected at D parallel to the axis of ordinates. In the

slope triangle PDE, PD represents to scale an infinitesmal increment cf distance ds, and DE the corresponding increment dV of the magnitude of the velocity. The velocity ordinates have been laid off to a scale such that 1 inch represents k_v feet per second, and the abscissae such that 1 inch represents k_s feet. From equation (23) Art. 16, the tangential component of acceleration is
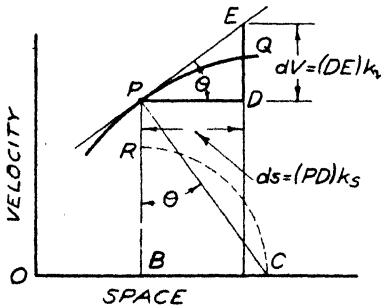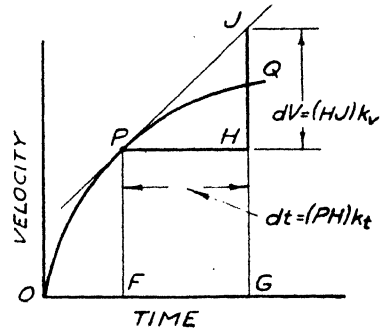
$$A^t = \frac{V\,dV}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

FIG. 97   FIG. 98

In Fig. 97 the magnitude of the velocity at the instant is $V = (BP)k_v$, the increment of velocity is $dV = (DE)k_v$, and the increment of space is $ds = (PD)k_s$. Substituting these values in equation (76) gives

$$A^t = \frac{(BP)(DE)k_v^2}{(PD)k_s} \quad . \quad . \quad . \quad . \quad . \quad (77)$$

But triangles PBC and PDE, since their sides are perpendicular, are similar, and

$$\frac{BC}{BP} = \frac{DE}{PD} \quad \text{or} \quad \frac{(BP)(DE)}{PD} = BC \quad . \quad . \quad . \quad (78)$$

Hence (77) may be written

$$A^t = (BC)\frac{k_v^2}{k_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Equation (79) states that for any point on a velocity-space graph, the tangential acceleration A' is equal to the product of the subnormal BC and the square of the velocity scale divided by the space scale. Hence, if the tangential accelerations are represented by ordinates equal in

length to the subnormals, the acceleration scale of the resulting acceleration-space graph is

$$1 \text{ in.} = k_a = \frac{k_r^2}{k_s} \text{ ft./sec./sec.} \quad . \quad . \quad . \quad . \quad (80)$$

In Fig. 97 the subnormal BC representing the tangential acceleration of point P at the instant is laid off as BR on the ordinate for P. The acceleration-space graph shown in Fig. 94 was obtained from the velocity-space graph by this subnormal construction. The acceleration scale of the graph is $1 \text{ in.} = \dfrac{k_r^2}{k_s} = \dfrac{(26.4)^2}{2} = 348.5 \text{ ft./sec./sec.}$

In any velocity-time graph the tangential acceleration at any point on the curve is proportional to the *slope* of the tangent to the curve at that point. This statement corresponds to equation (28) Art. 18, and to the definition of the tangential component of acceleration as the time rate of change of the magnitude of the velocity. In Fig. 98 let OPQ be any velocity-time curve, and let P be any point on the curve for which FP is the ordinate and PJ the tangent. Draw PH of some convenient length parallel to the axis of abscissae, and erect HJ parallel to the axis of ordinates. In the slope triangle PHJ, PH represents an infinitesimal increment of time dt, and HJ the corresponding increment of velocity dV. The velocity scale is such that 1 inch represents k_v feet per second, and the time scale is such that 1 inch represents k_t seconds. From equation (28) Art. 18, the tangential component of acceleration is

$$A^t = \frac{dV}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

In Fig. 98 the increment of velocity at the instant is $dV = (HJ)k_v$, and the increment of time is $dt = (PH)k_t$. Substituting these values in (81) gives

$$A^t = \frac{(HJ)k_v}{(PH)k_t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

If the same length of line is used for PH for each point on the velocity-time graph, then HJ would represent the acceleration for each point to a scale of

$$1 \text{ in.} = k_a = \frac{k_v}{(PH)k_t} \quad . \quad . \quad . \quad . \quad . \quad (83)$$

This plan was followed in Fig. 95 in deriving the acceleration-time graph from the velocity-time graph. In this graph the ordinates are

equally spaced $\frac{3}{16}$ in. apart. For each point two such spaces or a length of $\frac{3}{8}$ in. was used for PH. Hence the acceleration scale for the acceleration-time graph in Fig. 95 is

$$1 \text{ in.} = k_a = \frac{k_v}{(PH)k_t} = \frac{26.4}{(\frac{3}{8})0.159} = 443.5 \text{ ft./sec./sec.}$$

The accuracy of the results obtained by the application of the graphical methods just described will depend upon the accuracy and smoothness of the velocity graphs and upon how accurately the tangents and normals are drawn at each point of the velocity graphs. A degree of accuracy sufficient for general engineering work is, however, possible after a little practice. A square reflecting bar, a logarithmic spiral curve, or the use of the center of curvature of the curve at the point under consideration are of considerable assistance in drawing tangents and normals to the curves.
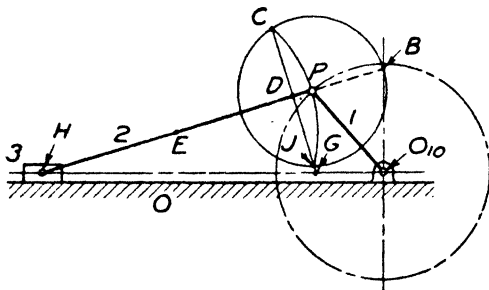
There are several special constructions for accurately determining the acceleration of the cross-head of an engine. The one most used is

FIG. 99

probably the Klein construction shown in Fig. 99. This construction is only applicable when the crank rotates at a uniform rate, and assumes that the constant linear velocity of the crank-pin center P is represented by the length of the crank, $O_{10}P$. A circle is first described with P as a center and PB as a radius. With E as a center and a radius equal to one-half the length of the connecting-rod, an arc is then drawn through P intersecting the circle at C and J. A line through C and J cuts the connecting-rod at D and the line of centers HO_{10} at G. The acceleration of the cross-head is proportional to the length GO_{10}. The acceleration

scale is 1 in. $= k_a = \dfrac{k_v^2}{k_s}$ ft./sec./sec., where k_s is the space scale and k_v

the velocity scale. The velocity scale, k_v, is the constant linear velocity of P in feet per second divided by the length $O_{10}P$ in inches. The proof of this construction will be found in Art. 80.

72. Analytical Determination of Velocities and Accelerations. The graphical methods of determining the velocity or acceleration of any point in a mechanism that have thus far been developed and that are

developed in the next chapter, are, in general, as easy to apply to one mechanism as to another. This will not be found to be true for any analytical method of attack. While analytical methods of attack are general in principle, the difficulty of applying any given method will be found to vary with the mechanism. Where a mathematical analysis is possible, the equations developed are often complicated and inconvenient to use.

The mathematical analysis of the slider-crank mechanism shown in Fig. 100 will be used as an illustration. The velocity, likewise the acceleration, of a point will be treated as the vector sum of its axial components. To make the analysis more general, a point P on the
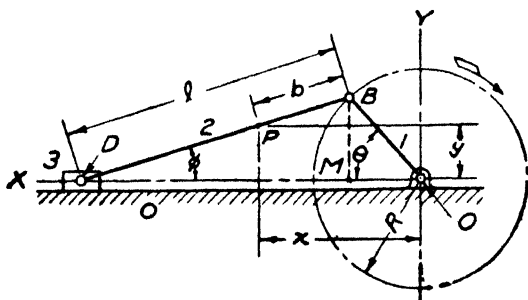
FIG. 100

connecting-rod will be used. For convenience the lengths and distances in Fig. 100 will be considered as measured in feet, and the angular velocity ω of the crank as expressed in radians per second. Let $n = \dfrac{l}{R}$, and $c = \dfrac{BP}{BD} = \dfrac{b}{l}$. Then, for any point P on the connecting-rod,

$$BM = R \sin \theta = l \sin \phi = nR \sin \phi$$

Hence,

$$\sin \phi = \frac{\sin \theta}{n} \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (84)$$

and

$$\cos \phi = \sqrt{1 - \sin^2 \phi} = \frac{1}{n}\sqrt{n^2 - \sin^2 \theta} = \frac{1}{n}\left(n - \frac{\sin^2 \theta}{2n}\right) \text{ nearly.}$$

The average error throughout a revolution as a result of taking $\sqrt{n^2 - \sin^2 \theta}$ equal to $\left(n - \dfrac{\sin^2 \theta}{2n}\right)$ is less than one-quarter of one per

cent for a value of $n = 4$, and rapidly decreases with an increase in the value of n.

$$x = R \cos \theta + b \cos \phi = R\left[\cos \theta + c\left(n - \frac{\sin^2 \theta}{2n}\right)\right]$$

$$y = (l - b) \sin \phi = R(1 - c) \sin \theta$$

Differentiating x and y with respect to time yields the velocity components parallel, respectively, to OX and OY:

$$V_{PO}^x = \frac{dx}{dt} = -R\left(\sin \theta + \frac{c}{2n} \sin 2\theta\right)\frac{d\theta}{dt} = -\omega R\left(\sin \theta + \frac{c}{2n} \sin 2\theta\right). \quad (85)$$

$$V_{PO}^y = \frac{dy}{dt} = R(1 - c) \cos \theta \frac{d\theta}{dt} = \omega R(1 - c) \cos \theta \ . \ . \ . \ . \ . \ . \quad (86)$$

Differentiating V_{PO}^x and V_{PO}^y with respect to time yields the acceleration components parallel, respectively, to OX and OY:

$$A_{PO}^x = \frac{dV_{PO}^x}{dt}$$

$$= -R\left(\sin \theta + \frac{c}{2n} \sin 2\theta\right)\frac{d^2\theta}{dt^2} - R\left(\cos \theta + \frac{c}{n} \cos 2\theta\right)\left(\frac{d\theta}{dt}\right)^2 . \quad (87)$$

$$A_{PO}^y = \frac{dV_{PO}^y}{dt} = R(1 - c) \cos \theta \frac{d^2\theta}{dt^2} - R(1 - c) \sin \theta\left(\frac{d\theta}{dt}\right)^2 \ . \ . \ . \ . \quad (88)$$

In these expressions for A_{PO}^x and A_{PO}^y, $\left(\frac{d\theta}{dt}\right)^2$ represents the square of the

angular velocity of the crank and $\frac{d^2\theta}{dt^2}$ the rate of change of the angular

velocity, or the angular acceleration, at the instant. If the angular velocity of the crank is not constant, its angular acceleration in terms of θ and t must be known in order to use equations (87) and (88). In many cases where this mechanism is used, especially in engines, the crank may be assumed to rotate at a uniform rate. This is accomplished by fitting the engine with a flywheel of proper size and weight. So far as balance is concerned, an engine that is in balance for a uniform rate of rotation of the crank is in balance for a non-uniform rate of rotation. If the angular velocity of the crank $\frac{d\theta}{dt}$, or ω, is assumed constant, $\frac{d^2\theta}{dt^2}$

becomes zero, and the components of acceleration, A_{PO}^x and A_{PO}^y, of P become

$$A_{PO}^x = -\omega^2 R\left(\cos \theta + \frac{c}{n} \cos 2\theta\right) \ . \ . \ . \ . \ . \quad (89)$$

$$A_{PO}^y = -\omega^2 R(1 - c) \sin \theta \ . \ . \ . \ . \ . \ . \quad (90)$$

The axial components of velocity and acceleration for any point P in the connecting-rod are given by equations (85), (86), (87), and (88) for a non-uniform rotation of the crank, and by equations (85), (86), (89), and (90) for a uniform rotation of the crank.

The connecting-rod has a varying angular velocity and, therefore, an angular acceleration, which may be obtained from its angular displacement. From equation (84) the angular displacement of the rod at any instant is

$$\phi = \sin^{-1}\left(\frac{\sin \theta}{n}\right) \quad \ldots \ldots \quad (91)$$

The angular velocity of the rod at any instant is obtained by differentiating ϕ with respect to time. Assuming the crank rotating at a uniform angular velocity ω, the angular velocity of the rod at any instant for any displacement θ of the crank is

$$\omega_{20} = \frac{d\phi}{dt} = \frac{\omega \cos \theta}{\sqrt{n^2 - \sin^2 \theta}} \quad \ldots \ldots \quad (92)$$

and the corresponding angular acceleration of the rod is

$$\alpha_{20} = \frac{d\omega_{20}}{dt} = \frac{\omega^2(1 - n^2) \sin \theta}{(n^2 - \sin^2 \theta)^{3/2}} \quad \ldots \ldots \quad (93)$$

When $b = l$ point P is at D and the value of c becomes unity. By substituting the value of unity for c in equations (89) and (90) expressions are obtained for A_{DO}^x and A_{DO}^y for the cross-head pin center D for uniform rotation of the crank. Hence, for the cross-head,

$$A_{DO}^x = -\omega^2 R\left(\cos \theta + \frac{\cos 2\theta}{n}\right) \quad \ldots \ldots \quad (94)$$

$$A_{DO}^y = \text{zero}$$

When $b = \text{zero}$, point P is at B and the value of c is zero. By substituting zero for c in equations (89) and (90), expressions are obtained for A_{BO}^x and A_{BO}^y for the crank-pin center B for uniform rotation of the crank. Hence, for point B,

$$A_{BO}^x = -\omega^2 R \cos \theta \quad \ldots \ldots \ldots \quad (95)$$

$$A_{BO}^y = -\omega^2 R \sin \theta \quad \ldots \ldots \ldots \quad (96)$$

It is to be noted that A_{BO}^x and A_{BO}^y are merely the vector components of the normal acceleration of a point moving in a circular path. Their vector sum is equal to $\omega^2 R$ in accordance with equation (29), Art. 18.

The mechanism shown in Fig. 101 is equivalent to a slider-crank chain having a connecting-rod of infinite length.   Point D is any point on the reciprocating slotted cross-head, or Scotch yoke, as it is frequently called.   The velocity and acceleration components for points B and D for this mechanism can be obtained as for the similarly designated points of the mechanism in Fig. 100.   The length of the connecting-rod
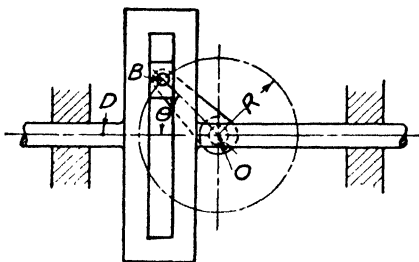

FIG. 101

being infinite, $n = \dfrac{l}{R}$ is infinite, and c for point D would equal unity. Substituting the value of c in equation (86) gives $V_{DO}^{v} =$ zero, and substituting the values of n and c in equations (85) and (94) gives

$$V_{DO} = - \omega R \sin \theta = - \omega R \sin \omega t \quad . \quad . \quad . \quad (97)$$

and

$$A_{DO} = - \omega^2 R \cos \theta = - \omega^2 R \cos \omega t \quad . \quad . \quad . \quad (98)$$

It is to be noted that these equations are the same as equations (24) and (25) of Art. 17.   Therefore, the slotted cross-head of Fig. 101 has simple harmonic motion if the driving crank rotates at a uniform speed.

CHAPTER IV

VELOCITY AND ACCELERATION VECTOR DIAGRAMS

73. Velocity and Acceleration. Methods of determining relative linear and angular velocities have already been shown to be of importance in analyzing the relative motions of the parts of mechanisms and machines. In Art. 71 it was shown that the variation of the tangential acceleration or the variation of the acceleration of a part having rectilinear motion can be determined from the velocity-time or the velocity-space graph. In order to determine the linear acceleration of any point or the angular acceleration of any link in a mechanism, it is first necessary to determine the relative velocities. Methods of determining relative velocities are, therefore, of great importance in making a complete analysis of the motions of the parts of a machine. Two methods have thus far been given. In this chapter a third method will be given before taking up the methods of determining accelerations.

The magnitude and direction of the forces acting on the moving parts of a machine at any instant are affected by friction and the inertia of the parts. As a rule the frictional resistances are small compared to the forces to be transmitted, and have little effect on the magnitude and direction of the resulting forces. The effect of the inertia of the parts depends upon the linear and angular accelerations of the moving parts or masses. For high-speed machinery the effects of the inertia of the parts may be very considerable. For slow-speed machinery the friction and inertia of the parts may have very little effect on the magnitude and direction of the resulting forces. Where these two factors may be neglected, the product of the absolute velocity and the force acting in its direction at one point in a machine will be equal to the product of the absolute velocity and the force acting in its direction at any other point in the machine. Application of this law of the conservation of energy makes it possible to approximate the magnitude and direction of the forces acting on the parts of slow-speed machinery. If only friction can be neglected, the magnitude and direction of the forces acting on the parts of a machine will be affected by the size, form, and mass of the parts and their linear and angular accelerations. Thus the force available for transmission at the cross-head of an engine

during the first part of the stroke is less than the net steam pressure on the piston by an amount equal to the product of the acceleration and the mass of the piston, piston-rod, and cross-head. Also, because of the rectilinear acceleration of the reciprocating parts, the linear and angular acceleration of the connecting-rod, and the normal acceleration of the crank, assumed to rotate at a uniform rate, the reactions at the crank-shaft bearings will be less at the instant than would correspond to the net steam pressure on the piston. As a result, the steam pressures on the cylinder heads of the engine are not balanced by the forces on the cross-head guides and at the crank-shaft bearings, and the engine would tend to rock and to vibrate on its foundation because of the unbalanced forces and moments. It is thus evident that to calculate inertia forces and moments and to find and to correct the unbalance of machines, it is necessary to determine the accelerations of the parts. Just how to determine the magnitude, direction, and line of action of the inertia force, the force necessary to accelerate a part at the rate it is being accelerated at the instant, is best shown by examples.
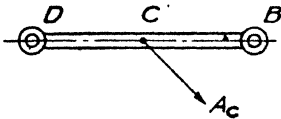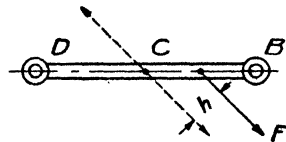
FIG. 102   FIG. 103

Suppose the rod of mass M, shown in Fig. 102, is found, as a part of a machine, to have linear but no angular acceleration, and that the linear acceleration of the center of gravity C of the rod is A_C feet per second per second in the direction shown. Obviously the force F necessary to overcome the inertia of the rod is

$$F = MA_C \qquad \qquad \qquad (99)$$

If the center of gravity C of the rod is midway between D and B, and the forces actually acting on the rod are applied at D and B, then the forces F_D and F_B necessary to overcome the inertia of the rod would act through D and B parallel to A_C, and

$$F_D = F_B \quad \text{and} \quad F_D + F_B = F = MA_C \quad . \quad . \quad (100)$$

Suppose, however, that in addition to C having a linear acceleration of A_C, the rod had an angular acceleration of α in a clockwise direction. Then, instead of the resultant force F acting through C, it would act parallel to and at a distance h to the right of A_C as shown in Fig. 103. Obviously, two opposing forces, equal and parallel to F, may be sub-

stituted at C without disturbing the system. Hence, the rod may be considered to be acted on by a force F through C producing linear acceleration, and a couple Fh producing angular acceleration. The turning moment Fh to produce an angular acceleration α of a part having a moment of inertia of mass I about its center of gravity is

$$Fh = I\alpha \qquad \ldots \ldots \ldots \ldots \quad (101)$$

Hence,

$$h = \frac{I\alpha}{F} = \frac{M\rho^2\alpha}{MA_C} = \rho^2\frac{\alpha}{A_C} \quad \ldots \ldots \quad (102)$$

where ρ is the radius of gyration of the mass about its center of gravity. If the actuating forces causing the motion of the rod are applied at B and D, the forces will be parallel to F, and their magnitudes can be found by the summation of forces and moments.

It is evident from the above illustrations that it is necessary to be able to determine linear and angular accelerations in order to solve problems in kinetics.

74. Velocity Vector Diagram. The use of velocity vector diagrams was first fully discussed by Professor R. H. Smith in the *Proceedings*
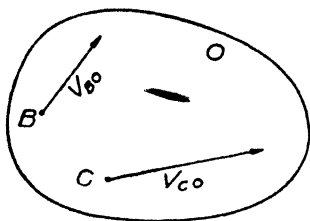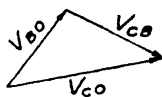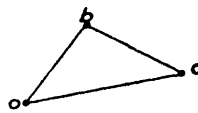
FIG. 104   FIG. 105   FIG. 106

of the Royal Society of Edinburgh in January, 1885. By this method of solution, a complete velocity vector diagram of a mechanism may be constructed and the velocity of any point relative to any other point determined by inspection.

In the construction of velocity vector diagrams capital letters will be used to represent the points whose velocities are known or are to be determined, and small letters will be used in the diagrams to designate the velocity vectors.

Consider first Fig. 104, where points B and C have the linear velocities V_{BO} and V_{CO} relative to body O. As was shown in Art. 5 and as illustrated in Fig. 105,

$$V_{CB} = V_{CO} \rightarrow V_{BO} \quad \text{or}, \quad V_{CO} = V_{BO} \rightarrowtail V_{CB} \quad \ldots \quad (103)$$

Because of its usefulness in this form of construction, vectors will be indicated as in Fig. 106. The line ob, parallel and equal to V_{BO} of Fig. 104, represents the velocity of point B relative to O, or the line bo represents the velocity of O relative to B. In the same manner the line oc represents V_{CO}, being parallel and equal to it. Drawing the line bc makes the two triangles, obc and $V_{BO}V_{CB}V_{CO}$, equal by construction, and the line bc will represent the velocity of C relative to B, and cb that of B relative to C. The vector diagram obc indicates the relative velocities of points B and C. It should be noted that in Fig. 105 the velocity vector representing the velocity of C relative to B is marked V_{CB}, the direction being indicated by an arrow-head, while in Fig. 106 the velocity vector representing the velocity of C relative to B is bc,
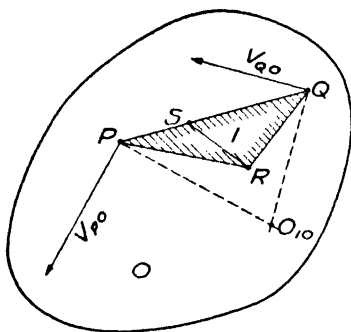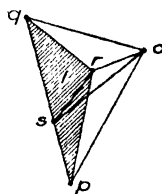
FIG. 107   FIG. 108

the direction being from b to c. Similarly, the velocity vector representing the velocity of B relative to C is cb, the direction being from c to b. Hence the diagram of Fig. 106 is more generally useful than that of Fig. 105.

In Fig. 107, let V_{PO} and V_{QO} represent the known velocities of the two points P and Q of the moving body 1 referred to body O. Fig. 108 is the velocity vector diagram in which op and oq are drawn parallel and equal to V_{PO} and V_{QO}, respectively, and in which pq represents the velocity of Q relative to P. Point R, being a part of body 1, cannot move toward or away from P along the line PR and, hence, can move relative to P only in a direction perpendicular to PR. By the same reasoning, R can move relative to Q only in a direction perpendicular to QR. *In other words, the motion of one point relative to any other point in the same rigid body must be at right angles to the line joining the two points. This obvious relation is the key to the construction of velocity vector diagrams.* Hence, in the vector diagram, Fig. 108, vector pr is drawn

perpendicular to its rotational radius PR in Fig. 107, and the vector qr perpendicular to QR. The intersection of these two lines locates r and determines the velocity vectors pr and qr and enables or to be drawn. Since point S lies on the line PQ, its velocities relative to P and Q must be perpendicular to the line PQ and proportional to the distances of S from P and Q. Hence s divides pq in the same proportion as S divides PQ. The point s could also have been located by remembering that S and R are both points on 1, and that the velocity vector of S relative to R must be perpendicular to RS. Hence, where a line from r perpendicular to RS intersects pq locates s. The figure $prqs$ is termed the *velocity image* of $PRQS$, P,R,Q, and S being selected points of body 1. Figure $prqs$ was constructed similar to the figure $PRQS$ of the configuration diagram but at right angles to it and of a size dependent upon the velocity scale used. Knowing this similarity between the image and the configuration diagram makes it possible easily to determine the velocity vectors of other points in a member such as 1.

It is to be noted by reference to Figs. 107 and 108 that the configuration diagram, and therefore the velocity image, may be extended to include other points. If the velocity image is extended to include the pole o, the corresponding point in body 1 extended is a point O_{10} which has no linear velocity relative to O. Such a point is, by definition, the instant center of the velocity relations between bodies 1 and O. This may be verified by the methods of Arts. 55 and 56. In connection with the study of the velocity vector diagrams of Figs. 110 and 113 it should be noted that each link has its velocity image, and that the location of the instant center of any link corresponds to the pole point o of the velocity image of the link.

Angular velocities are also obtainable from the information furnished by these velocity vector diagrams. Since 1 in Fig. 107 is a rigid body, the point Q can have only a motion of rotation relative to P. Hence, the velocity of Q relative to P is a measure of the angular velocity of PQ relative to O. The velocity of Q relative to P is represented to a definite scale in Fig. 108 by the line pq. Hence, from the fundamental relation $\omega = \dfrac{V}{R}$, the angular velocity of the line PQ relative to O is proportional to $\dfrac{pq}{PQ}$, direction counter-clockwise. If the velocity scale of Fig. 108 is 1 in. $= k_v$ feet per second and the space scale of Fig. 107 is 1 in. $= k_s$ feet, then, since the two figures are similar,

$$\omega = \left(\frac{pq}{PQ}\right)\frac{k_v}{k_s} = \left(\frac{pr}{PR}\right)\frac{k_v}{k_s} = \left(\frac{sr}{SR}\right)\frac{k_v}{k_s} = \text{etc.} \quad . \quad . \quad (104)$$

These equalities simply state that all the lines in a rigid body having plane motion relative to another body, have the same angular velocity.

75. Applications of the Velocity Vector Diagram. (*a*) *Four-Link Mechanism.* The four-link mechanism of Fig. 109 is that of Fig. 82. The articulations have been named O_{10}, B, C, and O_{30}. The linear velocity vector of point P relative to O being known, the velocities of R and Q are to be determined. This will be done by first finding the velocity of B, then C, then R, and finally Q, by means of a velocity vector diagram. Starting with the pole o, Fig. 110, draw op parallel and equal to the known vector V_{PO}. As before pointed out, B can have only a motion of rotation relative to any other point in the same rigid body. Hence, B must move perpendicular to PB relative to P and
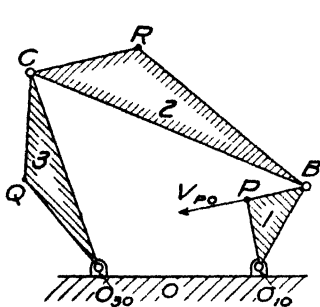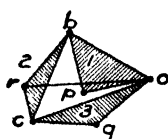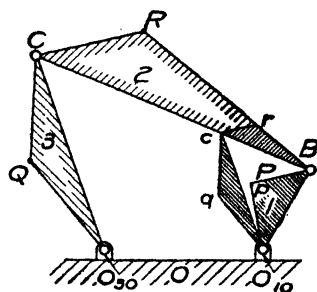
FIG. 109   FIG. 110   FIG. 111

perpendicular to $O_{10}B$ relative to O, and lines from p and o perpendicular to PB and $O_{10}B$ will intersect at b and determine the velocity vectors pb and ob. In a like manner lines from b and o perpendicular to BC and $O_{30}C$ intersect at c determining velocity vectors oc and bc. Lines from b and c perpendicular to BR and CR intersect at r and determine the velocity vectors br and cr, the desired velocity of R relative to O being determined by drawing the velocity vector or. Lines from o and c perpendicular to $O_{30}Q$ and CQ intersect at q and determine the desired vector oq, representing the velocity of Q relative to O. The velocity vectors or and oq may now be drawn from R and Q in Fig. 109 if desired.

It is to be noted that by the use of O_{20} the intersection r could have been determined as the intersection of lines from o and b or from o and c instead of from b and c. Also, since the triangles brc and BRC are similar, r could have been located on a line from o, b, or c by proportion.

As before pointed out, the velocity images *1*, *2*, and *3*, Fig. 110, are similar to links *1*, *2*, and *3*, Fig. 109, but are at right angles to them. If the scale of the vector diagram, Fig. 110, were such that vector op

representing the velocity V_{PO} was equal to $O_{10}P$, then the vector diagram, Fig. 110, could be revolved through 90 degrees and superimposed on the configuration diagram. Then, as shown in Fig. 111, image *1* would coincide with link *1*, and the other images would take the positions shown. Since vectors *or*, *oc*, and *oq* radiate from O_{10} and represent in magnitude the velocities of points R, C, and Q to the same scale that *op*, or $O_{10}P$, represents the velocity of P, the points *r*, *c*, and *q*, Fig. 111, may be considered as points in link *1* extended which have the same velocities as points R, C, and Q of the linkage. A diagram so constructed on the driving link of a mechanism is called a *phorograph*. It has the advantage over the ordinary vector diagram of being formed by lines parallel instead of perpendicular to the lines of the linkage, but has the disadvantage of representing the velocity directions revolved through 90 degrees. It is to be noted that the velocity scale of the phorograph is always equal to the known velocity of a point in the driving link divided by the rotational radius of the point. If, in Fig. 111, the velocity V_{PO} is the same for all positions of link *1*, then the velocity scale of the phorograph is the same for all phases of the linkage and equal to $\dfrac{V_{PO}}{O_{10}P}$.

If, however, driving link *1* rotates at a variable angular velocity, the velocity scale of the phorograph will vary from position to position of link *1* according to the variation of the angular velocity. The advantages of the phorograph may be realized in constructing velocity vector diagrams to any scale desired by considering all velocity directions displaced 90 degrees and using parallel instead of perpendicular lines. Where velocities are to be determined for various phases of a mechanism, this construction can be used without confusion, if, for the first phase, the ordinary velocity vector diagram is constructed.

By reference to Fig. 82 it will be found that the solution of the above problem by the method of instant centers is more direct and more easily followed than the solution just made by applying the velocity vector diagram. The reverse of this would be true had the points P, R, and Q been located on the lines joining the centers O_{10}, B, C, and O_{30}. The method to be preferred depends upon the character of the problem. Where accelerations are to be found, the vector diagram method of determining the velocities is always to be preferred. In the following application to a complex mechanism, solution by means of the velocity vector diagram is simpler than by the method of instant centers.

(b) *Marshall's Valve Gear.* Fig. 112 is a diagrammatic representation of Marshall's valve gear for steam engines. The frame of the engine is O, H is the center of the cross-head pin, P the center of the crank-pin, O_{10} the center of the crank-shaft, E the center of the eccentric, and G

the center of the pin connecting the rod DG to the stem of the slide valve. To reverse the engine center C is moved to the right. The power of the engine, whether going forward or backward, is controlled by the position of C, which is under the control of the operating engineer. Since for any given condition of running C is fixed with respect to the frame, it may be designated O_{50}.
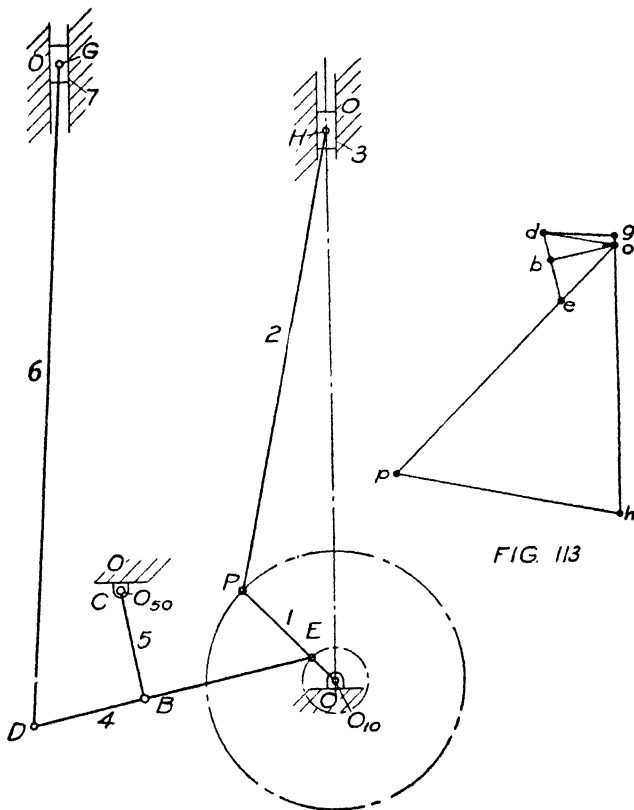

FIG. 113

FIG. 112

The interest in making a velocity analysis of a valve gear may be to find the corresponding velocities of the valve and cross-head or piston for various positions of the crank-pin which rotates at a known uniform rate, or, more to the purpose, such an analysis may be used as the basis for determining the accelerations of the parts of the gear. The given velocity op of the crank-pin center P is laid off from o in Fig. 113 perpendicular to the crank $O_{10}P$. Since the motion of H relative to O is vertical, and the motion of H relative to P is perpendicular to PH,

oh is vertical and ph is perpendicular to PII. Since E is on the center-line $O_{10}P$, e is in a position on op corresponding to the proportion $\dfrac{oe}{op} = \dfrac{O_{10}E}{O_{10}P}$. Point B moves perpendicular to EB relative to E and perpendicular to $O_{50}B$ relative to O_{50}. Hence, b is the intersection of lines through e and o respectively perpendicular to EB and $O_{50}B$. Since D is on EB, d is in a position on eb corresponding to the proportion $\dfrac{ed}{eb} = \dfrac{ED}{EB}$. The motion of G with respect to the frame O is vertical and is perpendicular to DG with respect to D. Hence, g is the intersection of a vertical through o and a line through d perpendicular to DG. The velocity vector diagram indicates, for the position of the crank shown, that the cross-head or piston is moving vertically downward at a velocity oh, while the valve stem or valve is moving vertically upward at a relatively small velocity og.

For any given condition of operation the mechanism consists of eight links for which there are 28 instant centers. For the above velocity relations, the direct transfer point between the cross-head and the valve is located at infinity, and several steps would be required to obtain the transfer point between crank and valve. A solution by the method of instant centers would, therefore, have proved more difficult than the solution above by means of the velocity vector diagram.

(c) *Direct Contact Mechanism.* In the direct contact mechanism shown in Fig. 114, link *2* is driven by link *1*. At the instant, point P_1 of link *1* is in contact with point P_2 of link *2*. From the angular velocity of the driver, link *1*, the linear velocity V_{P_1O} of point P_1 can be found. The problem is to find the velocity V_{P_2O}. Now

$$V_{P_1O} = V_{P_2O} + V_{P_1P_2} \quad \ldots \quad \ldots \quad (105)$$

V_{P_1O} and V_{P_2O} are known to be perpendicular, respectively, to $O_{10}P$ and $O_{20}P$. To draw the vector diagram, Fig. 115, we must know the direction of motion of P_1 relative to P_2. Since links *1* and *2* are rigid bodies, P_1 can move relative to P_2 only along the common tangent T, as there can be no relative motion of the contact points along the common normal N. Hence, op_1 is drawn perpendicular to $O_{10}P_1$, and p_2 is the intersection of a line through o perpendicular to $O_{20}P_2$ with a line through p_1 parallel to T.

See Art. 83 for a demonstration of the *three line construction* for the determination of velocities.

76. Acceleration Vector Diagram. Acceleration vectors may be used to form acceleration vector diagrams in a manner similar to the

use of velocity vectors in forming velocity vector diagrams. Since information from a velocity vector diagram is needed to construct an acceleration vector diagram, the notation for the latter should be consistent with that used for the former. Accordingly a line ob parallel to V_{BO} of Fig. 104 having been used in Fig. 106 to represent the velocity of point B relative to O, a vector $o'b'$ will be used to represent the total linear acceleration A_{BO} of point B relative to O. This total acceleration is usefully considered as the vector sum of its normal and tangential components, A_{BO}^n and A_{BO}^t. *The fact that the normal and tangential components of an acceleration are at right angles is the key to the construction of acceleration vector diagrams.* The point of intersection of the vectors representing the normal and tangential components of an acceleration such as A_{BO} will be designated (bo). Thus the normal and tangential components A_{BO}^n and A_{BO}^t of this acceleration would be represented by
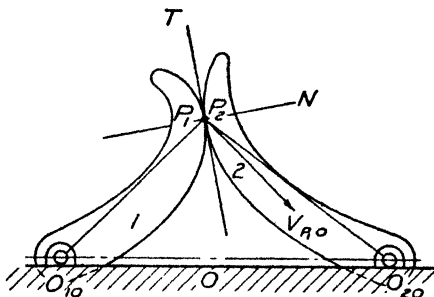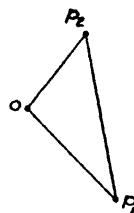
FIG. 114   FIG. 115

the component vectors $o'(bo)$ and $(bo)b'$. The normal and tangential acceleration components A_{OB}^n and A_{OB}^t of acceleration A_{OB} would be represented by the vectors $b'(ob)$ and $(ob)o'$. An example will make clear the notation to be used and the method of constructing acceleration vector diagrams.

In Figs. 116, 117, and 118, the configuration, velocity vector, and acceleration vector diagrams for an ordinary slider-crank chain are respectively represented. For this particular slider-crank chain, the crank rotates in a clockwise direction at a uniform rate at 126 revolutions per minute. The stroke is 24 inches, and the connecting-rod length is 60 inches. The space scale of Fig. 116 is $\frac{1}{2}$ in. = 12 in. or 1 in. = 2 feet, the velocity scale of Fig. 117 is 1 in. = 10 feet per second, and the acceleration scale of Fig. 118 is 1 in. = 120 feet per second per second.

In order to determine the relative accelerations, it is first necessary to determine the relative velocities. Since the linear velocity of the crank-pin center P can be computed, a velocity vector diagram giving

the relative velocities of P and H can be constructed by the methods of Art. 74. The linear velocity of P relative to the frame O is

$$V_{PO} = \pi D(\text{R.P.S.}) = \pi(\tfrac{24}{12})\tfrac{126}{60} = 13.2 \text{ ft./sec.}$$

The direction of motion of P at the instant is tangent to the crank-pin circle or perpendicular to $O_{10}P$. Hence, the linear velocity of P is fully represented in the velocity vector diagram, Fig. 117, by a line op perpendicular to $O_{10}P$ having a length of $\dfrac{13.2}{10} = 1.32$ inches. The direction of motion of H is horizontal relative to the frame O and is perpendicular

FIG. 116

SCALE 1"=10 FT/SEC.

FIG. 117

SCALE 1"=120 FT/SEC.²
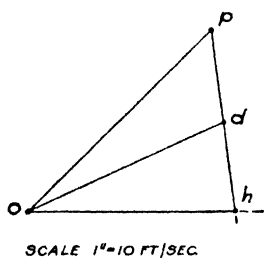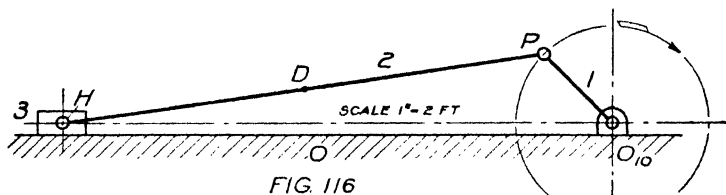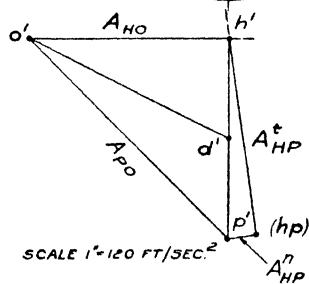
FIG. 118

to HP relative to P. Hence, the intersection h of a horizontal line through the pole o with a line through p perpendicular to HP completes the velocity vector diagram, Fig. 117, for the points P and H. The velocity V_{HP} of H relative to P is represented by the vector ph, and the velocity V_{HO} of H relative to the frame O by the vector oh to the scale of 1 in. = 10 feet per second. Measuring these vectors gives

$$V_{HP} = 9.45 \text{ ft./sec.}, \quad \text{and} \quad V_{HO} = 10.6 \text{ ft./sec.}$$

In constructing the acceleration vector diagram for P and H, Fig. 118, use is made of the fact that the normal and tangential components of the acceleration of a point are at right angles. Since the crank is rotating at a uniform rate, the acceleration of P has a normal but no tangential component. The normal acceleration component of P may be found by multiplying the velocity of P relative to O by the angular velocity of

$O_{10}P$, or by dividing the square of the velocity of P by the rotational radius $O_{10}P$. Hence,

$$A_{PO} = A_{PO}^n = \frac{(V_{PO})^2}{O_{10}P} = \frac{(13.2)^2}{\frac{12}{12}} = 174.24 \text{ ft./sec.}^2$$

Using the scale given above, A_{PO} would be represented by a vector having a length of $\dfrac{174.24}{120} = 1.45$ in. Since the acceleration of P acts from P towards O_{10} along PO_{10}, the acceleration vector $o'p'$ for point P in the diagram, Fig. 118, is drawn from the pole o' parallel to PO_{10} to a length of 1.45 in. The normal acceleration component of H relative to P acts along HP, and is

$$A_{HP}^n = \frac{(V_{HP})^2}{HP} = \frac{(9.45)^2}{60 \cdot 12} = 17.86 \text{ ft./sec.}^2$$

Hence, $p'(hp)$, representing this acceleration component in the diagram, is drawn from p' parallel to HP to a length of $\dfrac{17.86}{120} = 0.149$ in. The

tangential acceleration component of H relative to P is at right angles to the line $p'(hp)$. Since H has horizontal rectilinear motion relative to O, its acceleration acts along a horizontal line. Hence, the intersection h' of a horizontal line through o' with a line through (hp) perpendicular to $p'(hp)$ determines the vector $o'h'$ representing the acceleration of H relative to O and the vector $(hp)h'$ representing the tangential acceleration component of H relative to P. The line $p'h'$ joining p' and h' represents the acceleration of H relative to P in both magnitude and direction being from p' towards h'. Measuring these vectors to the scale of 1 in. = 120 feet per second per second gives

$$A_{HP}^t = 121.2 \text{ ft./sec.}^2$$

$$A_{HP} = 123.0 \text{ ft./sec.}^2$$

$$A_{HO} = 123.6 \text{ ft./sec.}^2$$

As ph is the velocity image of HP, so $p'h'$ is the acceleration image of HP. The velocity and acceleration for any point D on HP are represented by the rays drawn from the respective poles o and o' to points d and d' on ph and $p'h'$, such that

$$\frac{pd}{ph} = \frac{PD}{PH}, \quad \text{and} \quad \frac{p'd'}{p'h'} = \frac{PD}{PH}$$

The angular velocity ω_{20} of link *2*, the connecting-rod, relative to the frame O is equal to the velocity of H relative to P divided by the rotational radius PH. Hence,

$$\omega_{20} = \frac{V_{HP}}{PH} = \frac{9.45}{60/12} = 1.89 \text{ radians/sec. (counter-clockwise)}$$

This angular velocity could, of course, be obtained by dividing the velocity of any point in PH relative to any other point in PH by the proper rotational radius.

The angular acceleration α of link 2 relative to the frame O is equal to the tangential acceleration of H relative to P divided by the rotational radius PH. Hence,

$$\alpha_{20} = \frac{A'_{HP}}{PH} = \frac{121.2}{60/12} = 24.24 \text{ radians/sec.}^2 \text{ (clockwise)}$$

It is to be noted that while the connecting-rod is rotating counter-clockwise, the acceleration is clockwise; that is, the angular velocity of the rod at the instant is decreasing, not increasing.

An arbitrary choice of scales in the above problem made it necessary to calculate the value of each velocity and its corresponding normal acceleration component to be used in constructing the acceleration vector diagram. By properly relating the scales it is only necessary to calculate the initial values for a point on the driving link. However, whether the scales are arbitrarily chosen, or related as outlined in the article to follow, the given and calculated data should for convenience be neatly tabulated.
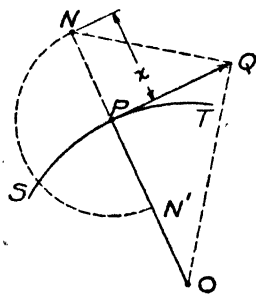
FIG. 119

77. Scale Relations. It is convenient in determining relative accelerations graphically to use a space scale of 1 in. = k_s feet, a velocity scale of 1 in. = k_v feet per second, and an acceleration scale of 1 in. = k_a feet per second per second. It can be shown that if these scales are properly related a complete graphical treatment is made possible, and the time and inconvenience of making intermediate calculations is saved.

In Fig. 119 let the space scale be 1 in. = k_s feet, and let the velocity of point P at the instant along its path ST be represented by the vector PQ in inches to a scale of 1 in. = k_v feet per second. The radius of curvature R of the path at point P is represented to the space scale by

OP in inches. Hence, the normal acceleration component of point P at the instant is

$$A_{PO}^n = \frac{V^2}{R} = \frac{\overline{k_v(PQ)}^2}{k_s(OP)} = \frac{\overline{PQ}^2}{OP} \times \frac{k_v^2}{k_s}$$

If A_{PO}^n is represented by x inches to a scale of 1 in. $= k_a$ feet per second per second, then

$$A_{PO}^n = xk_a = \frac{\overline{PQ}^2}{OP} \times \frac{k_v^2}{k_s}$$

If the three scales are so chosen that $k_a = \dfrac{k_v^2}{k_s}$, then

$$x = \frac{\overline{PQ}^2}{OP}, \quad \text{or} \quad \frac{x}{PQ} = \frac{PQ}{OP} \quad . \quad . \quad . \quad . \quad . \quad (106)$$

This means that if k_a is to equal $\dfrac{k_v^2}{k_s}$, the length of the velocity vector PQ must be a mean proportional between x and OP, in which case x, or PN, would be determined by drawing QN perpendicular to QO. The length PN in inches would then represent the magnitude of the normal acceleration component A_{PO}^n to a scale of 1 in. $= k_a = \dfrac{k_v^2}{k_s}$ feet per second per second. Since the normal acceleration is directed towards O, it can only be represented in direction as well as magnitude by PN', which is equal and opposite to PN. Hence, in determining relative accelerations graphically, the calculation of the normal accelerations and their vector lengths is eliminated by arbitrarily choosing k_s and k_v and making the acceleration scale

$$1 \text{ in.} = k_a = \frac{k_v^2}{k_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

or by arbitrarily choosing k_s and k_a and making the velocity scale

$$1 \text{ in.} = k_v = \sqrt{k_s k_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

If, for the case of a point P rotating about a fixed center at a uniform rate, the constant velocity of the point is represented by the length of the rotational radius, it is evident that angle OQP would be 45 degrees, likewise the angle PQN, and the uniform normal acceleration of the

point would be represented by a vector equal in length to the rotational radius, or the uniform velocity vector. Such a choice means that

$$k_s = \frac{R}{OP} \quad \text{and} \quad k_v = \frac{V}{OP}$$

or

$$k_a = \frac{k_v{}^2}{k_s} = \left(\frac{V}{OP}\right)^2 \frac{OP}{R} = \left(\frac{V}{R}\right)^2 k_s = \omega^2 k_s \quad . \quad . \quad . \quad (109)$$

78. Acceleration Vector Diagram, Complete Graphical Method. The configuration, velocity vector, and acceleration vector diagrams for a four-link chain are represented in Figs. 120, 121, and 122. The lengths in inches are, $O_{10}B = 1\frac{5}{8}$, $BC = 7\frac{1}{8}$, $O_{30}C = 4\frac{1}{4}$, $O_{10}O_{30} = 4\frac{1}{2}$, $BD = 4\frac{3}{4}$, etc. For simplicity, the driving link *1* is assumed to rotate at a uniform rate at 70 revolutions per minute. Hence, B has normal but no tangential acceleration. The linear velocity of B is

$$V_{BO} = 2\pi R(\text{R.P.S.}) = 2\pi\left(\frac{1.625}{12}\right)\frac{70}{60} = 0.993 \text{ ft./sec.}$$

Since the graphical determination of the accelerations of the points C, D, E, and F is of primary interest, the space and acceleration scales were chosen arbitrarily, and the velocity scale calculated in accordance with the preceding article. Using a space scale of 4 in. = 12 in., or 1 in. = $\frac{1}{4}$ foot, and an acceleration scale of 1 in. = 6 feet per second per second, k_s and k_a become $\frac{1}{4}$ and 6, respectively, and the velocity scale becomes 1 in. = 1.224 ft./sec., that is,

$$k_v = \sqrt{k_s k_a} = \sqrt{\tfrac{1}{4} \times 6} = 1.224 \text{ ft./sec.}$$

Hence, the velocity vector ob in Fig. 121 is drawn perpendicular to $O_{10}B$ to a length of $\dfrac{0.993}{1.224} = 0.81$ in. The velocity of C relative to O is perpendicular to $O_{30}C$, and the velocity of C relative to B is perpendicular to CB. Hence, in Fig. 121, the intersection of a line through o perpendicular to $O_{30}C$ with a line through b perpendicular to CB locates c and determines the vectors oc and bc representing the velocities V_{CO} and V_{CB}. The points e, d, and f may be located by proportion.

In accordance with the preceding article and the scales chosen, making the velocity vector BQ in Fig. 120 equal to the vector ob of the velocity vector diagram Fig. 121 and drawing QO' perpendicular to $O_{10}Q$ gives the length $O'B$ representing the normal, and in this case, the total acceleration of B relative to O to the scale chosen. Since this

normal acceleration acts from B towards O_{10}, the acceleration vector $o'b'$, Fig. 122, is drawn equal to $O'B$ and parallel to BO_{10}. In a similar manner, the normal acceleration component $O'C$ of C relative to O and the normal acceleration components $B'C$ and $B'D$ of points C and D relative to B can be obtained. Hence, in Fig. 122, the normal acceleration components $o'(co)$ and $b'(cb)$ are drawn parallel and equal to $O'C$ and $B'C$, respectively. The tangential acceleration components of C relative to O and of C relative to B are known to be perpendicular to

FIG 121

FIG 123

FIG 122

FIG 120
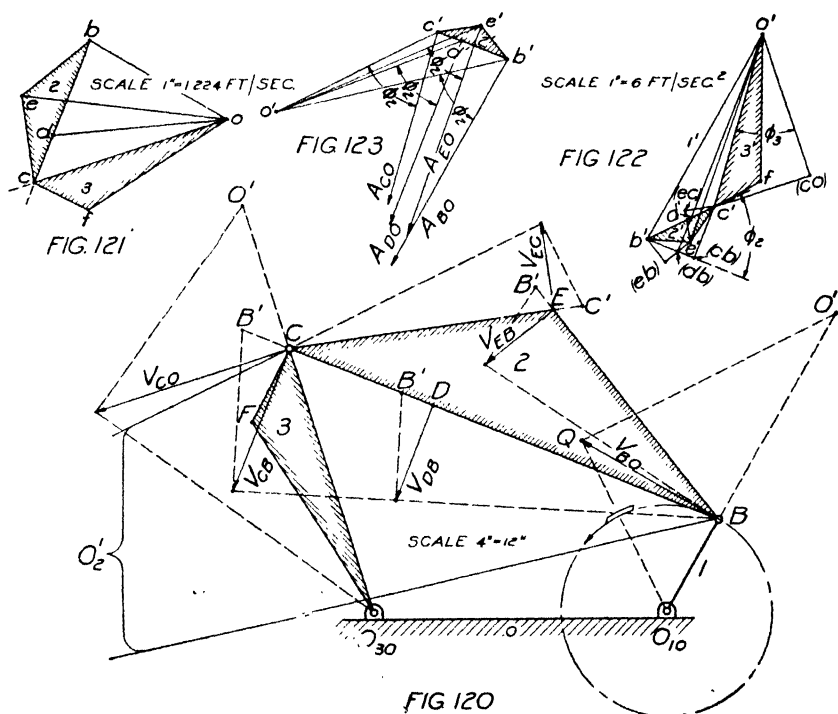
the normal acceleration components. Hence, the intersection of a line through (co) perpendicular to $o'(co)$ with a line through (cb) perpendicular to $b'(cb)$ determines c'' and the vector $o'c'$ representing the acceleration A_{CO} of C relative to O. The line $b'c'$ joining b' and c' determines the vector representing the acceleration A_{CB} of C relative to B. Since points C and D rotate about B with the same angular velocity and acceleration, the normal and tangential acceleration components for C and D must be proportional to their distances from B. Hence, it follows that the right triangles $b'(cb)c'$ and $b'(db)d'$ are similar, since their bases and altitudes are proportional. The points c' and d' and

b' are, therefore, in the same straight line.   Making vector $b'(db)$ equal to $B'D$ and drawing a perpendicular through (db) determines d' and the vector $o'd'$ representing A_{DO}, the acceleration of D relative to O.

In finding the acceleration of point D, it was pointed out that points C and D rotate about B with the same angular velocity and acceleration, and that, therefore, the normal and tangential acceleration components for C and D must be proportional to their distances from B.   Since this is true for any other point in link 2, the point e' may be found by proportion.   The point e' can also be found in a manner similar to that used for finding point c'.   The normal acceleration components $b'(eb)$ and $c'(ec)$ of E relative to rotation about B and C, respectively, are equal and parallel to $B'E$ and $C'E$.   Hence, the intersection of the tangential components through (eb) and (ec), perpendicular respectively to $b'(eb)$ and $c'(ec)$, determines e' and the acceleration vector $o'e'$ representing the acceleration A_{EO} of E relative to the fixed member O. Point f' may be located in a similar manner or by proportion.

Since c', d', b', and e' are the termini of the vectors $o'c'$, $o'd'$, $o'b'$, and $o'e'$, representing in magnitude and direction the accelerations of points C, D, B, and E of link 2, the figure $2'$ is properly termed the *acceleration image* of link 2 of the mechanism.   Thus also $3'$ is the acceleration image of link 3, and $1'$, or line $o'b'$, the acceleration image of link 1 of the mechanism.

• As explained in Art. 76, the angular velocity and angular acceleration of any link can be found by dividing the tangential velocity and tangential acceleration of any point relative to some other point in the link by the distance between the two points.

For applications of the acceleration vector diagram method to direct contact mechanisms see Arts. 82, 98, and 99.

79. Center of Acceleration.   In the preceding article it was shown in connection with Figs. 120 and 122 that the acceleration images $1'$, $2'$, and $3'$ for links 1, 2, and 3 are proportional to the respective links. Hence, the image $2'$ of Fig. 122 is proportional to link 2 of the mechanism, Fig. 120.   By construction, lines $b'(eb)$ and $b'(cb)$ are parallel to EB and CB, respectively.   Line $b'(cb)$ makes an angle ϕ_2 with $b'c'$, and line $b'(eb)$ makes the same angle with $b'e'$.   If the acceleration image $2'$ is rotated counter-clockwise about b' through an angle of $(180 - \phi_2)$, the line $b'c'$ will be parallel to line BC of the mechanism, image $2'$ will be parallel to link 2, and the origin o' will have moved to the position shown in Fig. 123.   Since, as previously shown, rays $o'c'$, $o'e'$, $o'd'$, and $o'b'$ are proportional to the accelerations of the points C, E, D, and B of link 2 of the mechanism, the origin o' corresponds to a point in link 2 having zero acceleration.   Having Fig. 123, the point in link 2 having

zero acceleration relative to the fixed member is easily located as the intersection O'_2 of lines through C and B, Fig. 120, parallel, respectively, to $c'o'$ and $b'o'$ of Fig. 123. This point is appropriately termed the *instant center of acceleration*. Just as the length of a ray from the instant center of velocity to any point in a link is proportional to the velocity of the point, so the length of a ray from the instant center of acceleration to any point in the link is proportional to the acceleration of the point. However, while the velocity vector for any point in a link always makes an angle of 90 degrees with the respective ray from the instant center of velocity, the acceleration vector for any point in a link makes an angle ϕ with the respective ray from the instant center of acceleration. The value of this angle varies with the angular velocity and angular acceleration of the link. It is an angle ϕ_2 for link *2* and a smaller angle ϕ_3 for link *3*. For convenience *2'* of Fig. 123 will be treated as link *2* of Fig. 120 to a different scale, in which case o' becomes the center of acceleration. Through b', e', d', and c', vectors have been drawn equal and parallel, respectively, to vectors $o'b'$, $o'e'$, $o'd'$, and $o'c'$ of Fig. 122 representing the accelerations A_{BO}, A_{EO}, A_{DO}, and A_{CO}. It is to be noted that the acceleration of each point in link *2* makes the same angle ϕ_2 with its respective ray.

It is to be noted that ϕ_3 is opposite in sense to ϕ_2, and therefore negative. Hence, the angle through which the image *3'* must be rotated about c' to bring it parallel to *3* is $(180 - \phi) = (180 + \phi_3)$. The acceleration vector for any point in link *3* will make an angle ϕ_3 with the ray drawn to the point from the center of acceleration of the link.

In Fig. 122 the vector $(cb)c'$ represents the tangential acceleration component A'_{CB} of C relative to B, and vector $b'(cb)$ the normal acceleration component A^n_{CB} of C relative to B. Hence,

$$\tan \phi = \frac{(cb)c'}{b'(cb)} = \frac{A'_{CB}}{A^n_{CB}} = \frac{BC\alpha_{20}}{BC\omega^2_{20}} = \frac{\alpha}{\omega^2} \quad . \quad . \quad . \quad (110)$$

The above equation shows that, to find the angle ϕ for any link, the angular velocity and angular acceleration of the link relative to the fixed member must be known, or the tangential and normal acceleration components of some point in the link relative to some other point in the link must be known or determined analytically or graphically. In addition to knowing ϕ, the directions of the accelerations of two points in the link relative to the fixed member must be known in order to locate the *center of acceleration*. There are special graphical constructions which enable the center of acceleration to be located if the direction

and magnitude of the accelerations of two points in the link are known. Obviously special methods are unnecessary where the acceleration vector diagram is available.

For further discussion of the methods of determining and using centers of acceleration the reader is referred to "Kinematics of Machinery" by A. W. Klein, or to "Mechanics of Machinery" by R. C. H. Heck.

80. Klein Construction. The Klein construction is a simple geometric construction for accurately determining the acceleration of the cross-head of a slider-crank chain. This construction is illustrated in Fig. 124 and is applicable only when the crank rotates at a uniform rate. A circle is first described with P as a center and PB as the radius. With E as a center and a radius equal to one-half the length of the connecting-rod, an arc is then drawn through P intersecting the circle at C and J. A line through C and J cuts the connecting-rod at D and the center-line HO_{10} at G. The acceleration of the cross-head is proportional to the length GO_{10}.

If, in accordance with Art. 77, the scales used in the velocity and acceleration vector diagrams, Figs. 125 and 126, are related thus, $k_a = \dfrac{k_v^2}{k_s}$, and the constant linear velocity of the center P of the crank-pin is represented by the length $O_{10}P$ of the crank, then the normal acceleration component A_{PO}^n of P relative to the frame will also be equal to $O_{10}P$. Hence, the scales are,

$$1 \text{ in.} = k_s \text{ ft.} = \frac{R \text{ in feet}}{O_{10}P \text{ in inches}}$$

$$1 \text{ in.} = k_v \text{ ft./sec.} = \frac{V_{PO} \text{ in ft./sec.}}{O_{10}P \text{ in inches}}$$

and

$$1 \text{ in.} = k_a \text{ ft./sec.}^2 = \frac{k_v^2}{k_s}$$

In the velocity vector diagram, Fig. 125, vector op is perpendicular and equal to $O_{10}P$, and oh and ph are respectively perpendicular to $O_{10}B$ and PB. Hence, the triangles $O_{10}PB$ and oph are equal, and PB represents V_{HP} to the same scale as ph. Laying off this vector from H and applying the right-angle construction determines the length $O'H$ or the length $p'(hp)$ representing the normal acceleration A_{HP}^n of H relative to P in the acceleration diagram, Fig. 126, to the same scale that

$o'p'$, or $O_{10}P$, represents the normal and, in this case, the total acceleration of P relative to the frame. Hence, referring to Fig. 124,

$$\frac{O'H}{HQ} = \frac{HQ}{HP} \text{ and } \frac{DP}{CP} = \frac{CP}{HP} \quad . \quad . \quad . \quad . \quad (111)$$

But CP and HQ are each equal to ph. Hence,

$$O'H = DP = p'(hp) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

The tangential acceleration of H relative to P is perpendicular to HP, and the motion of H relative to the frame is rectilinear. Hence, the intersection of a line through (hp) perpendicular to $p'(hp)$ with a line
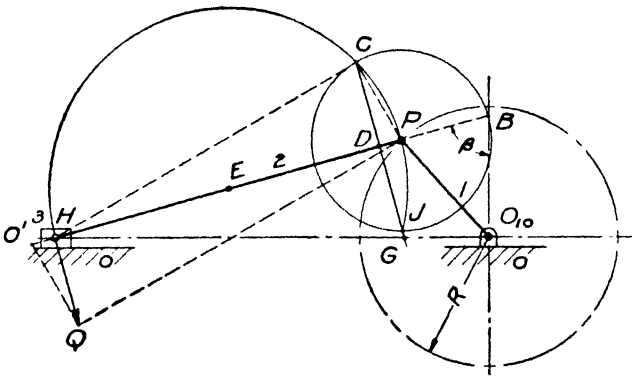
FIG. 125
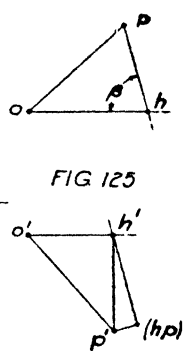
FIG. 124

FIG. 126

through o' parallel to HO_{10} determines h' and the vectors $o'h'$ and $p'h'$ representing, respectively, the accelerations A_{HO} and A_{HP}. By construction $o'p'$ and $p'(hp)$ are respectively parallel and equal to $O_{10}P$ and DP, and $(hp)h'$ and $o'h'$ are respectively parallel to DG and GO_{10}. Hence, figure $o'p'(hp)h'$ is equal to figure $O_{10}PDG$, and the former, if revolved through 180 degrees, could be superimposed on the latter. Therefore, GO_{10} represents the acceleration of the cross-head to the same scale as $o'h'$.

81. Coriolis' Law. If a point moves along a path as the path is *translated*, the acceleration of the point is the vector sum of the acceleration of the point relative to the path and the acceleration of the coincident point or any other point of the path. If, however, a point moves along a path which is *rotating*, the acceleration of the point, as will be shown, is the vector sum of the two vectors just mentioned plus a third vector; that is, the acceleration of the point will be the

vector sum of the acceleration of the point relative to the path, the acceleration of the coincident point of the path, and a complementary acceleration.

In the four-link mechanism shown in Fig. 127, links *1* and *3* have been extended so that they overlap. If link *1* is oscillated, the point D'

FIG. 129

FIG. 130

FIG. 131

FIG. 132

FIG 128

FIG. 127
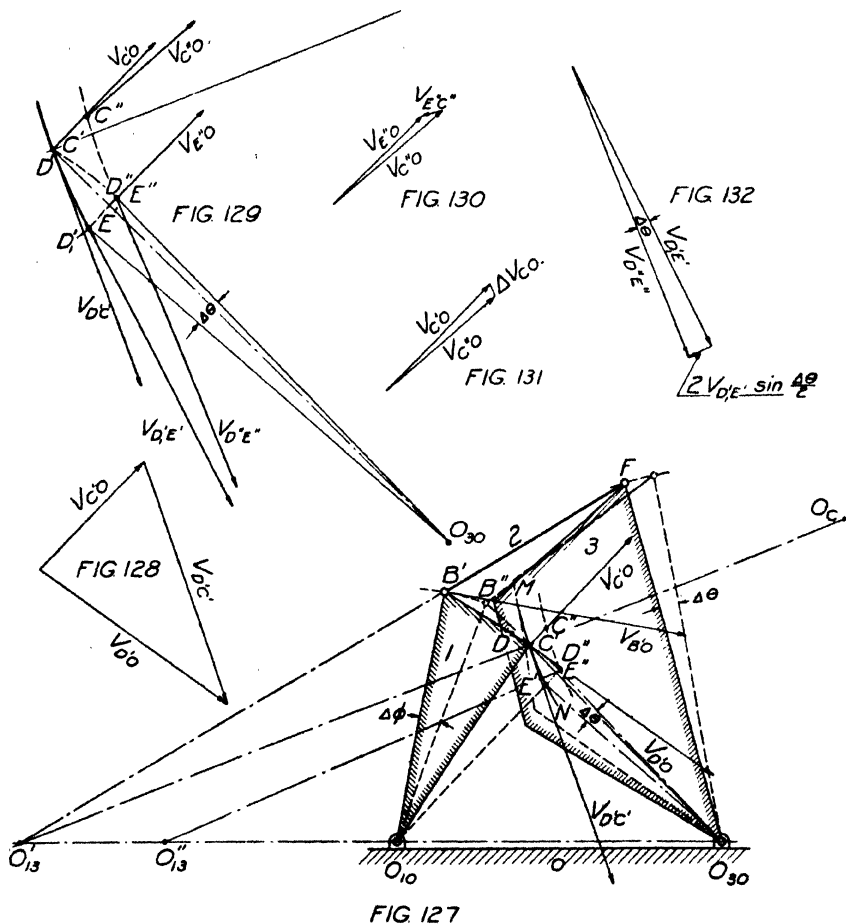
on link *1* will describe a curve MN on link *3*. For the initial position shown, point C' on *3* is coincident with D' on *1*. If *1* is moved through a small angle $\Delta\phi$, link *3* will be moved through a small angle $\Delta\theta$. During this rotation of *1*, point D' will have moved to D'', and the portion $C'E'$ of the curve on *3* will have rotated with link *3* to the position $C''E''$, E'' now being the point coincident with D'', the new position of point D.

As link *1* rotates through an angle $\Delta\phi$ in an interval of time Δt, it will be assumed that the linear velocity of B' changes from $V_{B'O}$ to another value $V_{B''O}$. Having these velocities, the linear velocities $V_{D'O}$ and $V_{D''O}$ of point D in *1* relative to the fixed member O can be found, also the velocities of $V_{C'O}$ and $V_{E''O}$ of the coincident points C' and E'' in link *3* relative to O. As previously shown,

$$V_{D'O} = V_{C'O} \nrightarrow V_{D'C'} \quad . \quad . \quad . \quad . \quad . \quad (113)$$

and

$$V_{D''O} = V_{E''O} \nrightarrow V_{D''E''} \quad . \quad . \quad . \quad . \quad (114)$$

Hence, as shown in Fig. 128, the velocity $V_{D'C'}$ of D' relative to the coincident point C' becomes known. Since D' is a point in *1* and C' is a point in *3*, the motion of D' relative to C' is about the instant center O'_{13}, and vector $V_{D'C'}$ is normal to the instant radius $O'_{13}D'$ and tangent to the curve MN at C'. Similarly, as shown in Figs. 127 and 129, vector $V_{D''E''}$ is normal to the instant radius $O''_{13}D''$ and tangent to the curve at E''.

The change in the velocity of D relative to the fixed member O in the interval of time Δt is, from equations (113) and (114);

$$V_{D''O} \rightarrow V_{D'O} = V_{E''O} \rightarrow V_{C'O} \nrightarrow V_{D''E''} \rightarrow V_{D'C'} \quad . \quad . \quad . \quad (115)$$

But, as shown in Fig. 130,

$$V_{E''O} = V_{C''O} \nrightarrow V_{E''C''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

Hence,

$$V_{D''O} \rightarrow V_{D'O} = V_{C''O} \nrightarrow V_{E''C''} \rightarrow V_{C'O} \nrightarrow V_{D''E''} \rightarrow V_{D'C'} \quad (117)$$

As shown in Figs. 129 and 131, $V_{C''O} \rightarrow V_{C'O} = \Delta V_{CO}$ is the change in the velocity of the coincident point C' as a fixed point on the curve due to the rotation of the curve. The velocity of a point along a curve at any instant is tangent to the curve at the position of the point. The magnitude of the velocity of the moving point is, relative to the curve, the same at any particular position on the curve, whether the curve is moving or is stationary relative to a reference body. Now points E' and E'' mark identically the same point on the two positions of the curve MN. Hence, $V_{D_1'E'}$ and $V_{D''E''}$ are, in magnitude, the same, but differ in direction, being respectively tangent to the curves at E' and E''. As shown in Figs. 127, 129, and 132, vectors $V_{D_1'E'}$ and $V_{D''E''}$ differ in direction by the angle $\Delta\theta$ through which link *3* has rotated in the interval Δt. Hence, Fig. 132,

$$V_{D''E''} = V_{D_1'E'} \nrightarrow 2V_{D_1'E'} \sin\frac{\Delta\theta}{2} \quad . \quad . \quad . \quad (118)$$

Substituting for $(V_{C''O} \to V_{C'O})$ and $V_{D''E''}$ in equation (117) gives

$$V_{D''O} \to V_{D'O} = \Delta V_{CO} \nrightarrow V_{E''C''} \nrightarrow V_{D_1'E'} \nrightarrow 2V_{D_1'E'} \sin\frac{\Delta\theta}{2} \to V_{D'C'} \quad (119)$$

But, considering the curve stationary, $V_{D_1'E'} \to V_{D'C'} = \Delta V_{DC}$, the change in the velocity of D in the interval Δt as it moves along the curve. Hence,

$$\Delta V_{DO} = V_{D''O} \to V_{D'O} = \Delta V_{CO} \nrightarrow \Delta V_{DC} \nrightarrow 2V_{D_1'E'} \sin\frac{\Delta\theta}{2} \nrightarrow V_{E''C''} \quad (120)$$

and

$$\frac{\Delta V_{DO}}{\Delta t} = \frac{\Delta V_{CO}}{\Delta t} \nrightarrow \frac{\Delta V_{DC}}{\Delta t} \nrightarrow \frac{2V_{D_1'E'}\sin\dfrac{\Delta\theta}{2}}{\Delta t} \nrightarrow \frac{V_{E''C''}}{\Delta t} \quad . \quad . \quad . \quad . \quad (121)$$

In the limit,

$$\frac{\Delta V_{DO}}{\Delta t} = A_{DO}$$

the acceleration of the describing point D relative to the fixed member O.

And, in the limit,

$$\frac{\Delta V_{CO}}{\Delta t} = A_{CO}$$

the acceleration of the coincident point C as a fixed point on the curve relative to the fixed member O due to the rotation of the curve about the instant center O_{30}.

And, in the limit,

$$\frac{\Delta V_{DC}}{\Delta t} = A_{DC}$$

the acceleration of the describing point D relative to the coincident point C due to the rotation of link 1 relative to link 3 about the instant center O'_{13}. The tangential component A'_{DC} of the acceleration A_{DC} is tangent to the curve at C, and the normal component A^n_{DC} is directed toward the center of curvature O_C of the curve and is equal to V^2_{DC} divided by the radius of curvature $O_C C'$.

Also, in the limit,

$$\frac{2V_{D_1'E'}\sin\dfrac{\Delta\theta}{2}}{\Delta t} = V_{DC}\frac{d\theta}{dt} = V_{DC}\omega_{30}$$

where V_{DC} is the linear velocity of the describing point D along the curve, and ω_{30} or ω_C is the angular velocity of the curve or coincident point C relative to the fixed member O.

It has been shown that, having the velocity of one point relative to another point in the same link, the angular velocity of the link is the relative velocity divided by the distance between the points. Hence,

$$V_{E''C''} = C''E''\omega_{30} \quad \ldots \ldots \ldots \quad (122)$$

In the limit, the distance between the points C'' and E'' is $V_{DC}dt$. Therefore,

$$\frac{V_{E''C''}}{\Delta t} = \frac{V_{DC}\omega_{30}dt}{dt} = V_{DC}\omega_{30}$$

Hence, by substitution in equation (121),

$$A_{DO} = A_{CO} +\!\!+ A_{DC} +\!\!+ 2V_{DC}\omega_C \quad \ldots \quad (123)$$

where, to be general, ω_C has been used instead of ω_{30} to represent the angular velocity of the curve or coincident point C.

Equation (123) is a mathematical statement of Coriolis' law. It is to be noted that if the curve described by the point D had a motion of translation, ω_C would be zero and A_{DO} would be equal to the sum of two vectors, A_{CO} and A_{DC}. This statement and equation (123) are in agreement with the statements made at the beginning of this article.

The compound supplementary acceleration $2V_{DC}\omega_C$ is such that if considered as a force acting at the terminus of the vector V_{DC} it would cause the vector to rotate about its origin in the same direction as the curve rotates about its instant center. Hence, for the mechanism discussed above, the supplementary acceleration $2V_{DC}\omega_C$ acts along the line from D' toward the instant center O'_{13}.

The compound supplementary acceleration $2V_{DC}\omega_C$ is due to the fact that the curve MN has a motion of rotation. Because of this motion the vector difference between $V_{E''O}$ and $V_{C''O}$ affects the acceleration or rate of change of velocity of point D. The acceleration of point D is also affected by the fact that while $V_{D_1'E'}$ and $V_{D''E''}$ are equal in magnitude they are different in direction. As before pointed out, if the curve MN had a motion of translation, these two factors would disappear.

It should be noted that for the mechanism used in the above demonstration, the acceleration of the coincident point C could be found by first finding the acceleration of point F in link 3 by applying the velocity and acceleration vector diagram method. This method of solution is illustrated in Figs. 133, 134, and 135. For convenience the essentials of Fig. 127 are repeated in Fig. 133, where $O_{10}B = 1.333$, $O_{10}D = 1.243$, $BF = 1.125$, $O_{30}F = 1.953$, $O_{30}C = 1.473$, and $O_{10}O_{30} = 1.75$ feet, the space scale being 1 in. = 1 foot or 1 in. = k_s = 1 foot. Link 1 is

assumed to have a uniform angular velocity ω_{10} of one radian per second; hence, the scale of Fig. 134 is 1 in. $= k_v = 1$ foot per second. The scale of the acceleration vector diagram, Fig. 135, is in accordance with the relation $k_a = \dfrac{k_v{}^2}{k_s}$ or 1 in. $= k_a = 1$ foot per second per second. In accordance with scale k_v, the known velocity of B is represented in Fig. 134 by vector ob equal in length to $O_{10}B$ and perpendicular to it. The velocities V_{DO} and V_{DB} are known to be perpendicular, respectively, to $O_{10}D$ and BD. Hence, the intersection of a line through o perpendicular to $O_{10}D$ with a line through b perpendicular to BD determines d and the velocity image 1 of link 1. In a similar manner the intersection of a line through b perpendicular to BF with a line through o perpendicular to $O_{30}F$ determines f and the velocity vector of The intersection of lines through o and f perpendicular, respectively, to $O_{30}C$ and FC determines c, the line joining c and d representing to scale the velocity V_{DC}.

Since the angular velocity of link 1 is uniform, the acceleration A_{DO} of point D relative to the fixed member O is directed from D towards O_{10}, and is represented to scale in Fig. 135 by a vector $o'd'$ equal in length to DO_{10}, or 1.243 in. The acceleration vectors $b'(fb)$ and $o'(fo)$ representing the normal acceleration components A_{FB}^n and A_{FO}^n are parallel to FB and FO_{30}, and equal to

$$b'(fb) = \frac{A_{FB}^n}{k_a} = \frac{V_{FB}^2}{(FB)k_s} \times \frac{1}{k_a} = \frac{0.\overline{62}^2}{1.125 \times 1} \times \frac{1}{1} = 0.341 \text{ in.}$$

and

$$o'(fo) = \frac{A_{FO}^n}{k_a} = \frac{V_{FO}^2}{(FO_{30})k_s} \times \frac{1}{k_a} = \frac{\overline{1.045}^2}{1.953 \times 1} \times \frac{1}{1} = 0.559 \text{ in.}$$

Point f' is determined by the intersection of lines through (fb) and (fo) perpendicular, respectively, to $b'(fb)$ and $o'(fo)$. The acceleration image $o'c'f'$ of $O_{30}CF$ and image $3'$ of link 3 may be constructed by proportion on $o'f'$, the image of $O_{30}F$, as a base. Vector $o'c'$ represents in magnitude and direction the acceleration of the coincident point C which was to be found, or

$$A_{CO} = (o'c')k_a = 0.933 \times 1 = 0.933 \text{ ft./sec.}^2$$

The determination of the acceleration of the coincident point C by the application of Coriolis' law is represented in Fig. 136. The acceleration A_{DO} of the describing point D relative to the fixed member O is given by equation (123) as,

$$A_{DO} = A_{CO} +\!\!\!+ A_{DC} +\!\!\!+ 2V_{DC}\omega_C$$

As shown above, the vector $o'd'$ representing the acceleration A_{DO} of the describing point is known in magnitude and direction. The relative

FIG. 133

FIG. 134

SCALE 1″= 1 FT/SEC.

FIG. 135

FIG. 136

FIG. 137

SCALE 1″= 1 FT/SEC.²
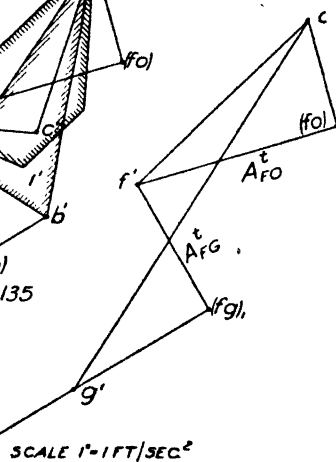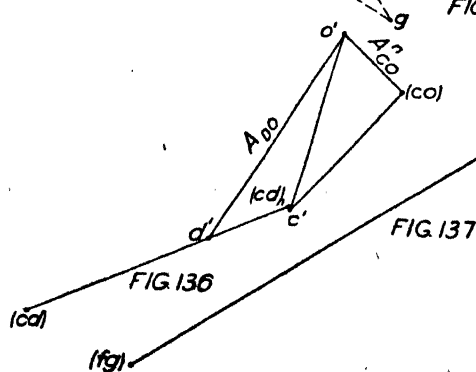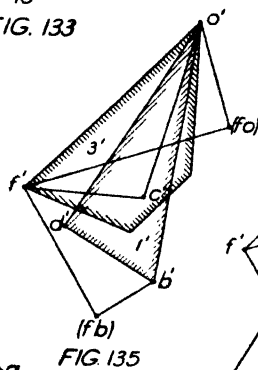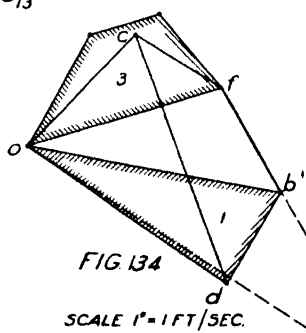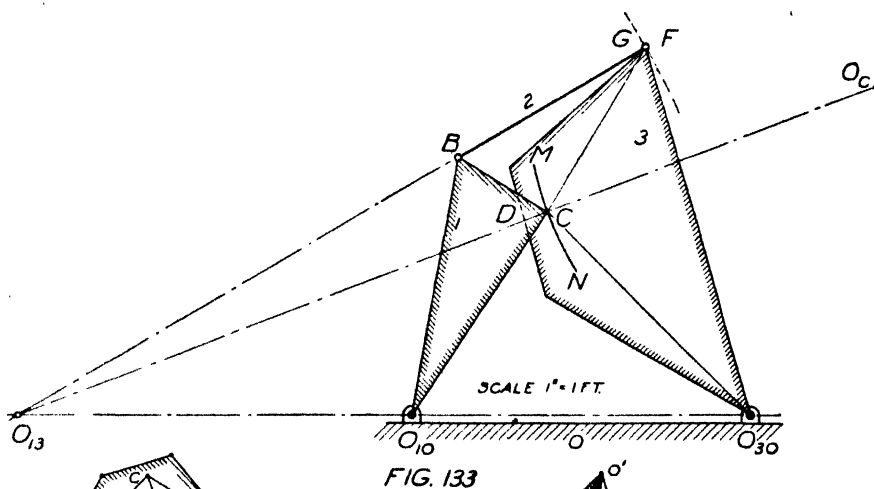
velocities V_{CO} and V_{DC} are also known from the velocity vector diagram, Fig. 134. The angular velocity ω_C, the supplementary term $2V_{DC}\omega_C$,

and the vectors $o'(co)$ and $c'(dc)$ representing the normal acceleration components A_{CO}^n and A_{DC}^n can therefore be computed:

$$\omega_C = \frac{V_{CO}}{(O_{30}C)k_s} = \frac{0.793}{1.473 \times 1} = 0.54 \text{ rad./sec.}$$

The vector representing $2V_{DC}\omega_C = \dfrac{2V_{DC}\omega_C}{k_a} = \dfrac{2 \times 1.353 \times 0.54}{1} = 1.46$ in.

$$o'(co) = \frac{A_{CO}^n}{k_a} = \frac{V_{CO}^2}{(O_{30}C)k_sk_a} = \frac{0.793^2}{1.473 \times 1 \times 1} = 0.427 \text{ in.}$$
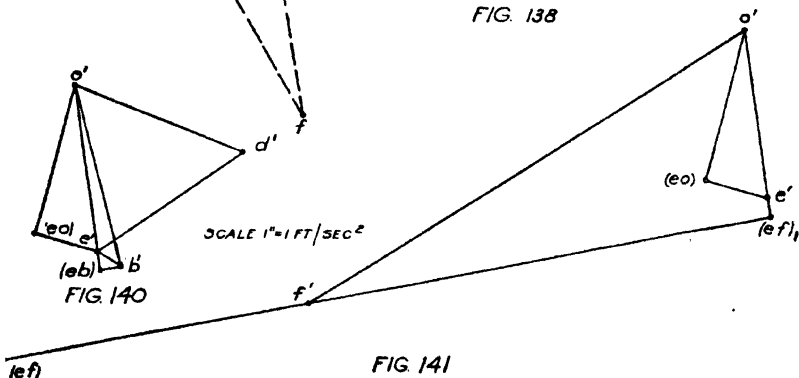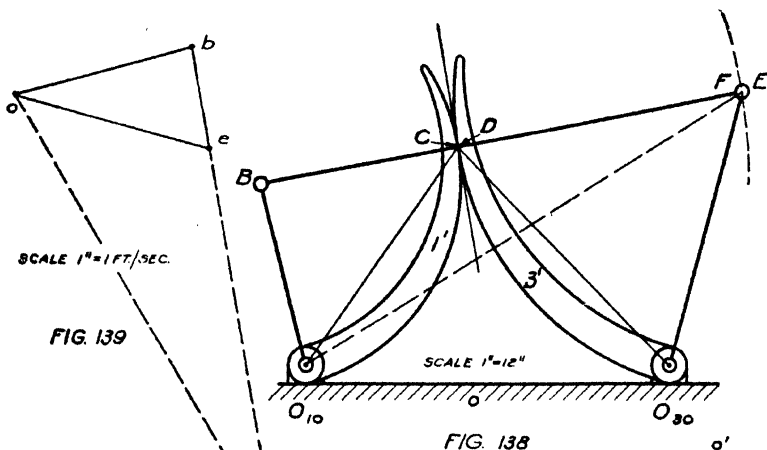
$$c'(dc) = \frac{A_{DC}^n}{k_a} = \frac{V_{DC}^2}{(O_CC)k_sk_a} = \frac{1.353^2}{1.813 \times 1 \times 1} = 1.01 \text{ in.}$$

where O_CC is the radius of curvature of the curve at point C. Unless the law of the curve is known, this radius must be determined by trial from a trace of the curve. Whether this can be done with acceptable accuracy depends upon the nature of the curve. In the present instance the first rough trial gave a radius of curvature 3.5 per cent too large. As will be shown, this uncertainty can often be avoided by a proper choice of the describing point.

It is known that vector $o'(co)$, Fig. 136, is parallel to CO_{30} and is directed as from C toward O_{30}, and that $c'(dc)$ is parallel to CO_C and is directed as from C toward the center of curvature O_C. The direction of the vector representing $2V_{DC}\omega_C$ is such that if acting as a force at the terminus of the vector cd, representing the velocity V_{DC}, it would rotate the vector in the same sense as the curve MN is rotating, that is, clockwise. Hence, the vector representing $2V_{DC}\omega_C$ acts parallel to O_CC and is directed as from O_C towards C, the opposite to that of vector $c'(dc)$. Regular procedure would require that vector $c'(dc)$ should be drawn from c', whose location is unknown. This difficulty is due to the fact that the describing point D was taken on the link whose motion was known. To facilitate the construction of the diagram, Fig. 136, we may write

$$A_{CD}^n = -A_{DC}^n \qquad \text{or,} \qquad d'(cd) = -c'(dc)$$

and

$$2V_{CD}\omega_C = -2V_{DC}\omega_C \qquad \text{or,} \qquad (cd)(cd)_1 = -(dc)(dc)_1$$

Hence $d'(cd) = 1.01$ in. is drawn to the left parallel to O_CC, and $(cd)(cd)_1 = 1.46$ in. is drawn to the right parallel to CO_C locating $(cd)_1$. Where a line through $(cd)_1$ perpendicular to $d'(cd)_1$ intersects a line through (co) perpendicular to $o'(co)$ locates c' and determines the

vector $o'c'$ representing in magnitude and direction the acceleration A_{CO} of the coincident point. It is to be noted that $o'c'$ of Fig. 136 is in agreement with $o'c'$ of Fig. 135.

Had the describing point been taken on link *3*, the curve being traced on link *1*, the difficulty encountered above would have been avoided. In general, the application of Coriolis' law is facilitated if the describing point is so chosen as to trace a curve on the member whose motion is given. The easiest way to determine the acceleration of a point on link *3* by the application of Coriolis' law is to extend link *1* to overlap point F as the describing point on link *3*, G being the coincident point on *1*. Obviously the curve traced on *1* is the arc of a circle having its center of curvature at B. In accordance with equation (123), the acceleration of the describing point F is

$$A_{FO} = A_{GO} + A_{FG} + 2V_{FG}\omega_G$$

Since the curve is traced on *1*, the angular velocity of the curve or the coincident point G is that of link *1*, or one radian per second. The velocity vectors og and gf representing the velocities V_{GO} and V_{FG} are determined by drawing from o and f in Fig. 134 lines perpendicular to $O_{10}G$ and $O_{13}F$, respectively, until they intersect at g. Hence,

$$o'g' = \frac{A_{GO}}{k_a} = \frac{A_{GO}^n}{k_a} = \frac{V_{GO}^2}{(O_{10}G)k_sk_a} = \frac{2.23^2}{2.23 \times 1 \times 1} = 2.23 \text{ in.}$$

$$o'(fo) = \frac{A_{FO}^n}{k_a} = \frac{V_{FO}^2}{(O_{30}F)k_sk_a} = \frac{1.043^2}{1.953 \times 1 \times 1} = 0.559 \text{ in.}$$

$$g'(fg) = \frac{A_{FG}^n}{k_a} = \frac{V_{FG}^2}{(BF)k_sk_a} = \frac{1.717^2}{1.125 \times 1 \times 1} = 2.627 \text{ in.}$$

$$(fg)(fg)_1 = \frac{2V_{FG}\omega_G}{k_a} = \frac{2 \times 1.717 \times 1}{1} = 3.434 \text{ in.}$$

Vector $o'g'$ is drawn parallel to GO_{10}, $o'(fo)$ parallel to FO_{30}, $g'(fg)$ parallel to GB, and $(fg)(fg)_1$ parallel to BG. The intersection of a line through (fo) perpendicular to $o'(fo)$ with a line through $(fg)_1$ perpendicular to $g'(fg)_1$ determines f' and the vector $o'f'$ representing in magnitude and direction the acceleration A_{FO} of the describing point F. It is to be noted that $o'f'$ of Fig. 137 is equal and parallel to $o'f'$ of Fig. 135. If the acceleration of any other point in link *3* such as C is desired, it can be found by proportion.

82. Applications of Coriolis' Law. (a) *Direct Contact Mechanism.* In Fig. 138 the curved link *3'* is driven by the curved link *1'*, D being

the contact point in $3'$ and C the coincident contact point in $1'$. As before pointed out, the line of transmission at the instant is the common normal to the curves at the point of contact. The centers of curvature B and E of the driving and driven curves at the point of contact must lie on this common normal. Center B as a point in $1'$ and center E as a point in $3'$ will have the same motion at the instant as though they

SCALE $1'' = 1\ FT./SEC.$

FIG. 139

SCALE $1'' = 12''$

FIG. 138

SCALE $1'' = 1\ FT/SEC^2$

FIG. 140

FIG. 141

were points of the four-link mechanism shown. Hence, the acceleration of point D in $3'$ may be found by first finding the acceleration of E as a point in the equivalent four-link mechanism by applying the acceleration vector diagram method or by applying Coriolis' law. These two solutions are illustrated in Figs. 138, 139, 140, and 141.

In Fig. 138, $O_{10}B = 1.00$, $O_{10}C = 1.425$, $BE = 2.69$, $O_{30}E = 1.5$, $O_{30}D = 1.655$, and $O_{10}O_{30} = 2.00$ feet, the space scale being 1 in. = 1 foot or 1 in. = k_s = 1 foot. Link 1 is assumed to have a uniform

angular velocity ω_{10} of one radian per second; hence, the scale of Fig. 139 is 1 in. $= k_v = 1$ foot per second. The scale of the acceleration vector diagrams Figs. 140 and 141 are in accordance with the relation $k_a = \dfrac{k_v{}^2}{k_s}$, or, 1 in. $= k_a = 1$ foot per second per second. In accordance with scale k_v, the known velocity of point B is represented in Fig. 139 by the vector ob equal in length to $O_{10}B$ and perpendicular to it. The velocities V_{EB} and V_{EO} are perpendicular, respectively, to BE and $O_{30}E$. Hence, e is determined by the intersection of lines through b and o perpendicular, respectively, to BE and $O_{30}E$. Since the angular velocity of link $1'$ or 1 is uniform, the acceleration A_{BO} of B relative to the fixed member O is directed from B towards O_{10}, and is represented to scale in Fig. 140 by a vector $o'b'$ equal in length to $O_{10}B$, or 1.00 in. The velocities V_{EO} and V_{EB} being known, the acceleration vectors $o'(eo)$ and $b'(eb)$ representing the normal acceleration components A_{EO}^n and A_{EB}^n can be computed:

$$o'(eo) = \frac{A_{EO}^n}{k_a} = \frac{V_{EO}^2}{(O_{30}E)k_sk_a} = \frac{\overline{1\cdot11}^2}{1.5 \times 1 \times 1} = 0.82 \text{ in.}$$

$$b'(eb) = \frac{A_{EB}^n}{k_a} = \frac{V_{BE}^2}{(BE)k_sk_a} = \frac{\overline{0.56}^2}{2.69 \times 1 \times 1} = 0.116 \text{ in.}$$

Hence, the intersection of lines through (eo) and (eb) perpendicular respectively to $o'(eo)$ and $b'(eb)$ determines e' and the vector $o'e'$ representing in magnitude and direction the acceleration A_{EO}. Construction by proportion on $o'e'$ as a base determines the location of d'. Hence,

$$A_{EO} = o'e' \times k_a = 0.895 \times 1 = 0.895 \text{ ft./sec.}^2$$

$$A_{EO}^t = (eo)e' \times k_a = 0.355 \times 1 = 0.355 \text{ ft./sec.}^2$$

$$\alpha_{EO} = \frac{A_{EO}^t}{(O_{30}E)k_s} = \frac{0.355}{1.5 \times 1} = 0.237 \text{ rad./sec.}^2$$

$$A_{DO} = o'd' \times k_a = 0.99 \times 1 = 0.99 \text{ ft./sec.}^2$$

In Fig. 141, the acceleration A_{EO} has been determined by the application of Coriolis' law by taking E as the describing point on 3 and F as the coincident point on 1 extended. Obviously, the curve described on 1 is the arc of a circle having B as its center of curvature. By equation (123),

$$A_{EO} = A_{FO} + A_{EF} + 2V_{EF}\omega_F$$

The motion of E relative to F is about the instant center O_{13}, which

is at the intersection of $O_{30}O_{10}$ and FB extended. Hence, V_{EF} is at right angles to BF. Point F in link 1 rotates about the instant center O_{10}, and its velocity V_{FO} is at right angles to $O_{10}F$. Since the motion of E relative to F is about the instant center O_{13}, and of F relative to the fixed member O is about the instant center O_{10}, velocities V_{EF} and V_{FO} are at right angles to FB and $O_{10}F$. Hence, in Fig. 139, the intersection of lines through o and e perpendicular to FB and $O_{10}F$ determines f. Having $\omega_F = \omega_{10}$, V_{EO}, V_{EF}, and V_{FO}, the vectors representing A^n_{FO}, A^n_{EF}, and $2V_{EF}\omega_F$ can be computed:

$$\omega_F = \omega_{10} = 1 \text{ rad./sec.}$$

$$o'f' = \frac{A_{FO}}{k_a} = \frac{A^n_{FO}}{k_a} = \frac{V^2_{FO}}{(O_{10}F)k_sk_a} = \frac{2.8^2}{2.8 \times 1 \times 1} = 2.8 \text{ in.}$$

$$o'(eo) = \frac{A^n_{EO}}{k_a} = \frac{V^2_{FO}}{(O_{30}E)k_sk_a} = \frac{1.11^2}{1.5 \times 1 \times 1} = 0.82 \text{ in.}$$

$$(ef)(ef)_1 = \frac{2V_{EF}\omega_F}{k_a} = \frac{2 \times 2.14 \times 1}{1} = 4.28 \text{ in.}$$

$$f'(ef) = \frac{A^n_{EF}}{k_a} = \frac{V^2_{EF}}{(BF)k_sk_a} = \frac{2.14^2}{2.69 \times 1 \times 1} = 1.70 \text{ in.}$$

where BF is the radius of curvature at F on the curve traced on link 1 by the describing point E. The acceleration A_{FO} being directed from F towards O_{10}, the vector $o'f'$ is drawn parallel to FO_{10}. The acceleration component A^n_{EF} is directed from F towards the center of curvature B, and the supplementary acceleration component $2V_{EF}\omega_F$ is so directed that if applied as a force at the terminus of the velocity vector fe representing V_{EF} it would rotate the vector in the same sense as the curve or link 1 is rotating. Hence, in Fig. 141, vector $f'(ef)$ is directed parallel to EB, and $(ef)(ef)_1$ is directed parallel to BE. The vectors representing the tangential components A^t_{EO} and A^t_{EF} are drawn through (eo) and $(ef)_1$ perpendicular to $o'(eo)$ and $f'(ef)_1$, respectively, their intersection e' determining the vector $o'e'$ representing in magnitude and direction the acceleration A_{EO}, which was to be determined. It is to be noted that $o'e'$ of Fig. 141 is equal and parallel to vector $o'e'$ of Fig. 140. If desired, d' can be located by proportion.

To determine A_{DO} by Coriolis' law without resorting to the equivalent four-link mechanism, it is necessary to find the curve MN traced on link $1'$ by the describing point D on link $3'$ as shown in Fig. 142. By Coriolis' law,

$$A_{DO} = A_{CO} \leftrightarrow A_{DC} \leftrightarrow 2V_{DC}\omega_C$$

The angular velocity of link $1'$ being known, vector oc in Fig. 143, representing to the scale of $k_v = 1$ the velocity V_{CO} of the coincident point C relative to the fixed member O, is known to be equal in length to $O_{10}C$ and perpendicular to it. Since V_{DO} and V_{DC} are perpendicular, respectively, to $O_{30}D$ and the common normal O_CC, lines drawn through o and c in Fig. 143 perpendicular, respectively, to $O_{30}D$ and O_CC deter-
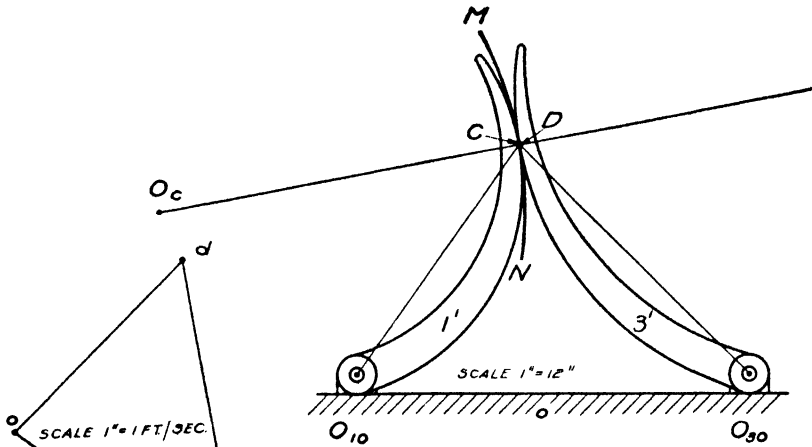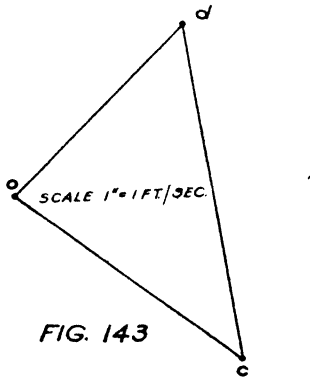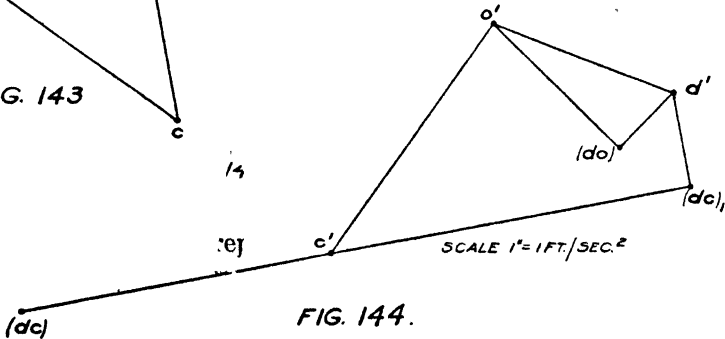

FIG. 142

FIG. 143

FIG. 144.

mine the location of d and the vectors od and cd representing the velocities V_{DO} and V_{DC}. The lengths of the vectors representing A_{DO}^n, A_{CO}, A_{DC}^n, and $2V_{DC}\omega_C$ can therefore be computed;

$$o'(do) = \frac{A_{DO}^n}{k_a} = \frac{V_{DO}^2}{(O_{30}D)k_sk_a} = \frac{\overline{1.22}^2}{1.655 \times 1 \times 1} = 0.90 \text{ in.}$$

$$o'c' = \frac{A_{CO}}{k_a} = \frac{A_{CO}^n}{k_a} = \frac{V_{CO}^2}{(O_{10}C)k_sk_a} = \frac{\overline{1.43}^2}{1.43 \times 1 \times 1} = 1.43 \text{ in.}$$

$$(dc)(dc)_1 = \frac{2V_{DC}\omega_C}{k_a} = \frac{2 \times 1.725 \times 1}{1} = 3.45 \text{ in.}$$

$$c'(dc) = \frac{A_{DC}^n}{k_a} = \frac{V_{DC}^2}{(O_C C)k_s k_a} = \frac{\overline{1.725}^2}{1.865 \times 1 \times 1} = 1.595 \text{ in.}$$

where $O_C C$ is the radius of curvature at C of the curve MN traced on link $1'$ by the describing point D on link $3'$. In Fig. 144, vector $o'c'$ = 1.43 in. is laid off parallel to CO_{10}, $o'(do)$ parallel to DO_{30}, $c'(dc)$ parallel to CO_C, and $(dc)(dc)_1$ parallel to $O_C C$. The intersection of lines through (do) and $(dc)_1$ perpendicular, respectively, to $o'(do)$ and $c'(dc)_1$, determines d' and the vectors $(do)d'$, $(dc)_1 d'$, and $o'd'$ representing A'_{DO}, A'_{DC}, and A_{DO}. It is to be noted that $o'd'$ of Fig. 144 is equal and parallel to $o'd'$ of Fig. 140.

For further applications to direct-contact mechanisms of Coriolis' law and the acceleration vector diagram see Arts. 98 and 99.

(b) *Shaper Mechanism.* In the shaper mechanism shown in Fig. 145 it is necessary to apply Coriolis' law or its equivalent to find the acceleration of a point on the vibrator, link *3*. As usual in such mechanisms, the driving crank *1* will be assumed to rotate at a uniform rate. In the present problem the crank, which is $5\frac{1}{2}$ in. long, has been assumed to rotate at 21 revolutions per minute. The crank carries a crank-pin which is fitted with a sliding block *2* which slides in or on the vibrator *3* as the crank rotates. The sliding block causes link *3* to oscillate about the fixed center O_{30} which imparts a reciprocating motion through link *4* to the slide, or ram, *5*. The cutting tool of the shaper is attached to the ram, link *5*.

Let C be the coincident point on the axi of the crank-pin in link *2* which is in contact with link *3*. Evident. as the crank rotates, describing point D will describe a curve, in this case a straight line, on link *2*. The problem is to determine the total acceleration of the describing point D on link *3* which is at the instant coincident with point C on the curve on link *2*. In solving the problem a space scale of 1 in. = 12 in. = 1 foot and a velocity scale of 1 in. = 1.10 feet per second will be used. Hence, $k_s = 1.0$, $k_v = 1.10$, and, for a complete graphical solution, $k_a = \dfrac{k_v^2}{k_s} = \dfrac{\overline{1.10}^2}{1.0} = 1.21$ feet per second per second.

The velocity of the coincident point C is

$$V_{CO} = \pi D(\text{R.P.S.}) = \pi\tfrac{11}{12}(\tfrac{21}{60}) = 1.008 \text{ ft./sec.}$$

Hence the length of the velocity vector oc is

$$oc = \frac{V_{CO}}{k_v} = \frac{1.008}{1.10} = 0.917 \text{ in.}$$

This vector is laid off from a convenient pole o perpendicular to CO_{10}, as shown in Fig. 146. The motion of D relative to C is seen by inspection to be along EO_{30}; hence, the instant center O_{23} is at infinity. The describing point D on link 3 rotates about O_{30}, which fixes the direction of the velocity of D relative to the fixed member O as perpendicular to EO_{30}. Hence, the intersection of a line through c parallel to EO_{30} with a line through o perpendicular to EO_{30} determines d and the length of

FIG 146
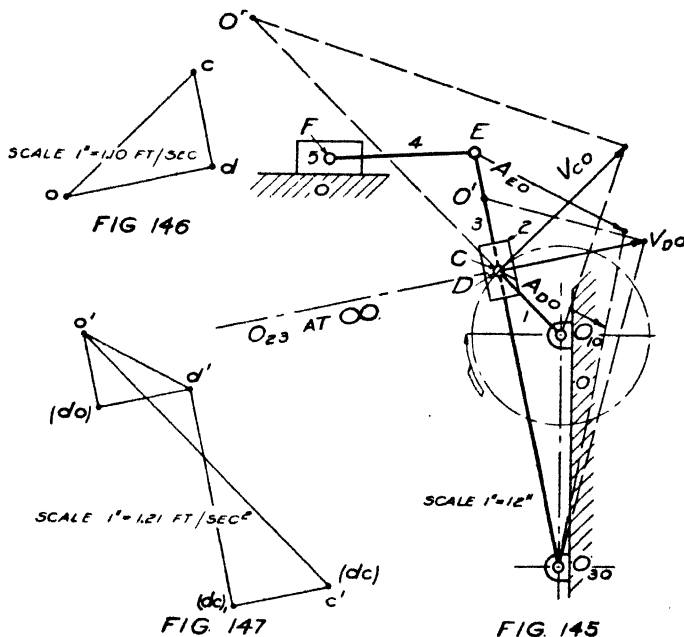
FIG. 147

FIG. 145

the vectors cd and od representing the velocities V_{DC} and V_{DO}. In accordance with equation (123)

$$A_{DO} = A_{CO} \rightarrowtail A_{DC} \rightarrowtail 2V_{DC}\omega_C$$

Placing the vectors representing V_{CO} and V_{DO} in Fig. 145, and applying the right-angle construction, determines the lengths $O'C$ and $O'D$ of the vectors $o'(co)$ and $o'(do)$ representing the normal acceleration components A_{CO}^n and A_{DO}^n in agreement with the scales chosen. Since link 1 rotates at a uniform rate, point C has no tangential acceleration, and A_{CO}^n represents the total acceleration of C relative to O. The normal acceleration component A_{DC}^n is zero, since D rotates relative to C about O_{23} at infinity. Hence, A_{DC}^t acting along EO_{30} is the acceleration A_{DC} of D relative to C. Summarizing,

$$o'c' = \frac{A_{CO}}{k_a} = \frac{A_{CO}^n}{k_a} = O'C = 1.83 \text{ in.}$$

$$o'(do) = \frac{A_{DO}^n}{k_a} = O'D = 0.38 \text{ in.}$$

$$c'(dc) = \frac{A_{DC}^n}{k_a} = \text{zero, making } c' \text{ and } (dc) \text{ coincident points.}$$

$$\omega_C = \omega_{20} = \omega_{30} = \frac{V_{DO}}{(O_{30}D)k_s} = \frac{0.847}{1.547 \times 1} = 0.547 \text{ rad./sec.}$$

$$(dc)(dc)_1 = \frac{2V_{DC}\omega_C}{k_a} = \frac{2 \times 0.55 \times 0.547}{1.21} = 0.497 \text{ in.}$$

$$(dc)_1 d' = \frac{A_{DC}}{k_a} \text{ acting parallel to } O_{30}E.$$

The direction of the vector $(dc)(dc)_1$ is such that if acting as a force at the end of the velocity vector cd it would rotate the vector in the same sense as link *2* or *3* is rotating, that is, clockwise. Hence, $(dc)(dc)_1$ acts parallel to CO_{23}. The acceleration vector diagram, Fig. 147, can therefore be constructed, since $o'c'$, $o'(do)$, and $(dc)(dc)_1$ are known in both magnitude and direction, and $(dc)_1d'$ and $(do)d'$ are known in direction, $(dc)_1d'$ and $(do)d'$ being parallel respectively to $O_{30}C$ and $O_{23}C$ and intersecting at d'. The line joining o' and d' determines the length of the vector $o'd'$ representing the acceleration A_{DO} of the describing point D on link *3* which was to be found.

$$A_{DO} = o'd' \times k_a = 0.625 \times 1.21 = 0.756 \text{ ft./sec.}^2$$

The velocities and accelerations of E and F relative to O may be found by extending the velocity and acceleration vector diagrams Figs. 146 and 147.

83. Three Line Construction for Velocities and Accelerations. The so-called three line construction is found of great assistance in completing the construction of velocity and acceleration vector diagrams for certain complex mechanisms. The scale of the mechanism shown in Fig. 148 is $\frac{1}{2}$ in. = 12 in. or 1 in. = 2 feet, hence $k_s = 2$. The velocities and accelerations of points B, C, D, E, and F for any motion of the driving link *1* are readily found by the ordinary vector diagram construction. A special construction, often referred to as the *three line construction*, is required to determine the velocities and accelerations of points G, H, and K of link *7*. The construction as applying to velocities will be dealt with first.

For this demonstration the driving link *1*, for the phase shown, is assumed to have an angular velocity of 15.4 radians per second but no angular acceleration. Hence, since radius $O_{10}C$ is 1.5 feet, the velocity of point C is

$$V_{CO} = \omega_{10}R = 15.4 \times 1.5 = 23.1 \text{ ft. per sec.}$$

This velocity is represented in Fig. 149 by vector *oc*, for which a convenient length of $\frac{7}{8}$ in. has been chosen. Therefore, the velocity scale is

$$1 \text{ in.} = k_v = 23.1 \div \tfrac{7}{8} = 26.4 \text{ ft. per sec.}$$

Velocities V_{EC} and V_{EO} are respectively perpendicular to CE and $O_{30}E$. Hence, the intersection *e* of a line through *c* perpendicular to CE, with a line through *o* perpendicular to $O_{30}E$, determines the velocity images *oc*, *ce*, and *oe* of links *1*, *2*, and *3*. Points *b*, *d*, and *f* may be located by proportion. The relative velocities V_{GB}, V_{HD}, and V_{KF} are known in direction but not in magnitude, being in direction perpendicular, respectively, to BG, DH, and FK. Hence, points *g*, *h*, and *k* of the velocity vector diagram Fig. 149 must lie on lines *G*, *H*, and *K* drawn through *b*, *d*, and *f* perpendicular, respectively, to BG, DH, and FK. The location of points *g*, *h*, and *k* on these lines must be such that lines joining them will form a triangle, or image, *ghk*, similar to GHK of Fig. 148 and at right angles to it. Through any point g_1 on line *G* draw a line perpendicular to GH of Fig. 148 intersecting line *H* at h_1. Through g_1 and h_1 draw lines perpendicular, respectively, to GK and HK. The chance that the intersection k_1 of these lines will fall on line *K* is very remote, since the location of g_1 on line *G* was arbitrarily chosen. However, the velocity image *ghk* must be parallel to $g_1h_1k_1$. Lines *G* and *H* intersect at *m*. Obviously the correct location of *k* on line *K* is where a line joining k_1 and *m* intersects *K*. Having *k*, the velocity image *7* of link *7* may be drawn and the velocity vector diagram completed by drawing the vectors *bg*, *dh*, *fk*, *og*, *oh*, and *ok*.

To admit of the various normal components of acceleration being determined graphically, the acceleration vector diagram of Fig. 150 is drawn to a scale of

$$1 \text{ in.} = k_a = \frac{k_v^2}{k_s} = \frac{\overline{26.4^2}}{2} = 348.5 \text{ ft. per sec.}^2$$

Since link *1* is without angular acceleration, the total acceleration of C is the normal acceleration represented by $O'C$, Fig. 148. From a convenient pole *o'* vector *o'c'* is drawn equal and parallel to $O'C$. The

vectors $c'(ec)$ and $o'(eo)$, representing the normal components of acceleration A_{EC}^n and A_{EO}^n of E relative to C and to O, are determined graphically by use of the velocity vectors ce and oe and the right angle construction. These vectors, $c'(ec)$ and $o'(eo)$, are drawn parallel respectively to EC and EO_{30}. Since the tangential are at right angles

FIG. 150

FIG. 149

FIG. 148
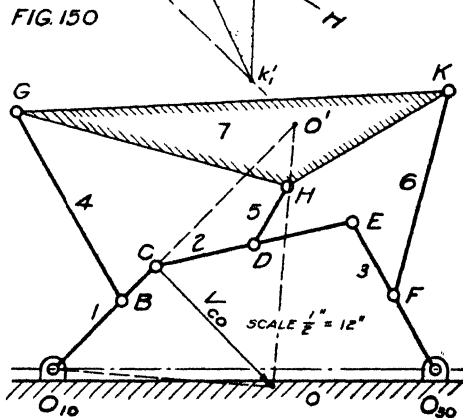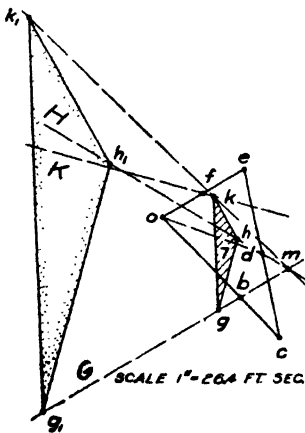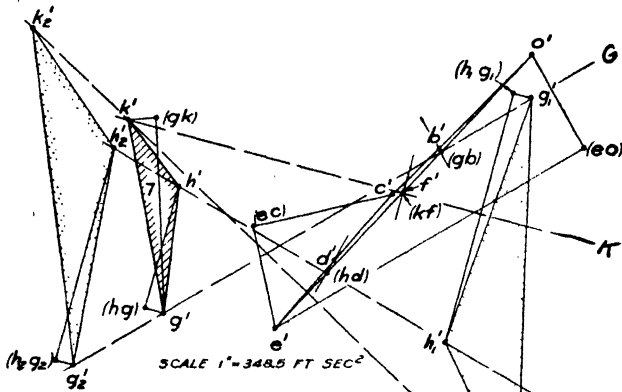
to the normal components of acceleration, e' is located at the intersection of lines through (ec) and (eo) perpendicular respectively to $c'(ec)$ and $o'(eo)$. Having e', the acceleration images $o'c'$, $c'e'$, and $o'e'$ can be drawn and the images b', d', and f' of points B, D, and F located by proportion. The length of the vectors $b'(gb)$, $d'(hd)$, and $f'(kf)$, representing the normal components of acceleration A_{GB}^n, A_{HD}^n, and A_{KF}^n, are determined graphically by using the velocity vectors bg,

dh, and *fk* and the right angle construction, and are drawn parallel, respectively, to *GB*, *HD*, and *KF*. It is to be noted that $f'(kf)$ is very short. Since the tangential and normal components of acceleration are at right angles, g' is somewhere on line G drawn through (gb) at right angles to $b'(gb)$ or to BG, Fig. 148. Similarly, h' and k' are somewhere on lines H and K drawn through (hd) and (kf) at right angles to $d'(hd)$ and $f'(kf)$ or to DH and FK. It is further known that lines joining g', h', and k' will form a triangle, or image, $g'h'k'$, similar to link GHK, and that this image will be at some unknown angle ϕ with its link. This image, correct as to size and position, will be located by constructing two trial or false images shown dotted in the figure. The normal component of acceleration A_{HG}^n of H relative to G, represented by vector $g'_1(h_1g_1)$ and found by using the velocity vector gh and the right angle construction, is first laid off from any point g'_1 in line G. Since the tangential is at right angles to the normal acceleration component, a line through (h_1g_1) at right angles to $g'_1(h_1g_1)$ or to HG will intersect H at h'_1. Having g'_1 and h'_1, the point k'_1 may be located by constructing on $g'_1h'_1$ a figure similar to GHK, or k'_1 may be located by using the vectors representing the normal components of acceleration A_{KH}^n and A_{KG}^n. Having found a trial or false position k'_1 of k', another point g'_2 on G is selected, and a second false position k'_2 of k' is determined. The intersection of line K with a line joining k'_1 and k'_2 locates the correct position of k'. By determining as already described the vector $k'(gk)$ representing A_{GK}^n and drawing through (gk) a line at right angles to $k'(gk)$, or to GK, the intersection g' is obtained. If h' is similarly found and the resulting image $g'h'k'$ comes out similar to GHK, the accuracy of the construction is confirmed. Obviously, line $k'_1k'_2$ is the locus of all the positions of k'. If the construction has been accurately done it is evident that where $k'_1k'_2$ intersects K is the true position of k'.

In the Stephenson's link mechanism shown in Fig. 151, the driving link *1* represents the crank and eccentrics of the engine, B and D being the centers of the two eccentrics. Rotation of link *1* imparts a reciprocating motion to the valve stem *8* through the links *2*, *3*, *4*, *6*, and *7*. The point of cut-off is determined by the position of the sliding block *6* relative to link *7*. By rotating bell crank *5* counter-clockwise, G of link *7* is moved nearer to center L of links *6* and *8*, and the point of cut-off is made later for counter-clockwise rotation of link *1*, or of the engine shaft. If *5* is rotated clockwise until the mid-point H between G and K is below L, the rotation of the engine is reversed. It is to be noted that points B, D, F, G, H, and K correspond to points B, D, F, G, H, and K of Fig. 148. The velocities and accelerations of B and D

are easily determined from the angular motion of the driving link *1*. For any given condition of operation, *F* is fixed relative to the frame and, therefore, has no motion. By the three line construction the velocities and accelerations of points *G*, *H*, and *K* of link *7* can be
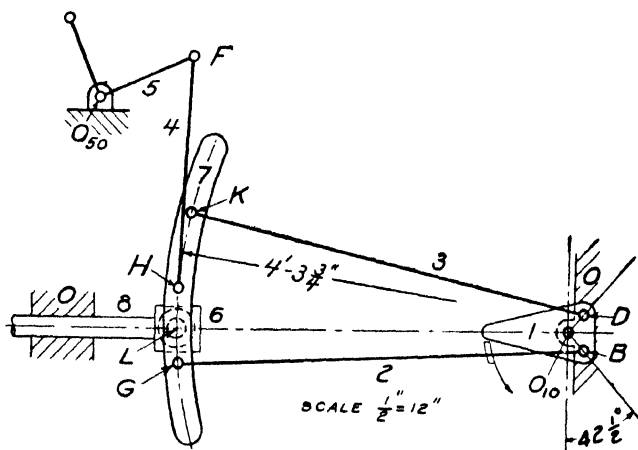

FIG. 151

found as for Fig. 148. Since *F* is fixed there is a certain amount of sliding between links *6* and *7*. The center of link *6* will therefore trace a circular arc on link *7*. Since the curve traced on *7* by the describing point *L* on link *6* has a motion of rotation, Coriolis' law would apply to the determination of the acceleration of the point *L* common to links *6* and *8*.

CHAPTER V

CAMS

84. Definitions. A *cam* is a machine element which is so formed that its rotation, oscillation, or reciprocation will impart a prescribed motion of reciprocation or oscillation to a *follower*. In cam mechanisms the cam is the *driver* and the driven element is the *follower*. In such mechanisms it is the given motion of the cam, or driver, and the prescribed motion of the follower that fixes the outline or shape of the cam. An *inverse* cam mechanism is one where the element corresponding to the follower of a cam mechanism is used as the driver. In such mechanisms it is the given motion of the follower, or driver, and the prescribed motion of the cam that fix the outline or shape of the cam. Cam mechanisms find a much wider field of application than inverse cam mechanisms.

Cam mechanisms are used wherever it would be impossible or very inconvenient by any other means to transform a given motion simple in character into a desired motion definitely prescribed and accurately timed. Such mechanisms are very much used in all kinds of automatic machinery such as textile machinery, shoe machinery, machine tools, cigarette-making machinery, etc.

85. Classification. Cams may be conveniently divided into five general classes:

    (*a*) Disk cams
    (*b*) Translation cams
    (*c*) Cylindrical cams
    (*d*) Conical cams
    (*e*) Spherical cams

86. Disk Cams. A disk cam is a shaped disk which by its rotation or oscillation imparts a purely or intermittently reciprocating or oscillating motion to a follower whose plane of motion coincides with or is parallel to that of the cam.

In Fig. 152 is shown a disk cam *C* which by its rotation imparts a reciprocating rectilinear motion to a roller follower *F*. The curved periphery of the disk constitutes the working surface of the cam, and

for this reason such cams are sometimes referred to as *periphery* cams. The follower shown is commonly called a *radial* roller follower since its axis or line of motion *DE* is a radial line perpendicular to the axis of the cam. If the line of motion of the follower were *GH* instead of *DE*, the follower would be referred to as an *offset* roller follower. For the same follower motion, such a shift of the follower stem and roller would require a change in the shape of the cam. It is to be noted that the rotation of the cam positively drives the follower *F* radially outward from *D* along the line *DE*, but that it does not positively control the inward motion of the follower. The roller *R* of the follower on the inward stroke along *ED* is kept in contact with the rotating cam either by the action of gravity or by the action of a spring. A spring is used
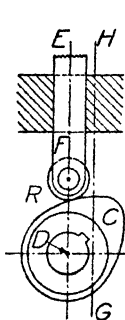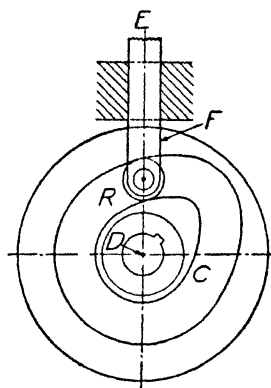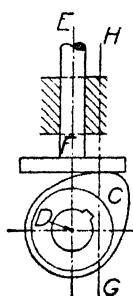
FIG. 152   FIG. 153   FIG. 154   FIG. 155
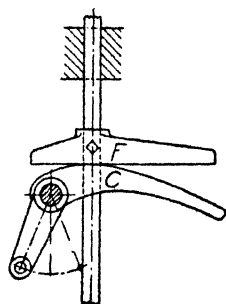
when the action of gravity would be insufficient or would be in the wrong direction to keep the roller of the follower in contact with the cam. The cam shown in Fig. 152 is therefore not a positive return cam.

In Fig. 153 is shown a *positive return* radial disk cam for the same follower motion as in Fig. 152. It is to be noted that it consists of a grooved disk, the groove being of proper form and of a width to accommodate the roller of the follower. Since the groove is cut in the face of the disk, such a cam is sometimes referred to as a *face* cam. The outline of the inner surface of the groove in Fig. 153 is identical with that of the working surface of the cam in Fig. 152, the outline of the outer surface of the groove being determined by the diameter of the roller of the follower. In order that the roller may be in contact with one side only of the groove and thus be free to roll, the groove should be slightly greater in width than the diameter of the roller. Because of this clearance the action may not for two reasons be satisfactory at

high speeds. When the motion or acceleration of the follower is reversed, the roller changes to the opposite side of the groove, and, on account of the clearance, strikes the opposite side with a sharp knock. In addition, the change in contact from one side of the groove to the other reverses the direction of rotation of the roller, which causes a grinding action. At slow or moderate speeds these effects may not be serious.

In Fig. 154 is shown a disk cam which by its rotation imparts a rectilinear motion to a flat-faced follower. The motion imparted will be the same whether the axis of the follower stem is DE or is offset to some such position as GH. A disk cam driving a flat-faced follower having an offset stem is shown in Fig. 216. If the axis DE of the follower stem does not coincide with the mid plane of the cam disk, the
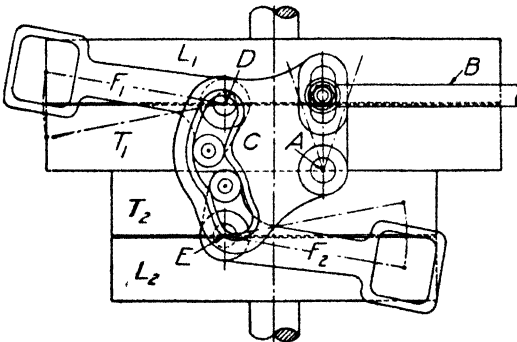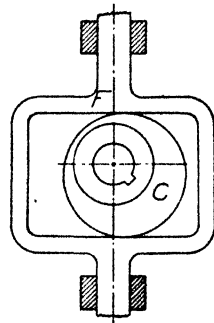
FIG 156   FIG 157

follower is offset axially. For a follower with a flat circular disk and a round stem, such an offset is an advantage, as the follower would rotate while reciprocating thus distributing instead of localizing the wear on the follower face. Such a follower is often called a *mushroom* follower.

Disk cams such as are shown in Figs. 152, 153, and 154 may be used to actuate oscillating as well as reciprocating followers. A rotating disk cam actuating an oscillating roller follower is shown in Fig. 217, and one actuating an oscillating flat-faced follower is shown in Fig. 218.

In Fig. 155 is shown a disk cam C which by its oscillation imparts a reciprocating motion to the flat-faced follower F. This combination is often called a *toe and wiper* cam because of the form of the cam and its follower and the action between them.

In Fig. 156 is shown an oscillating positive return disk cam actuating oscillating roller followers. The mechanism shown is a common type of belt shifter for metal planers. The loose pulleys are lettered L_1 and L_2 and the tight pulleys T_1 and T_2. In the phase shown the open belt is running on loose pulley L_1 and the cross belt on loose pulley L_2, the

table of the planer being stationary. To drive the table forward on its working stroke, the open belt must be shifted from L_1 to T_1 while the cross belt remains on L_2. To return the table to its original position, the open belt must first be shifted from T_1 to L_1 followed by the cross belt being shifted from L_2 to T_2. To get a quick return stroke, pulleys L_2 and T_2 are made smaller than L_1 and T_1. Only the middle portion of the slot in cam C is formed to impart motion to the shifters F_1 and F_2 about their centers D and E, the ends of the slot being circular about A as a center. To shift the belt for the working stroke, the slotted disk cam is rotated clockwise about center A by a pull in rod B. During this motion the follower roller for F_1 will be working in the mid portion of the slot and the follower roller for F_2 in the end of the slot, thus shifting the open belt from L_1 to T_1 without moving the cross belt
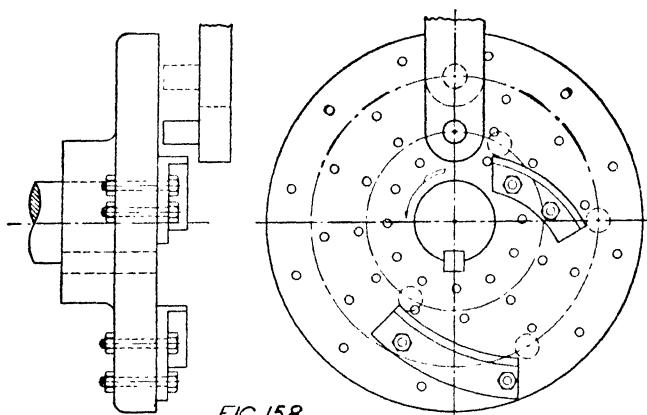
FIG 158

from L_2. To shift the belt for the return stroke, the cam is rotated counter-clockwise by a push on rod B. This motion will first shift the open belt from T_1 to L_1 and then, as the motion continues, will shift the cross belt from L_2 to T_2.

In Fig. 157 is shown a positive return disk cam which by its rotation imparts a reciprocating rectilinear motion to the flat-faced follower. Since the follower is in the form of a yoke enclosing the cam, the mechanism is frequently called a *yoke* cam. Since there is only one cam, the motion of the follower, as will be shown later, can be definite or prescribed only in one direction. Two disks must be used if the to and fro motions of the follower are to be independently prescribed. In Fig. 221 is shown a single disk cam driving a roller instead of a flat-faced yoke follower. Disk cams may drive oscillating as well as reciprocating yoke followers as shown in Figs. 224 and 225. It is to be satisfactory that

in Fig. 225 double disks are employed, the motion of the follower yoke being different in one direction from that in the other.

In Fig. 158 is shown a circular disk to whose face curved pieces are attached. These curved pieces form the working surfaces of the cam or cams carried by the disk. Since the form and position of these curved pieces may be changed, this type of cam is commonly spoken of as an *adjustable plate* cam. Either reciprocating or oscillating followers, usually fitted with rollers, may be used with this type of cam. It is a type well adapted for use in some automatic machines.

87. Translation Cams. A translation cam is a shaped plate which by its reciprocating rectilinear motion imparts a purely or intermittently reciprocating or oscillating motion to a follower whose plane of motion is parallel to that of the cam. In Fig. 159 is shown a translation

FIG 159   FIG 160
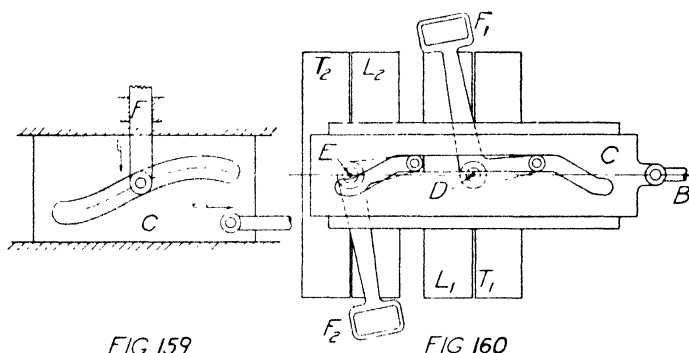
cam actuating a reciprocating roller follower. In Fig. 160 is shown a translation cam which by its intermittently reciprocating motion imparts an intermittently oscillating motion to the roller followers. Obviously, the purpose of the mechanism is identical with that of Fig. 156, and, being lettered the same, needs no further description.

88. Cylindrical Cams. A cylindrical cam is a cylinder whose surface carries a groove or a ridge or whose end is shaped, which, by its rotation or oscillation, imparts a purely or intermittently reciprocating or oscillating motion to a follower whose plane of motion is parallel to the axis of the cylinder.

In Fig. 161 is shown a cylindrical cam whose rotation imparts to a roller follower a rectilinear motion parallel to the axis of the cylinder. Since the working surface of the cam is the formed end of a hollow cylinder, this type of cam is frequently called an *end* cam. It is to be noted that such a cam is not a positive return cam since it can positively drive the follower only in one direction. By having the roller of the

follower run in a groove in the surface of the cylinder, the cam becomes a positive return cam. Such a positive return cam is shown in Fig. 162.
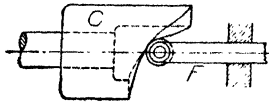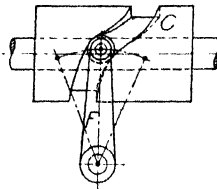

FIG. 161   FIG. 162

Instead of the groove, a curved ridge on the surface of the cylinder could be used with a follower carrying two rollers, one on each side of the ridge. The rotation of the cam in Fig. 162 imparts to the follower
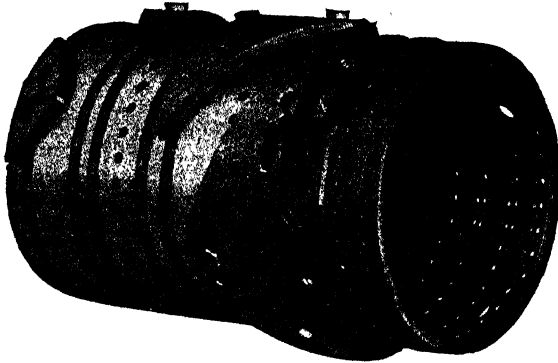

FIG. 163.

an oscillating motion in a plane parallel to the axis of the cam. It is evident by inspection that the angle of oscillation of the follower of such a cam mechanism is limited. In Fig. 163 is shown a cylinder designed to receive working cam surfaces of different shapes. This type of cam is often called a *drum* or *barrel* cam, and is a type much used in automatic lathes and other automatic machines. In Fig. 164 is shown another type of cylindrical cam often used in automatic machines. It consists of a circular disk
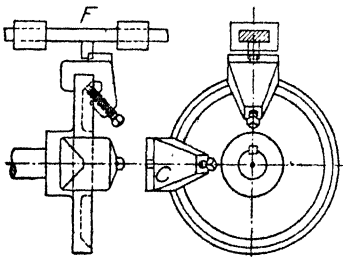
FIG 164

on whose periphery curved pieces are attached to impart a limited reciprocation or oscillation to the follower in a plane parallel to the

axis of the disk. It is therefore a cylindrical cam. Since the working cam surfaces attached to the disk are often for the purpose of actuating a trip follower, and are therefore "dogs," this type of cam is often called a *dog* cam.

89. Conical Cams. A conical cam is a cone whose surface carries a groove or ridge or whose end is shaped, which, by its rotation or oscillation, imparts a purely or intermittently reciprocating or oscillating motion to a follower whose plane of motion is parallel to an element of the cam cone. It is a useful form of cam where the desired motion of the follower is at an angle with the axis of rotation of the cam. Such a cam, actuating a reciprocating roller follower, is shown in Fig.
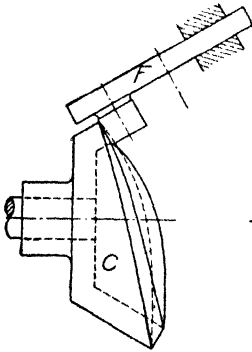

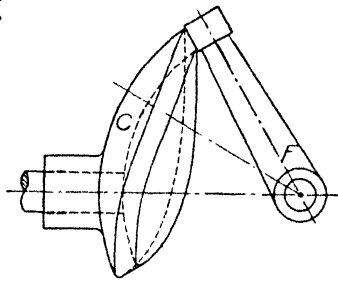FIG.165   FIG.166

165. As for cylindrical cams, the angle of oscillation for an oscillating follower is limited. For positive return of the follower a curved groove in the surface or a ridge on the surface of the cone may be employed as in cylindrical cams.

90. Spherical Cams. A spherical cam is a portion of a sphere employed to oscillate a follower about an axis through the center of the sphere at any angle with the axis of rotation of the sphere. Such a cam is illustrated in Fig. 166 with the cam and follower axes intersecting at right angles at the center of the sphere. To cause positive return of the follower the end of the follower must work in a curved groove or about a curved ridge on the inside surface of the sphere.

91. Design of a Cam. Of the five general classes of cams discussed above, disk cams and cylindrical cams are the two classes most widely used in machinery. Such cams, as a rule, rotate at a uniform rate. Of the various kinds of cams, certain forms of disk cams are the easiest to design. In Fig. 167 is shown a disk cam C whose clockwise rotation positively drives the knife-edged follower F radially outward along

the line *DE*. The follower during its return along *ED* is kept in contact with the working surface of the cam by the action of gravity or by the action of a spring.

In designing the cam shown in Fig. 167, the cam was assumed to rotate clockwise at a uniform rate. During the first 120-degree rotation of the cam, the *pitch point P* of the follower is to move from its initial position at *0* to its extreme position at *4*. During the next 120-degree rotation of the cam the pitch point *P* of the follower is to return from its extreme position at *4* to its initial position at *0* where it is to

FIG. 167   FIG. 168
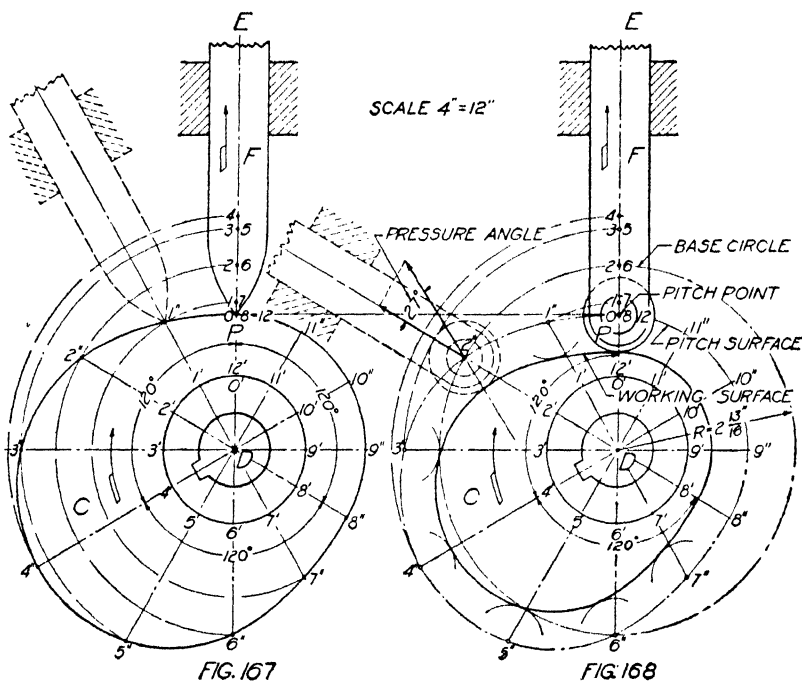
rest or dwell for the remainder of the revolution of the cam. The total distance moved through by the pitch point of the follower in either direction is called the *total displacement* of the follower. In the present problem this has been taken as 1½ inches. The outline of the cam is to be such that for each 30-degree rotation of the cam for the first 120 degrees the pitch point *P* of the follower is to move progressively from *0* to *1, 2, 3*, and *4*, and then to return from *4* to *5, 6, 7*, and *8* during each 30-degree rotation of the cam for the second 120 degrees. The determination of the corresponding points *1″, 2″, 3″*, etc., on the outline of the working surface of the cam is facilitated if the cam is

displacement with time was not mentioned. As a matter of fact, it was assumed that the follower would be uniformly accelerated for one-half its stroke and then uniformly retarded for the remaining half of its stroke, and that the return motion would be the same as the outward motion. Accordingly, the positions of the pitch point for each interval of time were so located, and the corresponding shape of the cam determined graphically.

In Fig. 169 is shown a displacement-time graph for uniformly accelerated motion from rest during one-half the total displacement

FIG. 169

FIG. 170
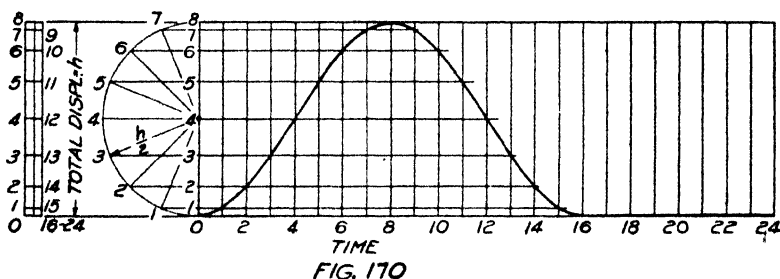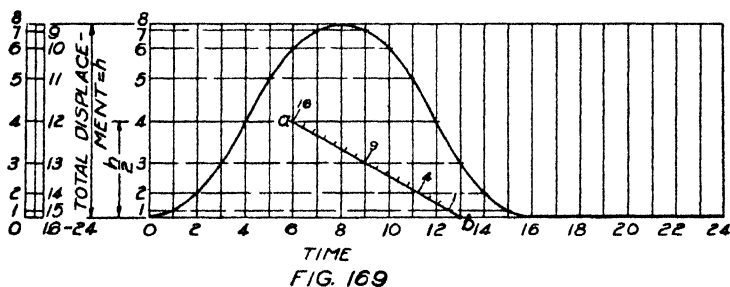
and uniformly retarded motion to rest for the remaining half of the displacement. For uniformly accelerated rectilinear motion from rest, the displacement or space covered in an interval of time t is

$$s = \tfrac{1}{2}At^2$$

Evidently, since the acceleration A is a constant, s varies directly as t^2. Hence, in 2 equal time units, 4 equal space or displacement units are covered; in 3 time units, 9 space units; in 4 time units, 16 space units; etc. This suggests an easy way of dividing one-half the total displacement, $\dfrac{h}{2}$, into parts corresponding to any convenient number of equal time units. These time units may be represented by any convenient length. In Fig. 169 the time required for one revolution of the

cam has been divided into 24 equal time intervals or units. The follcn-
is uniformly accelerated for 4 time periods, uniformly retarded ⸱r
4 periods, similarly accelerated and retarded on the return stroke, aɪ
dwells for the remaining 8 time periods. Hence, $\frac{h}{2}$ was divided intc
$(4)^2 = 16$ equal space units. In 1 time unit 1 space unit is covered;
in 2 time units $(2)^2 = 4$ space units; in 3 time units $(3)^2 = 9$ space
units; etc. If $\frac{h}{2}$ is of such a length that it cannot be conveniently
divided by the use of a scale into the required number of equal space
units, 16 in the present instance, a line ab of such a length that it can
be conveniently divided into 16 equal parts may be chosen and pro-
jections made from it as shown.

In Fig. 170 is shown a displacement-time graph for simple harmonic
motion of the follower. It has previously been pointed out that if a
point is moving in a circular path at a uniform speed, its projection
on a diameter of the circular path will have simple harmonic motion.
This suggests a simple method of plotting the displacement-time graph
in Fig. 170 for the same timing as in Fig. 169, but for simple harmonic
instead of uniformly accelerated and retarded motion of the follower.
The total displacement of the follower is to take place in 8 equal time
units. Hence the radius vector $\frac{h}{2}$ is to rotate at a uniform rate from 0
to 8 in eight equal units of time. Projections on the diameter, or h,
for each unit of time make it an easy matter to plot the displacement-
time graph as shown.

93. Disk Cam with Radial Roller Follower. A disk cam rotating
at a uniform rate is to impart motion to a radial roller follower
in accordance with the displacement-time graph shown in Fig. 171.
It is to be noted that the total displacement h is $1\frac{1}{4}$ in., and that the
time for one revolution of the cam has been divided into 16 equal in-
tervals. The follower is to move one-half the total displacement with
simple harmonic motion in 4 time units, is to dwell for 2 time units,
is to move to its maximum displacement with simple harmonic motion
in 4 time units, is to dwell 1 time unit, is to return with uniformly
accelerated and retarded motion in 4 time units, and is to dwell for the
last time unit of the revolution of the cam. Since there is to be a dwell
during the outward movement of the follower, the cam will be a *double-
step* **cam,** the total displacement taking place in two instead of one
uninterrupted step. The *displacement scale* for the pitch point P of the
follower is found by projecting from the displacement-time grapʻa, and

disp'own about the portion PE of the axis of the follower in Fig. 172. wa; diameter of the cam shaft and of the follower roller is 1 in. The ha meter of the hub has been taken 2 in. in accordance with a common st oportion for the diameter B of cast-iron hubs in terms of the diameter ,í the shaft A. For units in inches,

$$B = 1\tfrac{3}{4}A + \tfrac{1}{4} \quad \ldots \ldots \ldots \quad (124)$$

It is quite common to assume that the minimum radius of the cam shall not be less than the radius of the hub plus $\tfrac{1}{8}$ in. In accordance with this proportion the minimum radius of the cam in Fig. 172 has
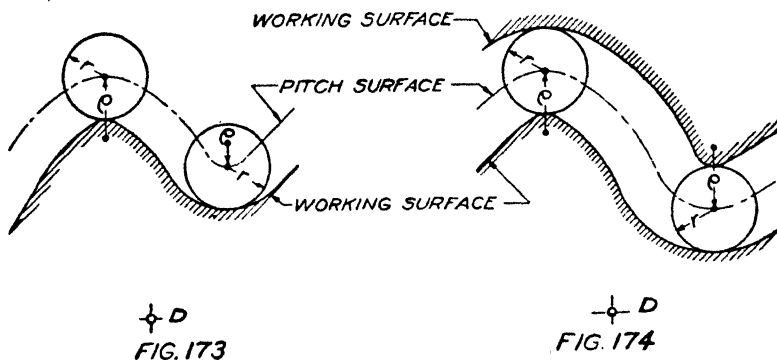

FIG. 171

SCALE 6"=12"

FIG. 172
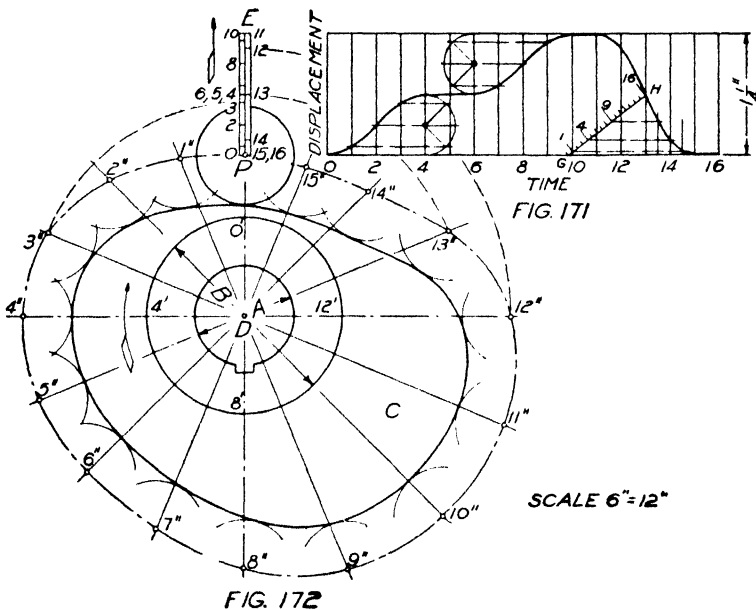
been taken as $1\tfrac{1}{8}$ in. Hence, the pitch point P in its lowest position is $1\tfrac{5}{8}$ in. above the center D of the cam shaft.

To avoid sacrifice of strength the keyway in the cam should not be located in the region where the radius of the cam is a minimum, and, in addition, should be so located as to easily establish the proper relative positions of follower, cam, and shaft in building and assembling the mechanism.

Having the displacement-scale PE and the center of the cam shaft D, the working surface of the cam is determined graphically by the method detailed in Art. 91. Since the displacement scale has been constructed on the basis of 16 equal time intervals for one revolution of

the cam, 16 radial lines equally spaced are drawn from the cam center
D. These radial lines are numbered to correspond to the numbered
positions of the pitch point of the follower, the numbers being primed
and set down in an order the reverse of that of the rotation of the cam.
To determine the pitch and working surfaces of the. cam, the same
expedient is employed as was used in Art. 91, the cam being fixed and
the frame and follower rotated in a direction the reverse of that for the
cam. Hence, radius D3 is rotated counter-clockwise to the radial position
3', thus determining the position of the roller center 3'' on the pitch
surface. Points 4'', 5'', 6'', etc., were similarly determined. With
these points as centers, circular arcs of a radius equal to the radius of

FIG. 173   FIG. 174

the roller were drawn, and the working surface of the cam was then
drawn tangent to these arcs as shown.

94. Diameter of Follower Roller. In Fig. 173 are shown the pitch
surface and the working surface of a cam whose center is at D. It is
to be noted that where the pitch surface is convex with respect to the
cam center D, the radius r of the follower roller must not be greater
than the minimum radius of curvature ρ of the pitch surface. The
maximum value of r is limited to the minimum value of ρ, for which
value the working surface would come to a sharp point. If r is taken
greater than ρ, the center of the roller will not be kept to the pitch
surface by the working surface, and the follower motion will not be as
planned. Where the pitch surface is concave with respect to the cam
center, the radius r of the roller may be greater than the radius of
curvature ρ of the pitch surface. Hence, it is the minimum radius of
curvature of the convex portion of the pitch surface that determines
the maximum diameter of the follower roller.

In Fig. 174 are shown the pitch surface and the working surfaces
of the groove in the face of a cam whose center is at D. It is to be

noted that whether the pitch surface is convex or concave with respect to the cam center, the maximum value of the radius r of the follower roller is limited to the minimum value of the radius of curvature ρ of the pitch surface. If r is greater than the minimum value of ρ for the pitch surface, the center of the follower roller will not be constrained by the working surfaces to follow the pitch surface, and the motion of the follower will not be as planned.

95. Radius of Base Circle. In Fig. 175 is shown, one-fifth full size, the pitch surface of the cam of Fig. 168 rectified on a line ad equal in length to the circumference of the base circle of the cam, the radial lines of the cam appearing as parallel vertical lines. As will be recalled, the base circle is a circle drawn through the point on the pitch surface where
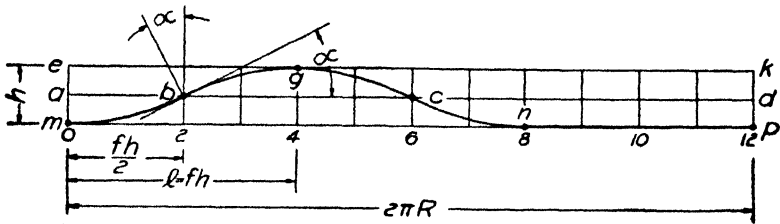

FIG. 175

the pressure angle is a maximum. The curve $mbgcnp$ is a *displacement-space* graph, and the completed figure may be called a *base diagram.* For this diagram, let

h = the total displacement of the radial roller follower,

β = the angle in degrees turned through by the cam during the total displacement;

R = the radius of the base circle of the cam;

l = the length of the portion of the base diagram corresponding to the maximum displacement;

f = the cam factor = $\dfrac{l}{h}$.

Hence,

$$l = fh = 2\pi R\left(\frac{\beta}{360}\right) \quad \ldots \ldots \ldots \ldots \quad (125)$$

or

$$R = \frac{360fh}{2\pi\beta} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (126)$$

It is to be noted that if the curve $mbgcnp$ were the working surface of a translation cam moving under a vertical knife-edged follower at a

uniform velocity corresponding to that of the base circle of the cam in Fig. 168, it would cause identically the same motion of the follower as the rotating cam. Furthermore, the pressure angle α for the knife-edged follower at m, b, g, c, and n would be the same as for the follower in Fig. 168 for the corresponding positions 0, 2, 4, 6, and 8 of the cam. Anywhere between m and b, however, the pressure angle is less than for the cam between 0 and 2, and anywhere between b and g it is greater than for the cam between 2 and 4, because of the rotation of the radial lines of the cam to parallelism in the process of rectification. For this reason the maximum pressure angle at the point of inflection of certain displacement-space graphs may not agree absolutely with the maximum pressure angle of the cam. Where there is a difference the error is so slight that it can be neglected. Hence, the cam factor f is dependent upon the displacement-space graph and the value of the maximum pressure angle chosen.

It was mentioned in Art. 92 that the cams shown in Figs. 167 and 168 were designed for uniformly accelerated motion from rest for the first half and uniformly retarded motion to rest for the last half of the total displacement of the follower. Hence, the displacement s in any interval of time t is,

$$s = \tfrac{1}{2}At^2$$

The length of the base circle arc corresponding to a displacement of $\dfrac{h}{2}$ of the follower is $\dfrac{fh}{2}$, as shown in Fig. 175. At this instant the pressure angle is a maximum, and its tangent is equal to the slope of the displacement-space graph at b. Since the cam rotates at a uniform rate, equal intervals of space represent equal intervals of time, or $t = kx$. Hence,

$$s = \tfrac{1}{2}At^2 = \tfrac{1}{2}Ak^2x^2$$

Now

$$s = h/2 \quad \text{when} \quad x = \frac{fh}{2}, \quad \text{or} \quad Ak^2 = \frac{4}{f^2h}$$

The first derivative of s equals the slope of the displacement-space graph and is a maximum for $x = \dfrac{fh}{2}$. Hence,

$$\frac{ds}{dx} = \frac{d}{dx}\left(\tfrac{1}{2}Ak^2x^2\right) = Ak^2x = \frac{4}{f^2h}\left(\frac{fh}{2}\right) = \frac{2}{f} = \tan\alpha$$

or

$$f = \frac{2}{\tan\alpha} \qquad (f = 3.93 \text{ for } \alpha = 27°)$$

Hence,

$$R = \frac{360fh}{2\pi\beta} = \frac{360 \times 3.93 \times 1.5}{2\pi 120} = 2\,\tfrac{13}{16}\ \text{in.}$$

which is in agreement with Fig. 168.

For the cam shown in Fig. 172 there are two points of inflection for the outward motion of the follower and one for the return motion. Hence, the pressure angle for the outward motion is a maximum either for position *2* or position *8*, Fig. 171, and is a maximum for the return motion for position *13*. It can be shown by equations (126) and (144) that the maximum pressure angle for position *13* is 35° 20′ and by equations (126) and (148) that the pressure angles for positions *2* and *8* are, respectively, 17° 50′ and 13° 45′.

In Table 1 below are given the values of the cam factor for various values of the maximum pressure angle for each of the five displacement-space graphs discussed in the next article, the values being obtained by the use of equations (127), (135), (144), (148), and (152).

TABLE 1

CAM FACTOR f FOR DIFFERENT VALUES OF THE MAXIMUM PRESSURE ANGLE α

Law of Displacement	Maximum Pressure Angle α					
	20°	25°	30°	35°	40°	45°
Straight line and two arcs of radius h as per Fig. 179 and equation (127)	3.10	2.59	2.27	2.03	1.89	1.83
Two arcs of radius H as per Fig. 182 and equation (135)	5.67	4.51	3.73	2.78	2.51	2.42
Uniformly accelerated and retarded motion as per Fig. 185 and equation (144)	5.50	4.29	3.47	2.86	2.38	2.00
Simple harmonic motion as per Fig. 188 and equation (148)	4.31	3.37	2.72	2.24	1.87	1.57
As per displacement graph Fig. 191 and equation (152)	5.50	4.29	3.47	2.86	2.38	2.00

96. Displacement, Velocity, and Acceleration Graphs.   There are two methods of designing cams.   The law of motion of the follower may be fixed and the corresponding outline of the cam determined; or the outline of the cam may be fixed for a given total displacement and the corresponding motion of the follower then determined.   The first method has been applied in Arts. 91 and 93, and the second method

will be illustrated presently. By the first method, the law of motion of the follower being known, the velocity and acceleration of the follower can be computed and the dynamic aspects of the problem be made apparent in advance of determining the outline of the cam. Furthermore, for disk cams with radial roller followers and for cylindrical cams with followers having a motion of translation parallel to the axis of rotation of the cam, the size of the cam may be computed in accordance with the preceding article for any given maximum pressure angle before-determining the outline of the cam.

In Art. 91 uniformly accelerated and retarded motions of the follower were assumed. An assumption of equal displacements in equal time units would have yielded a displacement graph as shown in Fig. 176;
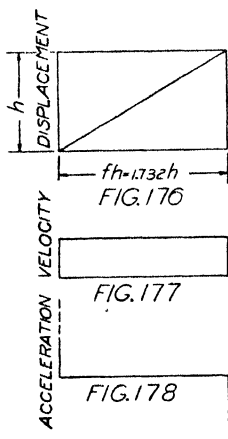
the velocity of the follower would have been uniform from the beginning to the end of its stroke as shown in Fig. 177; and the acceleration, except at the ends of the stroke, would have been zero. Such an assumption would require that the velocity of the follower be instantaneously increased from zero to a definite value and then instantaneously decreased. This would mean an indefinitely high instantaneous value of the acceleration at the beginning and at the end of the follower stroke as indicated in Fig. 178, and, hence, indefinitely high forces between the cam and follower. To avoid this very undesirable condition, the law of motion of the follower should be such as to give gradually increasing and then gradually decreasing displacements of the follower for equal time units. Five displacement graphs of this character are shown in Figs. 179 to 191, together with the corresponding velocity and acceleration graphs. For purposes of comparison all the graphs are drawn for a maximum pressure angle α of 30 degrees and for a total displacement of 1.5 inches while the cam rotates through an angle β of 120 degrees, the R.P.M. of the cam being 160. The same scales were used for all of the graphs; a space scale of 4 in. = 12 in., a velocity scale of 1 in. = 5 ft./sec., and an acceleration scale of 1 in. = 100 ft./sec.²

The displacement-space graph in Fig. 179 is seen to be a straight line tangent to two arcs of radius h for a maximum slope or maximum pressure angle α. For the straight portion of the graph the displacement of the radial follower is uniform, the velocity is constant, and the acceleration is zero as shown in Figs. 180 and 181. For the curved portion

of the graph, the motion of the follower is variable.   From inspection
of Fig. 179,

$$fh = 2h \sin \alpha + \frac{h - 2(h - h \cos \alpha)}{\tan \alpha} = \frac{h(2 - \cos \alpha)}{\sin \alpha} \qquad (127)$$

$$= 2.268h \text{ for } \alpha = 30°$$

The displacement for any value of x between zero and $(h \sin \alpha)$ is

$$s = h - \sqrt{h^2 - x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

For any value of x corresponding to an angular displacement ϕ of the
cam, ϕ and β being in radians,

$$\frac{x}{fh} = \frac{\phi}{\beta} \quad \text{or} \quad x = fh\left(\frac{\phi}{\beta}\right) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

Substituting the value of x from (129) in (128) gives

$$s = h - \sqrt{h^2 - \left(\frac{fh}{\beta}\right)^2 \phi^2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

The limiting value of ϕ is for $x = h \sin \alpha$ and is, from equations (129)
and (127),

$$\phi_{\text{max.}} = \frac{\beta h \sin \alpha}{fh} = \frac{\beta \sin^2 \alpha}{2 - \cos \alpha} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

$$= \frac{\beta}{4.536} \text{ for } \alpha = 30°$$

Differentiating (130) gives

$$V = \frac{ds}{dt} = \frac{h\omega\left(\frac{f}{\beta}\right)^2 \phi}{\sqrt{1 - \left(\frac{f}{\beta}\right)^2 \phi^2}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

where ω is the angular velocity of the cam.   Differentiating (132) gives

$$A = \frac{dV}{dt} = \frac{h\omega^2\left(\frac{f}{\beta}\right)^2}{\left[1 - \left(\frac{f}{\beta}\right)^2 \phi^2\right]^{3/2}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

The velocity and acceleration graphs are shown in Figs. 180 and 181
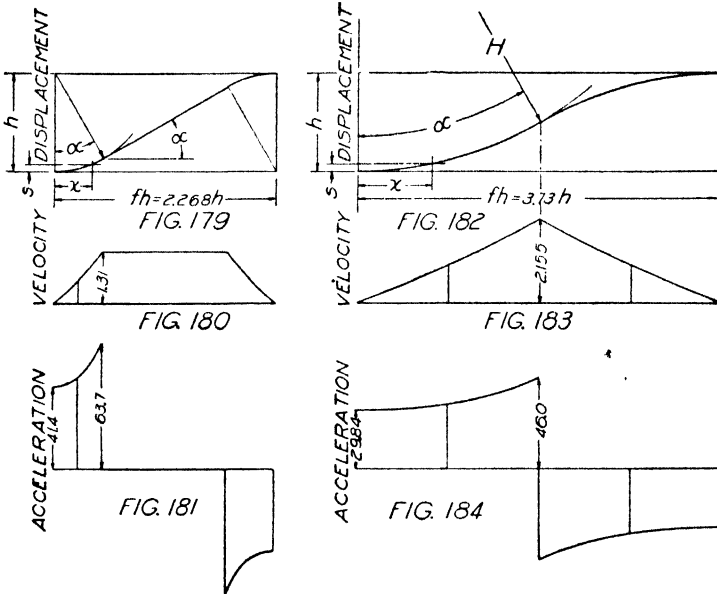for the data and scales mentioned above.

In Fig. 182 is shown a displacement-space graph composed of two arcs of radius H for a maximum slope or pressure angle α. The corresponding velocity and acceleration graphs of the radial follower are shown in Figs. 183 and 184. From inspection of Fig. 182,

$$\frac{fh}{2} = H \sin \alpha \quad \text{and} \quad \frac{fh}{2} = \left(H - \frac{h}{2} \right) \tan \alpha$$

Hence,

$$H = \frac{h}{2}\left(\frac{\tan \alpha}{\tan \alpha - \sin \alpha} \right) = \frac{h}{2 - 2 \cos \alpha} \quad \cdot \quad \cdot \quad \cdot \quad (134)$$

$$= 3.73h \quad \text{for} \quad \alpha = 30°$$

FIG. 179

FIG. 180

FIG. 181

FIG. 182

FIG. 183

FIG. 184

Substituting the value of H from (134) in the equation above, gives

$$fh = 2H \sin \alpha = \frac{h \sin \alpha}{1 - \cos \alpha} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (135)$$

$$= 3.73 h \quad \text{for} \quad \alpha = 30°$$

The displacement for any value of x between zero and ($H \sin \alpha$) is,

$$s = H - \sqrt{H^2 - x^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (136)$$

For any value of x corresponding to an angular displacement ϕ of the cam, ϕ and β being in radians,

$$\frac{x}{fh} = \frac{\phi}{\beta} \quad \text{or} \quad x = fh\left(\frac{\phi}{\beta}\right) \quad \ldots \ldots \quad (137)$$

Substituting the value of x from (137) in (136) gives

$$s = H - \sqrt{H^2 - \left(\frac{fh}{\beta}\right)^2 \phi^2} \quad \ldots \ldots \quad (138)$$

where the limiting value of ϕ corresponds to $\dfrac{fh}{2}$ and is $\dfrac{\beta}{2}$. Differentiating (138) gives

$$V = \frac{ds}{dt} = \frac{\omega\left(\dfrac{fh}{\beta}\right)^2 \phi}{\sqrt{H^2 - \left(\dfrac{fh}{\beta}\right)^2 \phi^2}} \quad \ldots \ldots \quad (139)$$

where ω is the angular velocity of the cam. Differentiating (139) gives

$$A = \frac{dV}{dt} = \frac{\omega^2\left(\dfrac{fh}{\beta}\right)^2 H^2}{\left[H^2 - \left(\dfrac{fh}{\beta}\right)^2 \phi^2\right]^{3/2}} \quad \ldots \ldots \quad (140)$$

In Fig. 185 is shown a displacement-space graph for uniform acceleration and retardation of the radial follower from rest to rest. The corresponding velocity and acceleration graphs are shown in Figs. 186 and 187. The displacement s at any instant during one-half the total displacement is

$$s = \tfrac{1}{2}At^2 \quad \ldots \ldots \ldots \ldots \quad (141)$$

$$V = \frac{ds}{dt} = At \quad \ldots \ldots \ldots \ldots \quad (142)$$

Since the cam rotates at a uniform rate, equal intervals of space represent equal intervals of time, or $t = kx$. Hence,

Now
$$s = \tfrac{1}{2}At^2 = \tfrac{1}{2}Ak^2x^2$$

$$s = h/2 \quad \text{when} \quad x = \frac{fh}{2}, \quad \text{or} \quad Ak^2 = \frac{4}{f^2h} \quad \ldots \quad (143)$$

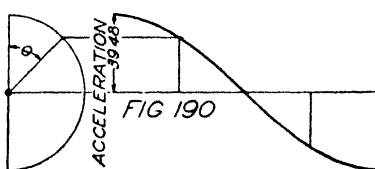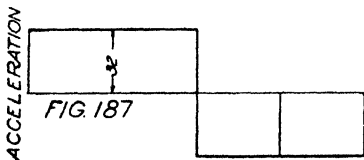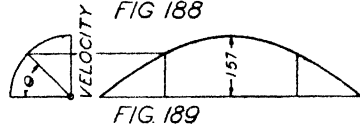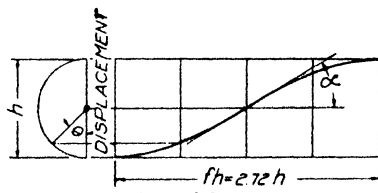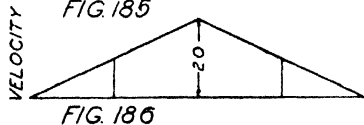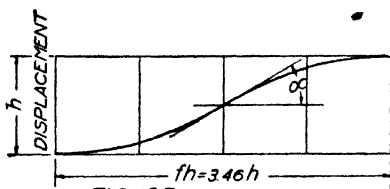The first derivative of s equals the slope of the displacement-space graph and is a maximum for $x = \dfrac{fh}{2}$. Hence,

$$\frac{ds}{dx} = \frac{d}{dx}\left(\tfrac{1}{2}Ak^2x^2\right) = Ak^2x = \frac{4}{f^2h}\left(\frac{fh}{2}\right) = \frac{2}{f} = \tan \alpha$$

or

$$f = \frac{2}{\tan \alpha} \quad (f = 3.46 \text{ for } \alpha = 30°) \quad \ldots \quad \ldots \quad (144)$$

FIG. 185

FIG. 186

FIG. 187

FIG 188

FIG. 189

FIG 190

In Fig. 188 is shown a displacement-space graph for simple harmonic motion of the radial follower. From Art. 17,

$$s = \frac{h}{2}(1 - \cos \theta) \quad \ldots \quad \ldots \quad \ldots \quad (145)$$

$$V = \frac{ds}{dt} = \omega_r \frac{h}{2} \sin \theta \quad \ldots \quad \ldots \quad \ldots \quad (146)$$

$$A = \frac{dV}{dt} = \omega_r{}^2 \frac{h}{2} \cos \theta \quad \ldots \quad \ldots \quad \ldots \quad (147)$$

The curves shown in Figs. 188, 189, and 190 may be plotted by projecting a rotating vector. In Fig. 188 it is the projection of a vector of length $\dfrac{h}{2}$ rotating through 180 degrees at a uniform rate in a period of

time corresponding to that required by the rotating cam during the total displacement h of the radial follower. To plot the velocity graph shown in Fig. 189 by the projection of a rotating vector, the length of the vector used must be made equal to $\omega_r\left(\dfrac{h}{2}\right)$, and equal to $\omega_r{}^2\left(\dfrac{h}{2}\right)$ for the acceleration graph shown in Fig. 190. From the data given above, the cam makes $\frac{160}{60} = \frac{8}{3}$ revolutions per second or one revolution in $\frac{3}{8}$ second. Since the cam makes $\frac{120}{360} = \frac{1}{3}$ of a revolution during the total displacement of the follower, the time occupied is $\frac{1}{3} \times \frac{3}{8} = \frac{1}{8}$ second. Hence,

$$\omega_r = 2\pi(\tfrac{180}{360}) \div \tfrac{1}{8} = 8\pi \text{ radians per second.}$$

Since the cam rotates at a uniform rate, equal intervals of space represent equal intervals of time, or $t = kx = \theta / \omega_r$. Hence, $\theta = \omega_r kx$, and, from equation (145), the slope of the displacement-space graph is

$$\frac{ds}{dx} = \frac{d}{dx}(1 - \cos\theta)\frac{h}{2} = \frac{h}{2}(\sin\theta)\,\omega_r k$$

which is a maximum when $\theta = \dfrac{\pi}{2}$, for which $x = \dfrac{fh}{2}$. Hence, from

$\theta = \omega_r kx$, $\omega_r k = \dfrac{\theta}{x} = \dfrac{\pi}{fh}$ and

$$\frac{ds}{dx} = \frac{h}{2}\left(\sin\frac{\pi}{2}\right)\frac{\pi}{fh} = \frac{\pi}{2f} = \tan\alpha$$

or

$$f = \frac{\pi}{2\tan\alpha} \qquad (f = 2.72 \text{ for } \alpha = 30°) \quad . \quad (148)$$

All the laws of displacement dealt with up to this point give abrupt changes in the acceleration of the follower and consequently abrupt changes in the unbalanced forces acting on the frame of the mechanism. To eliminate these abrupt changes during the cycle, the motion of the follower must be such that its acceleration and retardation on the out and return strokes will gradually increase from zero and then gradually decrease to zero. To attain this result, the following equations were devised for which the graphs shown in Figs. 191, 192, and 193 were plotted in accordance with the data previously used:

$$s = \frac{h}{2\pi}(2\theta - \sin 2\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (149)$$

$$V = \frac{ds}{dt} = \frac{h}{2\pi}(1 - \cos 2\theta)\frac{d(2\theta)}{dt} = \omega_r\frac{h}{2\pi}(1 - \cos 2\theta) \quad . \quad (150)$$

$$A = \frac{dV}{dt} = \omega_r\frac{h}{2\pi}\sin 2\theta\,\frac{d(2\theta)}{dt} = \omega_r{}^2\frac{h}{2\pi}\sin 2\theta \quad . \quad . \quad . \quad (151)$$

In accordance with equation (149) the displacement is a maximum and equal to h when $\theta = \pi$, the value of θ changing from zero to π while the cam rotates through an angle β. It is to be noted that while the displacement graph cannot be plotted by projecting a rotating vector, the velocity and acceleration graphs can be as shown in Figs. 192 and 193. For the data given above, the vectors shown rotate through
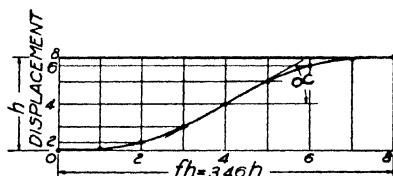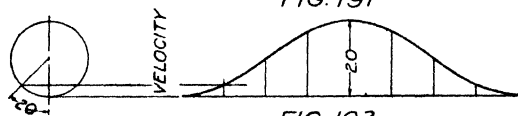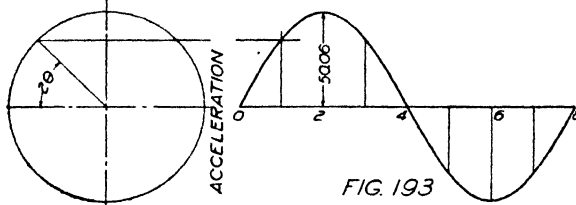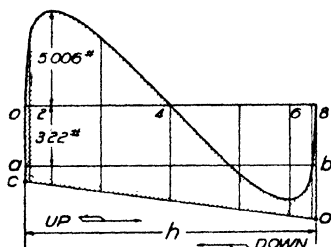

FIG. 191

FIG. 192

FIG. 193

FIG. 194

2π radians, that is, θ changes from zero to π while the cam rotates through an angle $\beta = 120°$ in 0.125 second. Hence $\omega_r = \dfrac{d(2\theta)}{dt} = \dfrac{2\pi}{0.125}$

$= 16\pi$ radians per second. Since the cam rotates at a uniform rate, equal intervals of space represent equal intervals of time, or $t = kx$; and, from equation (150), $\omega_r = \dfrac{2\theta}{t}$. Hence, $2\theta = \omega_r kx$, and, from equation (149), the slope of the displacement-space graph is,

$$\frac{ds}{dx} = \frac{h}{2\pi}(1 - \cos 2\theta)\frac{d(2\theta)}{dx} = \frac{h}{2\pi}(1 - \cos 2\theta)\omega_r k$$

which is a maximum for $\theta = \dfrac{\pi}{2}$ for which $x = \dfrac{fh}{2}$ and $s = \dfrac{h}{2}$. Hence,

from $2\theta = \omega_r k x$, $\omega_r k = \dfrac{2\pi}{fh}$ and

$$\frac{ds}{dx} = \frac{h}{2\pi}(1 - \cos \pi)\frac{2\pi}{fh} = \frac{2}{f} = \tan \alpha$$

or

$$f = \frac{2}{\tan \alpha} \quad (f = 3.46 \text{ for } \alpha = 30°) \ \ . \ \ . \ \ . \ \ . \ \ (152)$$

Of the five displacement graphs shown in Figs. 179, 182, 185, 188, and 191, the law of displacement assumed for Fig. 179 yields the smallest cam, and that assumed for Fig. 182 the largest cam. It is to be noted that the laws of displacement assumed for Figs. 185 and 191 yield cams of the same size. Each assumed law of displacement results in a satisfactory variation of velocity of the follower, as shown in Figs. 180, 183, 186, 189, and 192. The same cannot be said, however, of the corresponding variation of the acceleration. For the first four laws of displacement, as shown in Figs. 181, 184, 187, and 190 and as before mentioned, there are abrupt changes in the acceleration of the follower during its motion. As a result there will be abrupt changes in the unbalanced forces acting on the frame of the cam mechanism. The last assumption, as shown in Fig. 193 and as previously explained, is the only one of the five to give a gradual change in the acceleration and retardation of the follower throughout its motion.

97. Dynamics of a Cam Mechanism. For purposes of comparison, as before stated, the graphs shown in Figs. 179 to 193 inclusive were plotted for a radial roller follower driven by a disk cam rotating at a uniform rate at 160 R.P.M. A total displacement of 1.5 in. was assumed while the cam rotated through an angle β of 120 degrees, the return motion of the follower being the same as its outward motion. It will now be assumed that the follower moves vertically up and down, that the weight of the follower and roller is 3.22 pounds, and that the maximum pressure angle α is to be 30°. Assuming the law of displacement of the follower to correspond to equation (149) and Fig. 191, it is possible to determine the radius of the base circle of the cam and to investigate the dynamic aspects of the cam mechanism in advance of determining the outline of the cam. The radius R of the base circle of the cam for $\alpha = 30°$ is, by equation (126),

$$R = \frac{360 f h}{2\pi\beta} = \frac{360 \times 3.46 \times 1.5}{2\pi 120} = 2.48, \text{ say, } 2\tfrac{1}{2} \text{ in.}$$

The fact that θ varies from zero to π during the total displacement enables the displacement for any value of θ to be computed by equation (149). The lengths of the rotating vectors to be used in plotting the velocity and acceleration graphs are, according to equations (150) and (151), $\omega_r\left(\dfrac{h}{2\pi}\right)$ and $\omega_r{}^2\left(\dfrac{h}{2\pi}\right)$, respectively, where h is in feet and ω_r is the angular velocity of the rotating vectors in radians per second.

$$\beta = 120° = \frac{2\pi}{3}\ \text{rad.}$$

ω = angular velocity of the cam

$$= (\text{R.P.S.})\ 2\pi = \left(\frac{160}{60}\right)2\pi = \frac{16\pi}{3}\ \text{rad./sec.}$$

The vectors rotate through 2π radians during the total displacement of the follower or while the cam rotates through an angle β or in a period of time $t = \dfrac{\beta}{\omega} = \dfrac{2\pi}{3} \times \dfrac{3}{16\pi} = \dfrac{1}{8}$ second. Hence

$$\omega_r = \frac{2\pi}{t} = 16\pi\ \text{rad./sec.}$$

and

$$\omega_r\left(\frac{h}{2\pi}\right) = 16\pi \times \frac{0.125}{2\pi} = 1$$

$$\omega_r{}^2\left(\frac{h}{2\pi}\right) = (16\pi)^2\,\frac{0.125}{2\pi} = 16\pi$$

Fig. 191 is to a scale of 4 in. = 12 in.; Fig. 192 to a scale of 1 in. = 5 ft./sec.; Fig. 193 to a scale of 1 in. = 100 ft./sec.^{2}; and Fig. 194 to a scale of 1 in. = 10 pounds.

In Fig. 194 the total displacement h of the follower is represented full size by line 08. This line was divided into 8 parts, corresponding to the displacements shown in Fig. 191 for the eight equal intervals of time. On the vertical lines through the points so determined, the corresponding inertia forces of the follower were laid off, to a scale of 1 in. = 10 pounds. These forces, being equal to the product of the mass and acceleration of the follower, are proportional to the ordinates of the acceleration curve Fig. 193. Thus, for position 2, the magnitude of the inertia force of the follower is

$$F = MA = \frac{W}{g}\,A = \frac{3.22}{32.2} \times 50.06 = 5.006\ \text{pounds}$$

Since at the instant the follower is moving vertically upward, the vertical component of the pressure between the follower roller and the cam surface will be equal to 5.006 pounds plus the weight of the follower, 3.22 pounds. It is to be noted that just before position 5 is reached this pressure component has decreased to zero, and that beyond this the force of gravity would be insufficient to retard the follower as required to keep it in contact with the cam. Hence, if the assumed law of displacement is to be realized, the follower must be fitted with a spring. For satisfactory action the spring should have an initial compression when the follower is in its lowest position. This initial compression has been taken at 0.78 pound and is represented in Fig. 194 by the ordinate ac, making the vertical component of pressure at the beginning of the upward stroke of the follower $3.22 + 0.78 = 4.0$ pounds instead of 3.22 pounds. Assuming a helical spring, the downward force exerted by the spring on the follower will be proportional to the amount it is compressed. Representing the spring pressure at the end of the upward stroke by bd, and making this pressure 2.88 pounds, gives a minimum component of pressure near position 6 about equal to the initial compression of the spring. On this basis the spring pressure increases from 0.78 to 2.88 pounds during a compression of 1.5 in. The scale of the spring required is, therefore, $(2.88 - 0.78) \div 1.5 = 1.4$ pounds per inch of compression. Obviously the vertical component of the pressure between the follower roller and cam at any instant on the up or down stroke is represented to scale by the ordinate between cd and the curve, which ordinate varies from co at the beginning to $d8$ at the end of the up stroke and from $d8$ to co on the down stroke. During dwell the pressure between the roller and cam is evidently represented by co. It should be noted, however, that the magnitude of the vertical component of the unbalanced force acting on the frame of the cam mechanism at any instant is represented by the ordinate between 08 and the curve.

98. Automobile Cam with Radial Roller Follower. In a four stroke cycle automobile engine, the crank-shaft makes two revolutions to one of the cam shaft. Radial or offset disk cams with roller or mushroom followers are usually employed to operate the valves of such engines. In general such cams are designed in accordance with the second method of designing cams, the outline of the working surface of the cam being fixed for a given total displacement and the corresponding law of motion of the follower then determined. That such cams may be most easily produced and ground true to form, their outlines are usually made up of circular arcs or straight lines and circular arcs. Where straight lines tangent to circular arcs are employed, the cam is known as a *tangent*

cam, and as a *radial tangent* cam if the line of motion of the reciprocating follower passes through the center of the cam shaft.   This latter type is assumed in the present example.   In Fig. 195, let

β = the angle turned through by the cam during the total displacement;

δ = the angle turned through by the cam while the roller follower is actuated by the tangent surface, or flank, of the cam;
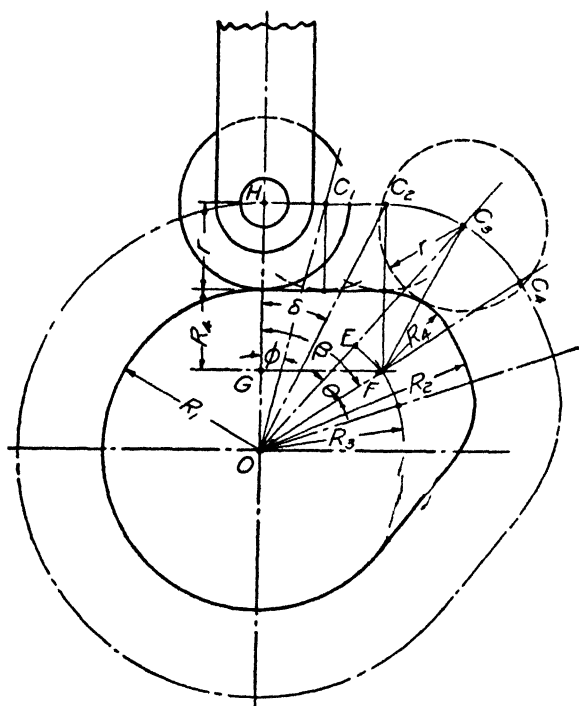

FIG. 195

ϕ = any angle turned through by the cam while the follower is in contact with the flank;

θ = the angle turned through by the cam from any position of follower roller on the nose of the cam to the position for total displacement;

R_1 = the minimum radius of the cam;

R_2 = the maximum radius of the cam;

R_4 = the radius of the nose of the cam;

$R_3 = R_2 - R_4$;

r = the radius of the follower roller;

h = the total displacement of the follower = $R_2 - R_1$;

s = the displacement of the follower at any instant;

ω = the angular velocity of the cam in radians per second.

In any given case the minimum radius R_1 of the cam, the total displacement h, and the corresponding angular displacement β of the cam are fixed by the conditions of service and accepted practice, likewise the radius r of the follower roller. Having these, the angle δ and the radius of the nose R_4 can be determined. Referring to Fig. 195,

$$h = R_3 + R_4 - R_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

but

$$\cos \beta = \frac{R_1 - R_4}{R_3} \quad \text{or} \quad R_3 = \frac{R_1 - R_4}{\cos \beta} \quad . \quad . \quad . \quad (154)$$

Substituting the value of R_3 from (154) in (153) gives

$$h = \frac{R_1 - R_4}{\cos \beta} + R_4 - R_1 \quad . \quad . \quad . \quad . \quad . \quad (155)$$

Solving (155) for R_4 gives

$$R_4 = R_1 - \frac{h \cos \beta}{1 - \cos \beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

also,

$$\tan \delta = \frac{HC_2}{OH} = \frac{R_3 \sin \beta}{R_1 + r} = \frac{(h + R_1 - R_4) \sin \beta}{R_1 + r} \quad . \quad (157)$$

For the roller in contact with the flank of the cam, the displacement of the pitch point of the follower for any angular displacement ϕ of the cam between zero and δ, is

$$s = OC_1 - (R_1 + r) = \frac{R_1 + r}{\cos \phi} - (R_1 + r) = (R_1 + r)(\sec \phi - 1) \quad (158)$$

Differentiating (158) with respect to time gives the velocity V of the follower:

$$V = \frac{ds}{dt} = (R_1 + r) \sec \phi \tan \phi \left(\frac{d\phi}{dt} \right) = \omega(R_1 + r) \frac{\sin \phi}{\cos^2 \phi} \quad . \quad . \quad (159)$$

Differentiating (159) gives the acceleration A of the follower:

$$A = \frac{dV}{dt} = \frac{d}{dt}[\omega(R_1 + r) \tan \phi \sec \phi] = \omega^2(R_1 + r) \left[\frac{2 - \cos^2 \phi}{\cos^3 \phi} \right] \quad (160)$$

For the nose of the cam for values of θ between zero and $(\beta - \delta)$,

$$s = OC_3 - (R_1 + r) = (OE + EC_3) - (R_1 + r)$$
$$= R_3 \cos \theta - (R_1 + r) + \sqrt{(R_4 + r)^2 - (R_3 \sin \theta)^2} \qquad . \quad . \quad (161)$$

Differentiating (161) gives the velocity V of the follower:

$$V = \frac{ds}{dt} = -\omega \left[\frac{R_3{}^2 \sin \theta \cos \theta}{\sqrt{(R_4 + r)^2 - R_3{}^2 \sin^2 \theta}} + R_3 \sin \theta \right] \quad . \quad . \quad . \quad (162)$$

Differentiating (162) gives the acceleration A of the follower:

$$A = \frac{dV}{dt} = -\omega^2 \left\{ \frac{R_3{}^2 [R_3{}^2 \sin^4 \theta + (R_4 + r)^2 (1 - 2 \sin^2 \theta)]}{[(R_4 + r)^2 - R_3{}^2 \sin^2 \theta]^{3/2}} + R_3 \cos \theta \right\}$$
$$(163)$$

The design of a cam of the type discussed above for an automobile engine whose maximum speed of rotation is 3600 R.P.M. will now be taken as an example. It will be assumed that the exhaust valve is to have a total displacement, or lift, of $\frac{11}{32}$ in.; that it opens 40 degrees before the crank reaches the bottom dead center position, and that it closes 12 degrees after the crank reaches the top dead center position when the engine is cold. The valve is to have a dwell in the extreme lift position for a period corresponding to a 30 degree rotation of the crank. To allow for expansion as the engine warms up, there is to be a clearance of 0.008 in. between the follower roller and the cam when the engine is cold. The minimum radius of the cam is to be $\frac{13}{16}$ in. and the radius of the roller $\frac{7}{16}$ in.
Hence,

$$R_1 = \tfrac{13}{16} \text{ in., } r = \tfrac{7}{16} \text{ in., } h = \tfrac{11}{32} \text{ in., and } R_2 = R_1 + h = 1\tfrac{5}{32} \text{ in.}$$

$$\omega = \text{angular velocity of the cam} = \tfrac{1}{2}(\tfrac{3600}{60})2\pi = 60\pi \text{ rad./sec.}$$

It remains to find the nose radius R_4 and the angle β through which the cam turns while the valve is lifted $\frac{11}{32}$ in. when the engine has warmed up. Before finding β it is necessary to find the angle ϕ through which the cam must turn for a lift corresponding to the clearance of 0.008 in. between the roller and the cam when the engine is cold. By reference to Fig. 195, it is evident that

$$OC_1 = R_1 + r + 0.008 \text{ in.}$$

and

$$\cos \phi = \frac{R_1 + r}{OC_1} = \frac{R_1 + r}{R_1 + r + 0.008} = \frac{\tfrac{13}{16} + \tfrac{7}{16}}{\tfrac{13}{16} + \tfrac{7}{16} + 0.008} = 0.99364$$

or

$$\phi = 6.5°$$

Hence, during the lift of the valve, the cam must rotate through an angle,

$$\beta = \frac{1}{2}\left(\frac{180 + 40 + 12 - 30}{2}\right) + 6.5 = 57°$$

From equation (156) the radius of the nose is

$$R_4 = R_1 - \frac{h \cos \beta}{1 - \cos\beta} = \frac{13}{16} - \frac{11}{32}\left(\frac{\cos 57°}{1 - \cos 57°}\right) = 0.4014$$

Whence

$$R_3 = R_2 - R_4 = R_1 + h - R_4 = 0.755$$

From equation (157)

$$\tan \delta = \frac{(h + R_1 - R_4) \sin \beta}{R_1 + r} = \frac{(\frac{11}{32} + \frac{13}{16} - 0.4014) \sin 57°}{\frac{13}{16} + \frac{7}{16}} = 0.50655$$

or

$$\delta = 26° 52'$$

and

$$\beta - \delta = 30° 8'$$

The values given in Table 2 below were computed by the use of equations (158), (159), (160), (161), (162), and (163). For convenience, δ and $(\beta - \delta)$ were each divided into two equal parts corresponding to positions 0, 2, 4, 6, and 8 in Figs. 196 to 199 inclusive.

TABLE 2

|	$\delta = 26°-52'$				$(\beta - \delta) = 30°-8'$		
Position...........	0	2	4		4	6	8
ϕ.................	0	13°-26'	26°-52'	θ	30°-8'	15°-4'	0
s in inches..........	0	0.0352	0.1510	s	0.1510	0.2945	0.3437
V in ft./sec........	0	4.92	11.18	V	11.18	5.83	0
A in ft./sec.^{2}.......	3704	4245	6295	A	−3485	−4060	−4245

Evidently the point of inflection of the displacement graph at which the pressure angle is a maximum is for position 4 when the displacement is 0.151 in. Hence the radius of the base circle is

$$R = R_1 + r + s = \frac{13}{16} + \frac{7}{16} + 0.151 = 1.401 \text{ in.}$$

and by equation (126) the cam factor is,

$$f = \frac{2\pi\beta R}{360h} = \frac{2\pi 57 \times 1.401}{360 \times \frac{11}{32}} = 4.07$$

By reference to Fig. 195, the maximum pressure angle is seen to be equal to δ, or 26° 52′. Substituting this pressure angle in equation (152) yields a cam factor of 3.95, showing that an assumed law of

FIG. 197

FIG. 198

FIG. 199

FIG. 196
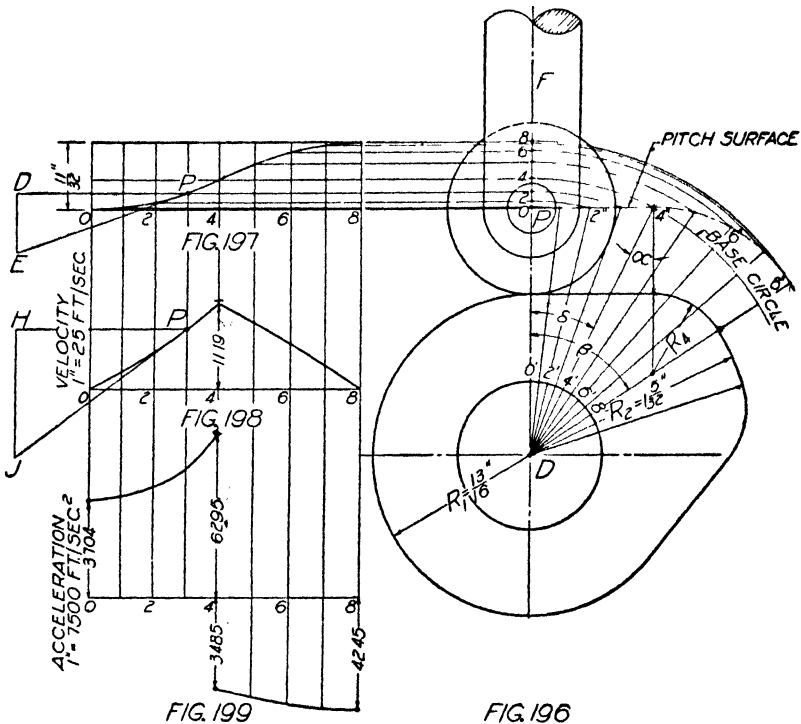
displacement in accordance with equation (149) would have resulted in a somewhat smaller cam.

Instead of attacking the problem mathematically as above, the displacement, velocity, and acceleration graphs for the follower may be found graphically as shown in Figs. 196, 197, 198, and 199. Having found β, δ, and R_4 from the given data, the cam and follower may be drawn as shown in Fig. 196, and the displacement graph, Fig. 197, found in the usual way. The velocity graph may then be obtained by graphically differentiating the displacement graph in accordance with Art. 70 and the acceleration graph obtained by graphically differentiating

the velocity graph in accordance with Art. 71. In Fig. 197 the total displacement is represented full size, and the corresponding time while the cam rotates through the angle β is represented by a line 1.393 in. long. Hence the space and time scales are

$$1 \text{ in.} = k_s = \tfrac{1}{12} \text{ ft.}$$

and

$$1 \text{ in.} = k_t = \frac{1}{\text{R.P.S.}}\left(\frac{\beta}{360}\right)\frac{1}{1.393} = \frac{60}{1800}\left(\frac{57}{360}\right)\frac{1}{1.393} = 0.00379 \text{ sec.}$$

In accordance with Art. 70 a constant length must be used for PD if the intercepts DE are to be to the same scale and used as ordinates of the velocity graph. In the present instance a constant length of 0.88 in. was used for PD, making the velocity scale

$$1 \text{ in.} = k_v = \frac{k_s}{(PD)k_t} = \frac{^1/_{12}}{0.88 \times 0.00379} = 25 \text{ ft./sec.}$$

In differentiating the velocity graph a constant length $PH = PD$ = 0.88 in. was used. Hence, in accordance with Art. 71, the intercept HJ at any instant represents the acceleration of the follower to a scale of

$$1 \text{ in.} = k_a = \frac{k_v}{(PH)k_t} = \frac{25}{0.88 \times 0.00379} = 7500 \text{ ft./sec.}^2$$

By the method just outlined and by drawing three times full size, values of the acceleration of the follower may be obtained that are in error by not more than 5 per cent. The evident difficulty with the method is in accurately drawing the tangent lines to the curves.

The most accurate graphical method of determining the velocities and accelerations is by use of velocity and acceleration vector diagrams. In applying these diagrams it is necessary to deal with equivalent mechanisms. Evidently the equivalent mechanism, when the follower roller is in contact with the flank, is as shown in Fig. 200, and as shown in Fig. 203 when the follower roller is in contact with the nose of the cam. In Fig. 200 it is obvious that the pitch point D, common to the follower block *2* and stem *3*, will trace a straight line on the rotating cam *1*. Coriolis' law must therefore be applied to determine the acceleration of D, D being the describing point on the follower and C the coincident point on the rotating cam. The position shown corresponds to $\phi = 13° 26'$, or to position *2'* Fig. 196. The space scale of Fig. 200

is 1 in. $= k_s = \frac{1}{12}$ ft.; the velocity scale of Fig. 201 is 1 in. $= k_v = 5\pi$ ft./sec.; and the acceleration scale of Fig. 202 is 1 in. $= k_a = \dfrac{k_v{}^2}{k_s}$ $= 300\pi^2$ ft./sec.^{2}. As a point in the cam, C rotates about O_{10} at a velocity

$$V_{CO} = \frac{2\pi(O_{10}C)}{12}(\text{R.P.S.}) = \frac{2\pi 1.285}{12}\left(\frac{1800}{60}\right) = 6.425\pi \text{ ft./sec.}$$

FIG 201

FIG. 200

FIG 202

Hence vector oc in Fig. 201 is drawn normal to $O_{10}C$ to a length of

$$oc = \frac{V_{CO}}{k_v} = \frac{6.425\pi}{5\pi} = 1.285 \text{ in.}$$

It is evident from the instant centers that V_{DO} is directed normal to $O_{30}D$ and that V_{DC} is directed normal to $O_{13}D$. Hence vectors through

o and c normal to $O_{30}D$ and $O_{13}D$, respectively, intersect at d and determine the velocity vectors od and cd. Hence,

$$V_{DO} = (od)k_v = 2.696\pi \text{ ft./sec.}$$

$$V_{DC} = (cd)k_v = 6.60\pi \text{ ft./sec.}$$

By Coriolis' law, equation (123),

$$A_{DO} = A_{CO} +\!\!\!+ A_{DC} +\!\!\!+ 2V_{DC}\omega_C$$

Since the cam rotates at a uniform rate,

$$A_{CO} = A_{CO}^n = \frac{V_{CO}^2}{(O_{10}C)k_s} = \frac{(6.425\pi)^2}{1.285 \times \frac{1}{12}} = 385.5\pi^2 \text{ ft./sec.}^2$$

and vector $o'c'$, parallel to CO_{10}, is of a length

$$o'c' = \frac{A_{CO}}{k_a} = \frac{385.5\pi^2}{300\pi^2} = 1.285 \text{ in.}$$

Since D relative to the frame O and relative to C, or to the cam, traverses straight-line paths, the radius of curvature of the path in each case is infinite, and A_{DO}^n and A_{DC}^n are therefore zero. Hence $A_{DO} = A_{DO}^t$ and $A_{DC} = A_{DC}^t$ and $o'd' = (do)d'$ and $c'd' = (dc)_1 d'$. These vectors are respectively parallel and normal to $O_{10}D$ and $O_{13}D$. Since the curve traced by the describing point D is on the cam, the angular velocity ω_C of the curve is equal to the angular velocity ω of the cam, or,

$$\omega_C = 2\pi(\text{R.P.S}) = 2\pi(\tfrac{1800}{60}) = 60\pi \text{ rad./sec.}$$

Hence,

$$2V_{DC}\omega_C = 2(6.6\pi)60\pi = 792\pi^2 \text{ ft./sec.}^2$$

The direction of the vector representing $2V_{DC}\omega_C$ is such that if applied as a force at the terminus of the velocity vector cd it would rotate the vector in the same sense as the cam carrying point C is rotating, that is, counter-clockwise. Hence the vector is parallel to $O_{13}D$ and of a length equal to

$$(dc)(dc)_1 = \frac{2V_{DC}\omega_C}{k_a} = \frac{792\pi^2}{300\pi^2} = 2.64 \text{ in.}$$

As shown above, vectors $o'c'$ and $(dc)(dc)_1$ in Fig. 202 are known in magnitude and direction, and $(dc)_1d'$ and $b'd'$, or $(do)d'$, are known in direction, hence the intersection d' is determined, and

$$A_{DO} = (o'd')k_a = 1.44 \times 300\pi^2 = 4250 \text{ ft./sec.}^2$$

which is almost identically the same as the computed value given in Table 2 for the same position.

By reference to Fig. 203, the equivalent mechanism for the follower
roller in contact with the nose of the cam is seen to be the familiar slider-
crank chain.   The position shown corresponds to $\theta = 15° 4'$, or to
position 6, Fig. 196.   The space, velocity, and acceleration scales for
Figs. 203, 204, and 205 are the same as used for Figs. 200, 201, and
202.   It is to be noted that EF is the radius of curvature of the pitch
surface of the cam at point F.   In the present instance this radius is

FIG. 204

FIG. 205

FIG. 203
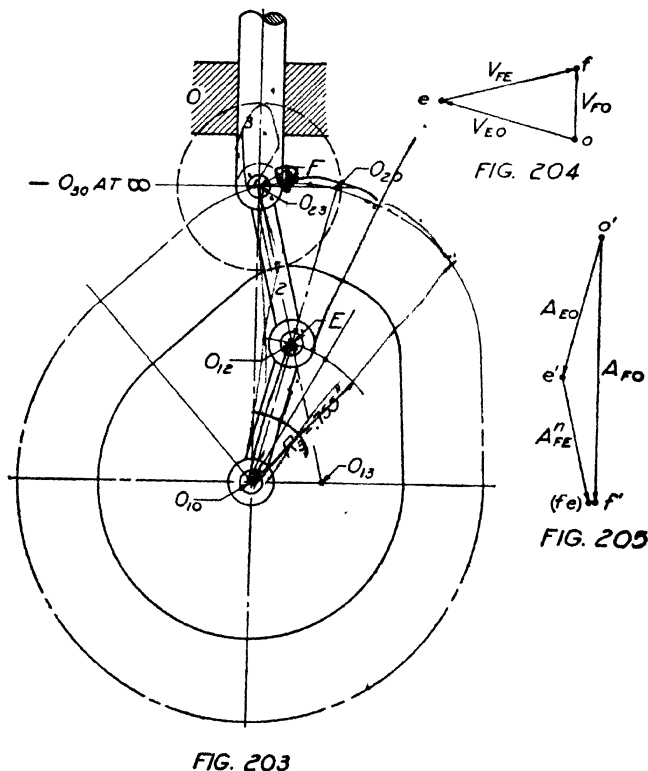
known by construction to be constant and equal to $(R_4 + r) = 0.8389$ in.
throughout contact between the roller and the nose of the cam; other-
wise it would be necessary to determine this radius by trial for each
position of the cam and follower.   Obviously, since link 1 rotates with
the cam, the velocity V_{EO} at the instant of point E relative to the frame
is normal to $O_{10}E$, and has a magnitude of

$$V_{EO} = \omega(O_{10}E) = 60\pi\left(\frac{0.755}{12}\right) = 3.775\pi \text{ ft./sec.}$$

Hence vector oe in Fig. 204 is drawn normal to $O_{10}E$ to a length of

$$oe = \frac{V_{EO}}{k_v} = \frac{3.775\pi}{5\pi} = 0.755 \text{ in.}$$

Velocities V_{FO} and V_{FE} are parallel and normal, respectively, to $O_{10}F$ and EF. Hence the intersection at f of vectors through o and e, parallel and normal, respectively, to $O_{10}F$ and EF, determines the velocity vectors of and ef. Hence,

$$V_{FO} = (of)k_v = 0.375 \times 5\pi = 1.875\pi \text{ ft./sec.}$$

and

$$V_{FE} = (ef)k_v = 0.75 \times 5\pi = 3.75\pi \text{ ft./sec.}$$

Since the angular velocity of $O_{10}E$ is uniform,

$$A_{EO} = A_{EO}^n = \frac{V_{EO}^2}{Rk_s} = \frac{(3.775\pi)^2}{0.755 \times \frac{1}{12}} = 2235 \text{ ft./sec.}^2$$

and the length of the vector in Fig. 205 representing A_{EO} is

$$o'e' = \frac{A_{EO}}{k_a} = \frac{2235}{300\pi^2} = 0.755 \text{ in.}$$

This vector is directed parallel to EO_{10} from E towards O_{10}. Now,

$$A_{FE}^n = \frac{V_{FE}^2}{(EF)k_s} = \frac{(3.75\pi)^2}{0.8382 \times \frac{1}{12}} = 1990 \text{ ft./sec.}^2$$

and the length of the vector representing A_{FE}^n is

$$e'(fe) = \frac{A_{FE}^n}{k_a} = \frac{1990}{300\pi^2} = 0.67 \text{ in.}$$

Vector $(fe)f'$ representing A_{FE}^t is known to be directed normal to $e'(fe)$, and vector $o'f'$ representing $A_{FO} = A_{FO}^t$ is known to be parallel to FO_{10}. Hence the intersection of lines through (fe) and o', respectively normal and parallel to $e'(fe)$ and FO_{10}, determines f' and the vector $o'f'$ representing A_{FO}, or

$$A_{FO} = (o'f')k_a = 1.37 \times 300\pi^2 = 4055 \text{ ft./sec.}^2$$

which is almost identically the same as the computed value given in Table 2 for the same position.

As before pointed out, the displacements and corresponding accelerations of the follower may be used to design a suitable spring to keep the follower in contact with the cam. To this end, let

W = the weight of the follower, valve, spring retainer, and half the spring = 1.00 lb.;

D = diameter of the valve = $1\frac{3}{4}$ in.;

R = mean radius of the coils of the spring = $\frac{5}{8}$ in.

For a helical spring,

$$PR = p\frac{\pi d^3}{16} \qquad \cdots \cdots \cdots \cdots \cdots \cdots \quad (164)$$

and

$$PR = \frac{\delta d^4 E}{64R^2 n} \qquad \cdots \cdots \cdots \cdots \cdots \cdots \quad (165)$$

where

d = the diameter of the wire in inches;

p = the maximum induced shear stress in pounds per square inch;

P = the axial load in pounds compressing or elongating the spring;

δ = the deflection, or amount in inches that the spring is compressed or elongated;

n = the number of coils;

E = the shear modulus of elasticity of the spring material in pounds per square inch

= 12,000,000 for steel wire.

FIG. 206

In Fig. 206 the total displacement of the follower or valve is represented four times full size by the line *08*. This line is divided to correspond to the displacements shown in Fig. 197. The ordinates at *0, 2, 4*, etc., were computed from the corresponding ordinates of Fig. 199, and are shown in Fig. 206 to a scale of 1 in. = 200 pounds. At position *2*, for example, the acceleration of the follower from Fig. 199 is 4245 ft./sec.^{2}, and the corresponding inertia force is

$$F = MA = \frac{1.00}{32.2} \times 4245 = 131.7 \text{ lbs.}$$

Length of ordinate at $2 = \dfrac{131.7}{200} = 0.6585$ in.

It is to be noted that at position *4* there is a sudden change from accel-
eration to retardation, and that for the remainder of the lift a downward
force is required to keep the follower roller in contact with the cam.
Since the action of gravity is only one pound, it is insufficient to give
the required retardation, and a spring must therefore be used. In fact,
the action of gravity is so small compared to the required retarding
force that it will be neglected. To represent it on the diagram would
require a line 0.005 in. below and parallel to *08*. For satisfactory action
the spring pressure at any instant from position *4* to *8* should be some-
what in excess of the required retarding force. Hence, in Fig. 206,
the pressure line *cd* of the spring is drawn below and approximately
parallel to *ab*, the ordinates of the shaded portion of the diagram giving
the variation of the vertical component of the pressure between the
roller and cam during the up and down motion of the follower and valve.
The scale of the spring required is

$$\frac{8d - oc}{h} = (144 - 100)\tfrac{32}{11} = 128 \text{ lbs./inch of compression,}$$

and the total deflection is

$$\delta = \tfrac{144}{128} = 1.125 \text{ in.}$$

From equation (164) for an allowable stress $p = 65000$,

$$d = \sqrt[3]{\frac{16PR}{\pi p}} = \sqrt[3]{\frac{16 \times 144 \times 0.625}{\pi 65000}} = 0.192 \text{ or No. 6 wire,}$$

and from equation (165)

$$n = \frac{\delta d^4 E}{64PR^3} = \frac{1.125(0.192)^4 12000000}{64 \times 144(0.625)^3} = 8.15, \text{ say 8.}$$

Hence a helical spring of number 6 steel spring wire having 8 effective
coils and set with an initial compression of 100 pounds will be satis-
factory.

98. Automobile Cam with Mushroom Follower. In the preceding
example, the motion of the follower for a cam outline of known form
was first analyzed mathematically, next the method of determining
velocity and acceleration graphs by graphical differentiation was
employed, and finally the method of determining the velocity and
acceleration of the follower for any position of the cam by applying
velocity and acceleration vector diagrams was illustrated. The present
example will be treated by the first and last methods only. In Fig.
207, let

β = the angle turned through by the cam during the total displacement;

δ = the angle turned through by the cam while the follower is in contact with the flank of the cam;

ϕ = any angle turned through by the cam while the follower is in contact with the flank;

θ = the angle turned through by the cam from any position of the follower in contact with the nose of the cam to the position for total displacement;
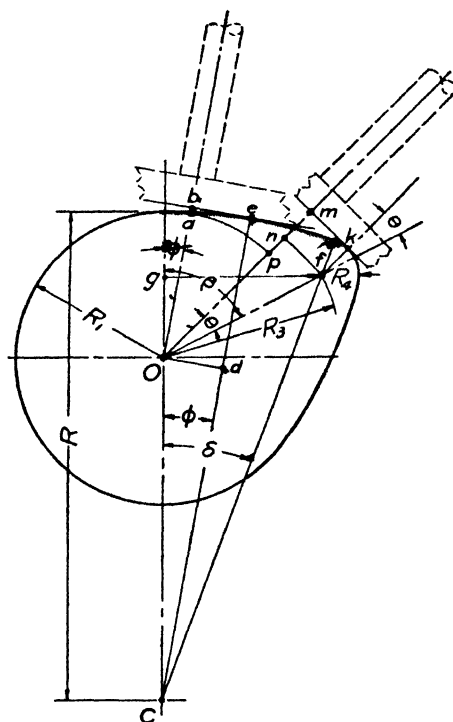

FIG. 207

R_1 = the minimum radius of the cam;

R_2 = the maximum radius of the cam;

R_4 = the radius of the nose of the cam;

$R_3 = R_2 - R_4$;

h = the total displacement of the follower = $R_2 - R_1$;

s = the displacement of the follower at any instant;

ω = the angular velocity of the cam in radians per second.

In any given case the minimum radius R_1, the total displacement h, the corresponding angular displacement β of the cam, and the radius of the nose R_4 are fixed by the conditions of service and accepted practice. Having these, the angle δ and the outline radius R can be determined. Referring to Fig. 207,

$$\overline{cf}^2 = \overline{fg}^2 + \overline{cg}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

But

$$cf = R - R_4, \quad fg = R_3 \sin \beta, \quad \text{and} \quad cg = R - R_1 + R_3 \cos \beta$$

Substituting these values of cf, fg, and cg in (166) and solving for R gives

$$R = \frac{R_3{}^2 - R_4{}^2 + R_1{}^2 - 2R_3R_1 \cos \beta}{2(R_1 - R_4 - R_3 \cos \beta)} \quad . \quad . \quad (167)$$

also,

$$\sin \delta = \frac{fg}{cf} = \frac{R_3 \sin \beta}{R - R_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

For the follower in contact with the flank of the cam, the displacement of the follower for any angular displacement ϕ of the cam between zero and δ, is

$$s = ab = ob - R_1 = de - R_1 \quad . \quad . \quad . \quad . \quad . \quad (169)$$

But

$$\cos \phi = \frac{cd}{co} = \frac{R - de}{R - R_1} \quad \text{or} \quad de = R - (R - R_1) \cos \phi$$

Substituting the value of de in (169) gives

$$s = (R - R_1)(1 - \cos \phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

Differentiating (170) with respect to time gives the velocity V of the follower:

$$V = \frac{ds}{dt} = \omega(R - R_1) \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

Differentiating (171) with respect to time gives the acceleration A of the follower:

$$A = \frac{dV}{dt} = \omega^2(R - R_1) \cos \phi \quad . \quad . \quad . \quad . \quad . \quad (172)$$

For the nose of the cam for values of θ between zero and $(\beta - \delta)$,

$$s = om - R_1 = on + nm - R_1 = R_3 \cos \theta + R_4 - R_1 \quad . \quad (173)$$

Differentiating (173) gives

$$V = \frac{ds}{dt} = - \omega R_3 \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

Differentiating (174) gives

$$A = \frac{dV}{dt} = - \omega^2 R_3 \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

The design of a cam of the above type for an automobile engine whose maximum speed of rotation is 3600 R.P.M. will now be taken as an example. It will be assumed as before that the exhaust valve is to have a lift of $\frac{11}{32}$ in., that it opens 40 degrees before the crank reaches the bottom dead center position, and that it closes 12 degrees after the crank reaches the top dead center position. To allow for expansion as the engine warms up there is to be a clearance angle of the cam shaft of 4.5 degrees when the engine is cold. The minimum radius of the cam is to be $\frac{3}{4}$ in. and the radius of the nose $\frac{3}{16}$ in., hence,

$$R_1 = \tfrac{3}{4} \text{ in.}, \quad R_4 = \tfrac{3}{16} \text{ in.}, \quad h = \tfrac{11}{32} \text{ in.};$$

$$R_2 = R_1 + h = 1\tfrac{3}{32} \text{ in.}, \quad \text{and} \quad R_3 = R_2 - R_4 = \tfrac{29}{32} \text{ in.};$$

$$\omega = \text{angular velocity of cam} = \tfrac{1}{2}(\tfrac{3600}{60})2\pi = 60\pi \text{ rad./sec.};$$

$$\beta = \frac{1}{2}\left(\frac{180 + 40 + 12}{2}\right) + 4.5 = 62° 30'$$

Substituting in equations (167) and (168) gives

$$R = 2.5024 \text{ in.} \quad \text{and} \quad \delta = 20° 20'$$

Hence,

$$(\beta - \delta) = 42° 10'$$

The values given in Table 3 below were computed by the use of equations (170) to (175), inclusive. For convenience δ and $(\beta - \delta)$ were each divided into two equal parts corresponding to positions *0, 2, 4, 6,* and *8* in Figs. 208 to 211, inclusive.

In Fig. 209 the total displacement is represented full size, and the corresponding time while the cam rotates through the angle β is represented by a line 1.74 in. long. Hence the space and time scales are

$$1 \text{ in.} = k_s = \tfrac{1}{12} \text{ ft.}$$

and

$$1 \text{ in.} = k_t = \frac{1}{\text{R.P.S}}\left(\frac{\beta}{360}\right)\frac{1}{1.74} = \frac{60}{1800}\left(\frac{62.5}{360}\right)\frac{1}{1.74} = 0.003325 \text{ sec.}$$

The velocity scale of Fig. 210 is 1 in. $= k_v = 25$ ft./sec., and the acceleration scale of Fig. 211 is 1 in. $= k_a = 7500$ ft./sec.^{2}.

TABLE 3

Position	$\delta = 20°\text{-}20'$				$(\beta - \delta) = 42°\text{-}10'$		
Position	0	2	4		4	6	8
ϕ	0	10°-10'	20°-20'	θ	42°-10'	21°-5'	0
s in inches	0	0.0275	0.1090	s	0.1090	0.285	0.3437
V in ft./sec.	0	4.85	9.56	V	9.56	5.27	0
A in ft/sec.^{2}	5186	5100	4860	A	1987	2506	2585.5

In applying the velocity and acceleration vector diagrams it is necessary, as before stated, to deal with an equivalent mechanism. The equivalent mechanism for the follower in contact with the flank of the cam is shown in Fig. 208. Obviously, if a pin, located at the center of curvature of the outline of the flank, were fitted with a sliding block 2, rotation of the pin with the cam would impart the same motion to the slotted follower as the cam imparts to the mushroom follower. For the follower in contact with the nose of the cam, the equivalent mechanism would be similar, the pin being located at the center of curvature of the nose. In Fig. 208, position 2 has been chosen to illustrate the application of the vector diagram method. At the instant point F in 3 is coincident with C, a point common to links 1 and 2. Point C as a point in 1 rotates about the fixed center O_{10}. Hence velocity V_{CO} and its vector oc in Fig. 212 are normal to $O_{10}C$.

$$V_{CO} = \omega(O_{10}C)k_s = 60\pi(1.75)\tfrac{1}{12} = 8.75\pi \text{ ft./sec.}$$

$$oc = \frac{V_{CO}}{k_v} = \frac{8.75\pi}{5\pi} = 1.75 \text{ in.}$$

Velocity V_{FO} is directed vertically upward, and velocity V_{CF} horizontally to the right. Hence the intersection of a vertical line through o with a horizontal line through c determines f and the velocity vectors of and fc.

$$V_{FO} = (of)k_v = 1.725 \times 5\pi = 8.625\pi \text{ ft./sec.}$$

$$V_{CF} = (fc)k_v = 0.308 \times 5\pi = 1.54\pi \text{ ft./sec.}$$

Since the cam rotates at a uniform rate $A_{co} = A_{co}^n$, and

$$A_{co} = A_{co}^n = \frac{V_{co}^2}{(O_{10}C)k_s} = \frac{(8.75\pi)^2}{1.75 \times \frac{1}{12}} = 525\pi^2 \text{ ft./sec.}^2$$

Adopting an acceleration scale of 1 in. $= k_a = \dfrac{k_v^2}{k_s} = \dfrac{(5\pi)^2}{\frac{1}{12}} = 300\pi^2$

ft./sec.^{2}, the length of acceleration vector $o'c'$ in Fig. 213 becomes

$$o'c' = \frac{A_{co}}{k_a} = \frac{525\pi^2}{300\pi^2} = 1.75 \text{ in.}$$

FIG. 209

FIG 210

FIG 213

FIG 211

FIG 212

FIG 208
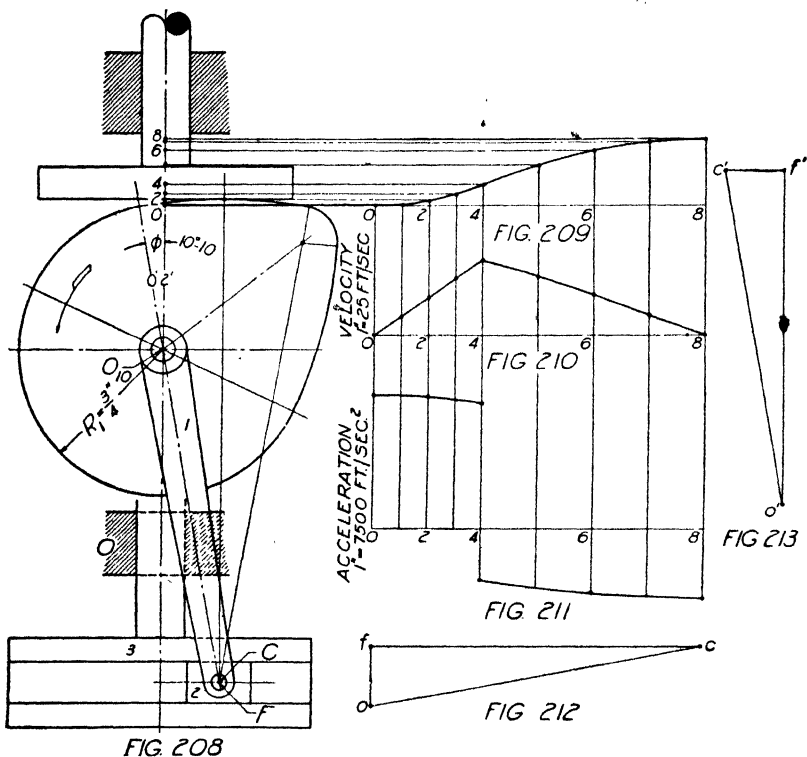
This vector is directed parallel to CO_{10}. Since F moves vertically in a straight line relative to the frame and moves horizontally in a straight line relative to C, accelerations A_{FO} and A_{CF} have no normal components. Hence the intersection at f' of a vertical line through o' with a horizontal line through c' determines the acceleration vectors $o'f'$ and $c'f'$. Hence,

$$A_{FO} = (o'f')k_a = (1.725)300\pi^2 = 5107.5 \text{ ft./sec.}^2$$

It is to be noted that this value is in very close agreement with the value in Table 3 for the same position. It is also to be noted that the equivalent mechanism is the familiar Scotch yoke, the follower therefore having harmonic motion as is plainly indicated by equations (170) to (175), inclusive.

100. Disk Cam with Reciprocating Flat-Faced Follower. The first method of designing cams was illustrated and fully discussed in Arts. 91 and 93, and the second method in Arts. 96, 98, and 99. The cam constructions in this and the remaining articles of this chapter, with the exception of Arts. 101 and 105, will be based on the first method of design, which requires the working surface of a cam to be determined from the assumed law of displacement of its follower.

It is required in the present problem to determine the working surface of a disk cam to impart a motion to a reciprocating flat-faced follower in accordance with the displacement-time graph shown in Fig. 214. The total displacement of the follower has been taken as $1\frac{1}{4}$ in., and the time for one revolution of the cam in a clockwise direction at a uniform rate has been divided into 16 equal time intervals. As for the problem in Art. 93, the follower is to move one-half the total displacement with simple harmonic motion in 4 time units, is to dwell for 2 time units, is to move to its maximum displacement with simple harmonic motion in 4 time units, is to dwell 1 time unit, is to return with uniformly accelerated and retarded motion in 4 time units, and is to dwell for the last time unit of the revolution of the cam. In Fig. 215 the axis of the **stem** of the flat-faced follower intersects the axis of the cam. It is to be noted that this is not essential, since the axis of the stem could be in any other location parallel to that shown without affecting the cam outline or the motion of the follower. To determine the outline of the cam it is necessary to construct a displacement scale for some point on the follower. While any point in the follower might be used, the most convenient point is D, where a perpendicular through O intersects the contact surface of the follower. Having the X axis of Fig. 214 in line with D, the displacement scale DS is easily constructed by projection. Since the zero position of the follower is its lowest position, the center of the cam must be a distance below the contact surface of the follower equal to the minimum radius of the cam. For reasons which will be explained later, this radius was taken $1\frac{3}{8}$ in., or $\frac{1}{4}$ in. greater than for the cam in Art. 93. As the cam rotates clockwise, point D on the contact surface of the follower advances to positions *1, 2, 3,* etc.; or, as the follower is rotated counter-clockwise with the cam stationary, point D moves to the radial positions *1″, 2″, 3″,* etc. Hence, say for the second position, the intersection of an arc

of radius *O2* with the radial line *O2'* of the cam determines the position
2" of point *D* of the contact surface of the follower. Since the contact
surface of the follower is perpendicular to the stem of the follower, a
line through *2"* perpendicular to *O2'* determines one line to which the
working surface of the cam must be tangent. Lines representing the
contact surface of follower having been similarly located for the re-
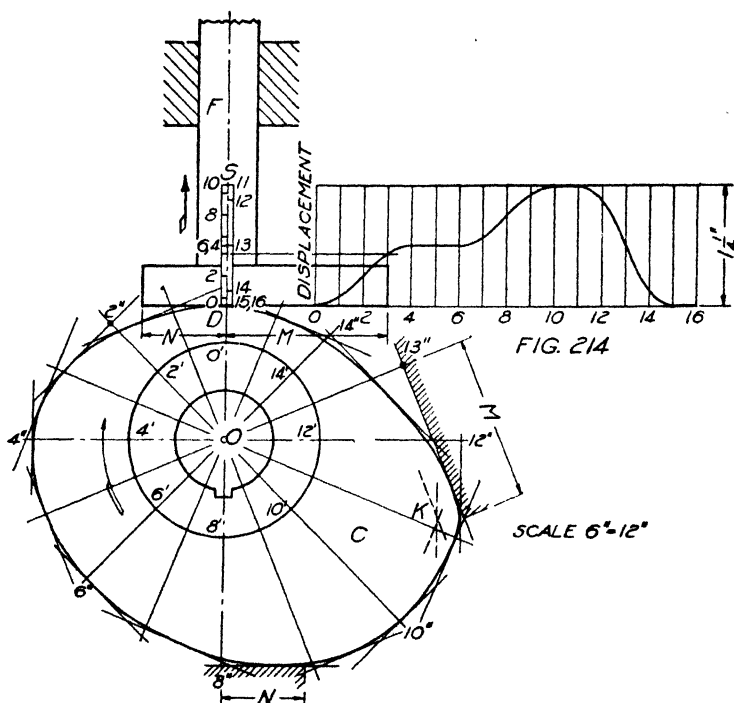
FIG. 214

FIG. 215

maining positions, the working surface of the cam is drawn tangent
to them as shown.

Had the minimum radius of the cam been taken $1\frac{1}{8}$ in. as for the
cam in Art. 93, the lines for positions *11*, *12*, and *13* would have inter-
sected as shown at *K*, thus requiring the working surface of the cam to
come to quite a sharp point. This defect was remedied in the usual
way by increasing the minimum radius of the cam, in this instance,
to $1\frac{3}{8}$ in. It is to be noted that a satisfactory working surface for a
minimum radius of 1 in. is impossible, since the line for position *12*
would fall beyond the intersection of the lines for positions *11* and *13*.

The point of contact between cam and follower is a maximum

distance to the right of D for position *13*, and a maximum to the left of D for position *8* as shown. To allow for wear and for initial variation in form, the contact surface should extend somewhat beyond the apparent points of contact. As shown in Fig. 215, M and N have been made about $\frac{1}{4}$ in. greater than the maximum distances to the points of contact. If the follower stem were round and its axis located as shown but offset from the midplane of the cam, the friction between the cam and follower face would cause the follower to rotate while reciprocating, thus requiring the follower face to be in the form of a circular disk of radius M. Such a follower is known as a mushroom follower. The rotation of the follower distributes the wear and assists lubrication.

For a reciprocating flat-faced follower the pressure between the cam and follower, friction neglected, is at right angles to the follower face. The pressure angle is therefore constant and equal to 90 degrees minus the angle between the follower face and stem. Since in the present instance the angle between the face and stem is 90 degrees, the pressure angle is zero.

For this and the cams to follow, the velocity of sliding between cam and follower at any instant may be determined, as previously explained, by the method of instant centers of Art. 65 or by the vector diagram methods of Arts. 47 and 75(c).

101. Stamp Mill Cam. In Fig. 216 is shown a reciprocating flat-faced follower F actuated by a rotating disk C in the form of a double involute cam. This cam mechanism is used in stamp mills for pulverizing crushed ore such as gold-bearing quartz. The follower stem or rod E with a stamp head and steel shoe attached to its lower end, is called the stamp. On being raised and released by the rotating cam, the stamp drops, and the ore in the mortar is crushed between the stamp shoe and the die or anvil. It is to be noted that the stamp is raised and dropped twice during each revolution of the double cam. Furthermore, since a certain period of time is required for the drop of the stamp, the cam must be so constructed that the stamp is raised during a rotation of less than 180 degrees. As a rule a stamp mill consists of a battery of 5 or 10 stamps, the cams being mounted on the same shaft in proper angular relation to give the desired sequence for the rise and fall of the stamps.

A common outline for the working surfaces of stamp mill cams is an involute curve $PP_1P_2P_3$ generated by a line PL rolling on a base circle B. Hence P_1G is equal in length to the arc PG, and KM to the arc PGM. Therefore, as OG and the cam rotate from a position parallel to the follower face to the position shown, the follower or stamp is raised a distance equal to the difference between MK and GP_1. Since GP_1

and MK are proportional to the angular displacement of the cam, it follows that the stamp would be lifted at a uniform velocity provided the cam rotates at a uniform rate. The follower face is in its lowest position when the stamp shoe is on the die, and in this position it should be a sufficient distance, say GP_1, above the center of the cam shaft to clear the hub of the cam. Assuming the lowest position of the follower face to be a distance GP_1 above the center of the cam, and P_3 to be the terminus of the involute, then the lift due to the action of the involute is

$$h = HP_3 - GP_1 = \left(\frac{\theta}{360}\right)2\pi R \ \ . \ \ \ . \ \ \ . \ \ \ . \ \ \ (176)$$

Assuming the hub radius to be $1\frac{1}{2}$ in., $GP_1 = 1\frac{11}{16}$ in., $h = 2\frac{1}{2}$ in., and $\theta = 105°$, then

$$R = \frac{h}{2\pi}\left(\frac{360}{\theta}\right) = \frac{2.5}{2\pi}\left(\frac{360}{105}\right) = 1.363 \text{ in., say } 1\frac{3}{8} \text{ in., or } \theta = 104.5°$$

It is to be noted that throughout involute action the normal through the point of contact of cam and follower is tangent to the base circle B.

SCALE 3"=12"
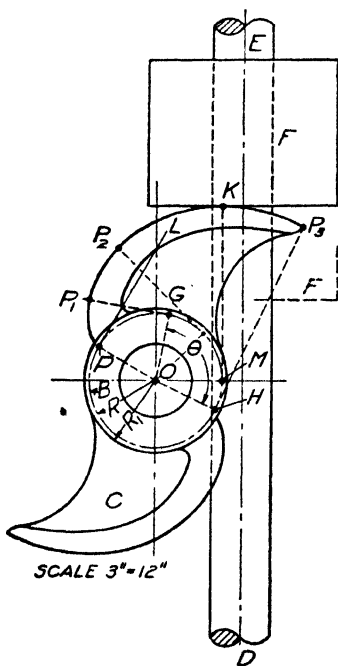
FIG. 216

Hence the point of contact K remains at a constant distance from the axis of the stamp stem. By making the axis of the stem tangent to the base circle, the amount the point of contact is offset with respect to the stem axis is somewhat reduced and the lifting action thereby improved. The additional lift of the stamp after involute action ceases is given by the difference between OP_3 and HP_3. The pressure between the cam and the follower as the cam slides off the follower is relieved by the tendency of the stamp to keep on moving upward. The distance in feet that the stamp would move without help from the cam after involute action ceases would be $s = V^2 \div 2g$. Evidently a uniform velocity of lift from the instant contact between cam and follower begins is impossible, as it would require an accelerating force of infinite magnitude. Owing, however, to the flexible belt drive

and the elasticity and yield of the members involved, a short interval elapses before uniform velocity is established, and the initial pressure between cam and follower is in consequence reduced in magnitude to operative values, provided, of course, it is not attempted to rotate the cams at too high speed.

102. Disk Cam with Oscillating Roller Follower. In Fig. 217 the disk cam C which rotates clockwise at a uniform rate is to impart a motion of oscillation to the roller follower F which is pivoted at P_0.
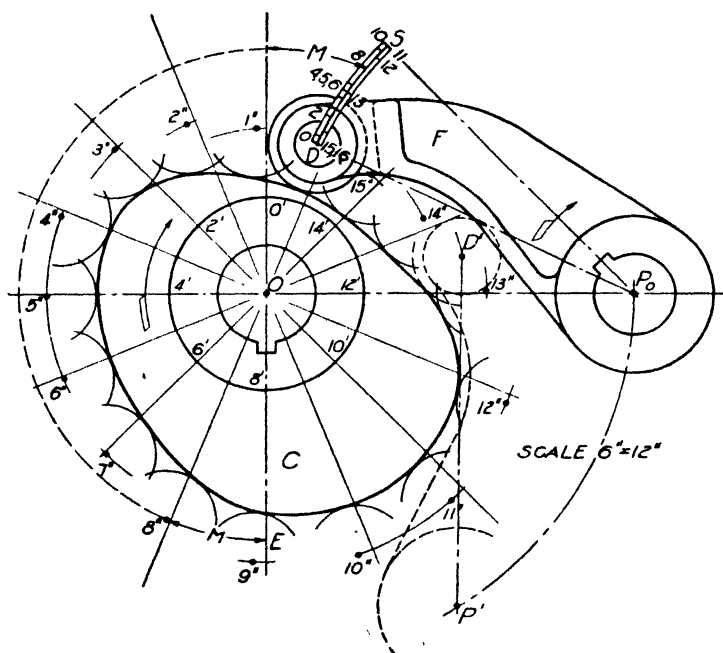
FIG 217

The total angular displacement of the follower is to be 20 degrees for an arc of travel of the pitch point D of $1\frac{1}{4}$ in. Hence, the follower radius P_0D must be,

$$P_0D = 1.25\left(\frac{180}{20\pi}\right) = 3.58 \text{ in.}$$

The law of displacement of D along the circular arc DS is to be the same as for the problem of Art. 93. To construct the displacement scale for Fig. 217, it is therefore required to divide the arc DS, which is $1\frac{1}{4}$ in. long, to correspond exactly to the displacement scale of Fig. 172, for which the total rectilinear displacement is $1\frac{1}{4}$ in. One method is to transfer the divisions of the straight line of the same length as the arc

to the edge of a strip of paper and then, after bending it to conform to the arc, to transfer the divisions from the edge of the paper to the arc. Having the displacement scale DS in accordance with the desired law of displacement of the pitch point D, the working surface of the cam may most easily be determined by considering the cam stationary and rotating the follower counter-clockwise. As before explained, this gives the same relative motion of follower and frame as clockwise rotation of the cam. Since 16 equal intervals of time were used in constructing the displacement scale, 16 radial lines equally spaced are drawn through the center of the cam. While any one of these lines might be used as a reference, the most convenient line is the one nearest to but wholly on one side of the displacement scale DS. This line, a vertical line in the present instance, is numbered o', and the others numbered consecutively in the direction of rotation of the follower. The method of finding the location of the pitch point for each interval will be illustrated by considering position 8. With $O8$ as a radius, an arc is drawn until it intersects $O8'$ at E. This corresponds to a rotation of the follower of eight intervals plus the offset M. Since a rotation of eight intervals only is desired, the rotated position $8''$ of 8 is back of E a distance $E8''$ equal to M. All other positions of the pitch point, such as $1''$, $2''$, $3''$, etc., were found in a similar manner. With each point as a center, arcs of the roller radius were struck, and the working surface of the cam drawn tangent to them as shown.

If the follower arm oscillates in the plane of the cam, it must be so shaped as not to interfere with the cam. With a sheet of celluloid on which the centers O, D, P_0, and the radius P_0D are marked, the position for which the cam would be the nearest to or overlap P_0D the farthest may easily be found by revolving the sheet of celluloid about center O. Lacking a sheet of celluloid, a transparent triangle may be used. It is to be noted that for the position of the pitch point D' about midway between $13''$ and $14''$, the follower radius $P'D'$ is nearest to the working surface of the cam. The required shape of the follower arm to safely clear the cam is shown dotted. To simplify the pattern work the outlines of the follower arm should be composed of straight lines and circular arcs.

Since the path of the follower roller does not depart so widely from a radial line through the center of the cam, the approximate size of the cam for any given pressure angle could be computed by the method outlined in Art. 95.

103. Disk Cam with Oscillating Flat-Faced Follower. In Fig. 218 the disk cam C which rotates clockwise at a uniform rate is to impart a motion of oscillation to a flat-faced follower F which is pivoted at P_0.

The total angular displacement and the law of displacement of the follower are to be the same as for the problem in the preceding article. Point D on the face of the follower was taken the same distance, 3.58 in., from the pivot P_0 as in Fig. 217, in order that the same displacement scale DS might be used; otherwise any other convenient point on the face might have been used. As in the previous problem, the most convenient expedient in determining the outline of the working surface
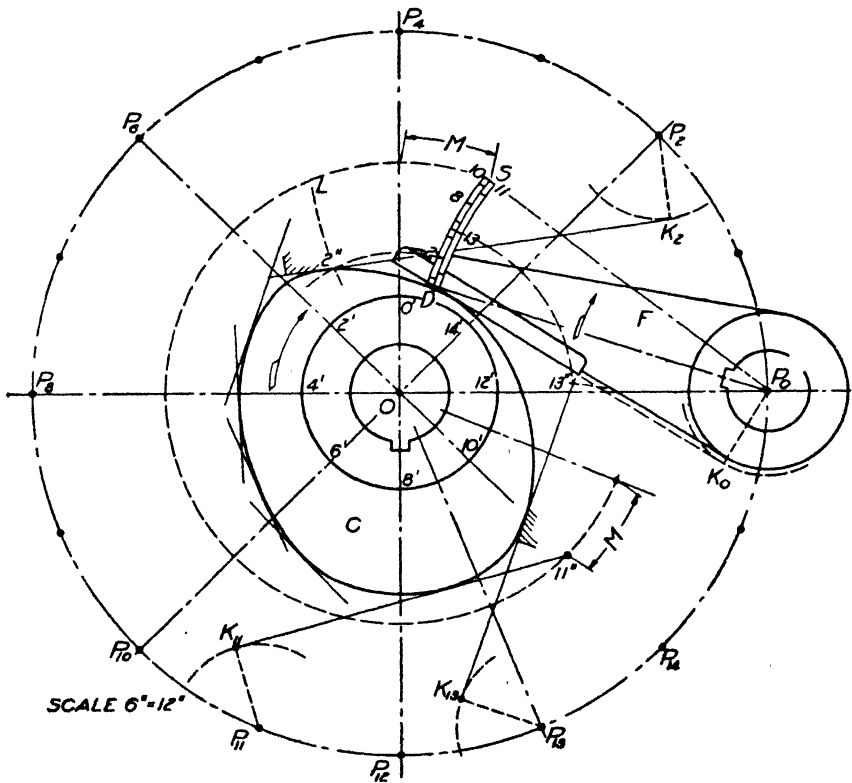
FIG 218

of the cam is to consider the cam stationary and to rotate the follower and frame in the counter direction. This was done in the previous problem to determine the location of the center of the follower roller for each interval. In the present instance it is required to find the location of a line representing the follower face for each interval. Hence two convenient points such as D and P must be located for each interval, and a line drawn from D tangent to an arc of radius P_0K_0 having the revolved position of P as a center. In eleven intervals, D has moved

to position *11* and has revolved to position *11″*, a distance *M* back of the radial line *11′*, and P_0 has revolved to position P_{11}. A line from *11″* tangent to an arc of radius P_0K_0 having P_{11} as a center locates the position of the cam face for position *11*. All the other positions of the cam face were located in a similar manner, and the outline of the working surface of the cam drawn tangent to them as shown. Since *D* is at a constant distance from *P*, the revolved positions of *D* could have been located by the *locus* instead of the *offset* method. Thus *2″* is at a distance from P_2 equal to P_0D, and hence at the intersection of two arcs of radii *O2* and P_0D having their respective centers at *O* and P_2.

To fix the length of the follower face it is necessary to determine the contact points nearest to and farthest from the pivot point. In the present instance these points occur for positions *13* and *2*, respectively. To allow for wear and variations in form, the follower face should be made a suitable amount longer than the distance between the two contact points determined.

For a given law of displacement and distance between *O* and P_0, a satisfactory cam outline for cam mechanisms of this type may be possible for rotation in one direction while not possible for rotation in the opposite direction. A sharp-pointed cam or an impossible outline can be avoided only by having a suitable law of displacement and distance OP_0.

104. Positive Return Single-Disk Cams with Reciprocating Yoke Followers. With the exception of Fig. 153, Art. 86, it was assumed in connection with the cam mechanisms previously dealt with that the follower on its return stroke was kept in contact with the cam by the action of gravity or by the action of some form of spring. In Fig. 220 is shown a rotating single-disk cam which positively controls the motion of the reciprocating flat-faced yoke follower in both directions. For this type of cam mechanism the return motion of the follower must be the same but opposite in direction to that of the outward motion. This is true since that part of the working surface which causes motion in one direction causes the motion in the opposite direction, the other half of the working surface merely acting to positively constrain the motion to that prescribed. Hence such a cam mechanism could not be designed for a displacement graph such as shown in Fig. 171, but must be designed for a prescribed motion in one direction which is acceptable for the motion in the opposite direction. The mechanism is therefore not suitable for situations where the outward and return motions must be different.

The cam in Fig. 220 is designed in accordance with the displacement graph shown in Fig. 219, the outward and return motions to be uniformly

accelerated and retarded for 4 and 2 periods, respectively. The displacement scale DS is easily obtained by projection. Having the displacement scale, the outline of the working surface of the cam is determined in the same manner as for the cam in Art. 100. Since such cams are at all times tangent to the faces of the yoke, they are of constant

FIG. 219

FIG. 220
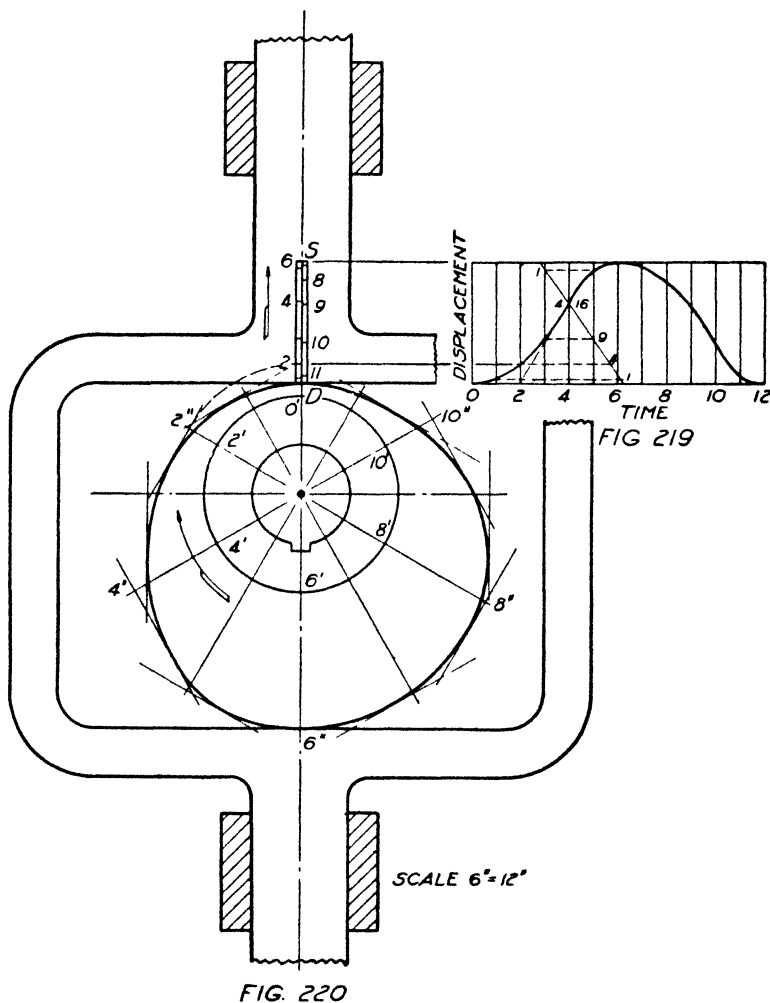
breadth and are referred to as *constant-breadth cams*. Hence, having located *2″*, the distance to the opposite point *8″* on the same radial line is equal to the distance between the faces of the follower.

The single-disk cam with a reciprocating roller yoke follower shown in Fig. 221 is subject to the same limitation, so far as the motion of the

follower is concerned, as the mechanism shown in Fig. 220. The cam
in Fig. 221 was designed for the same motion of the follower as assumed
in Fig. 220, and the outline of the working surface of the cam was
determined by the same method as used for the roller follower cam in
Art. 93. Since the
distance from any
point on the *pitch
surface* to the one
opposite on the same
radial line of the
cam is equal to the
distance between the
centers of the fol-
lower rollers, such
cams are referred to
as *constant diameter
cams.* The diameter
of this type of posi-
tive return cam may
be computed by the
method outlined in
Art. 95 for any given
maximum pressure
angle.

**105. Positive Re-
turn Single-Disk
Circular Arc Cams.**
In Figs. 222, 223,
and 224 are shown
three positive return
single-disk cams the
outlines of whose
working surfaces are
made up of circular
arcs. Such cams are
designed for a given
total displacement

SCALE 6"=12"
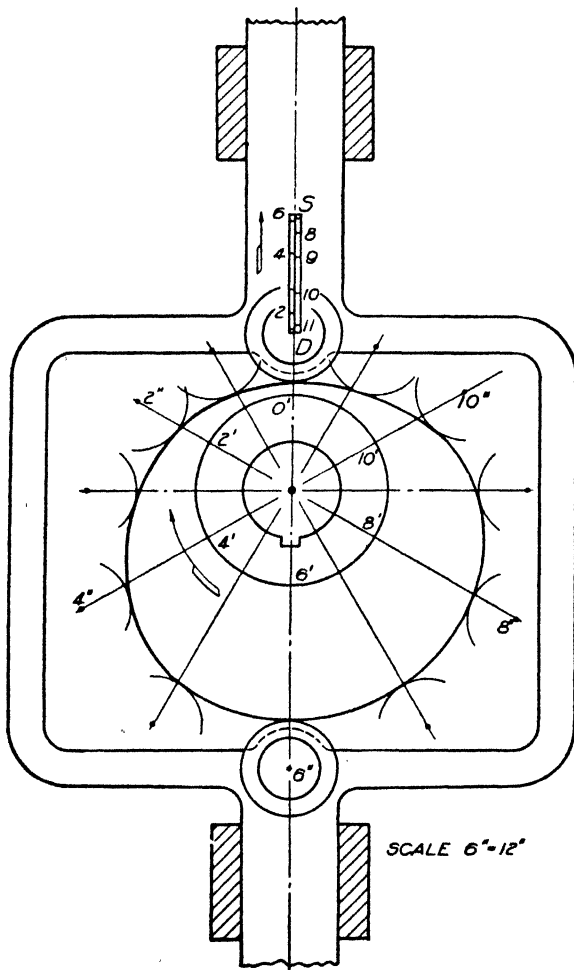
FIG. 221

of the follower, the law of displacement following the assigned outline
of the cam.

The outline of the cam in Fig. 222 is made up of arcs of two different
radii. In order that the cam may be of constant breadth, the centers,
O, B, and C must be at the apices of an equilateral triangle. It is
evident by inspection that the total displacement h is equal to $(R - R_1)$

or $R = h + R_1$. Obviously there is a dwell of the follower at either end of its stroke while the cam rotates through 60 degrees, and the follower travels its total displacement in either direction while the cam rotates through 120 degrees.   In Art. 99 the law of motion of a reciprocating flat-faced follower operated by a circular arc disk cam was treated mathematically.   From that treatment the law of motion of the follower in Fig. 222 may be easily obtained.   From equation (170) for any displacement ϕ of the cam between zero and $\delta = 60$ degrees while the

FIG. 222

follower is in contact with the flank of the cam of radius R, the displacement of the follower is, since $R_3 = R - R_1$,

$$s = R_3(1 - \cos \phi) \quad \ldots \ldots \quad (177)$$

Hence

$$V = \frac{ds}{dt} = \omega R_3 \sin \phi \quad \ldots \ldots \quad (178)$$

and

$$A = \frac{dV}{dt} = \omega^2 R_3 \cos \phi \quad \ldots \ldots \quad (179)$$

From equation (173) for any displacement θ of the cam between $(\beta - \delta) = 60$ degrees and zero while the follower is in contact with the nose of the cam of radius $R_4 = R_1$, the displacement of the follower is

$$s = R_3 \cos \theta \quad \ldots \ldots \ldots \quad (180)$$

Hence,

$$V = \frac{ds}{dt} = - \omega R_3 \sin \theta \quad \ldots \ldots \quad (181)$$

and

$$A = \frac{dV}{dt} = - \omega^2 R_3 \cos \theta \quad \ldots \ldots \quad (182)$$

It is evident by inspection of the constant-breadth cam shown in Fig. 223 that $B = R_3 + R_1$, that $h = R_3 - R_1$, and that $R = R_3 + R_1$ $= 2R_3 \sin \dfrac{\alpha}{2}$, from which

$$\sin \frac{\alpha}{2} = \frac{R_3 + R_1}{2R_3} = \frac{h + 2R_1}{2h + 2R_1} \qquad . \quad . \quad . \quad (183)$$

and

$$R_3 = \frac{R_1}{2 \sin \dfrac{\alpha}{2} - 1} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

The required angle α may therefore be computed for any given values of the minimum radius R_1 of the cam and the total displacement h of
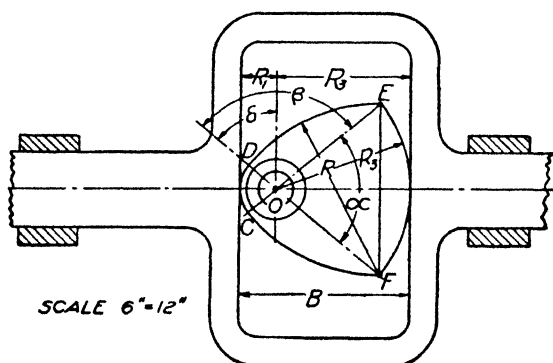
FIG. 223

the follower. Having R_1 and α, the radius R_3 may be computed. For $\alpha = 60°$, R_1 must be zero when $h = R_3$; for $\alpha = 180°$, R_3 must be equal to R_1 and $h = 0$. Obviously there is a dwell of the follower at either end of its stroke while the cam rotates through an angle α, and the follower travels its total displacement in either direction while the cam rotates through an angle $\beta = (180 - \alpha)$. For any displacement ϕ of the cam between zero and $\delta = 90 - \dfrac{\alpha}{2}$ while the follower is in contact with the flank of the cam of radius R, equations (177), (178) and (179) apply. From equation (173) for any displacement θ of the cam between $(\beta - \delta) = \left(\beta - 90 + \dfrac{\alpha}{2}\right)$ and zero while the follower is in contact with the sharp nose of the cam of radius $R_4 = 0$, the displacement of the follower is

$$s = R_3 \cos \theta - R_1 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

Hence

$$V = \frac{ds}{dt} = - \omega R_3 \sin \theta \quad \ldots \ldots \quad (186)$$

and

$$A = \frac{dV}{dt} = - \omega^2 R_3 \cos \theta \quad \ldots \ldots \quad (187)$$

In Fig. 224 is illustrated a circular arc cam operating an oscillating flat-faced follower. The cam outline is a complete circle of radius R whose center E is a distance CE from the center of rotation C of the cam. Such a cam is known as an *eccentric*. As the cam rotates, its center E describes a circle, and the center line of the follower moves from a position of tangency at E to the position at F. It is to be noted that the follower moves downward through the angle θ while the cam rotates through an angle $\alpha = (180 - \theta)$, and moves upward through the angle θ while the cam rotates through the angle $\beta =$

SCALE 6"-12"

FIG. 224

(180 + θ). It is this difference between α and β that makes difficult the design of positive return single-disk cams to operate oscillating followers. For any given distance between the centers C and O and desired oscillation θ, the eccentric throw CE must be
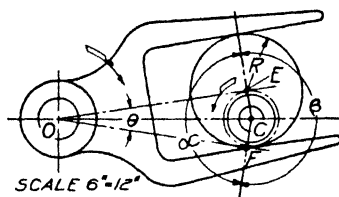
$$CE = (CO) \sin \frac{\theta}{2} \quad \ldots \ldots \quad (188)$$

106. Positive Return Double-Disk Cams. As before mentioned, it is necessary for positive return single-disk cams to have the return motion of the follower the same but opposed to that of the outward motion. By using two disk cams, the return motion and the outward motion may be different and separately prescribed. A positive return double-disk cam operating an oscillating roller yoke follower is shown in Fig. 225. The two cams are mounted side by side on the same shaft, and the centers of the follower rollers are maintained a constant distance apart equal to that between the centers O_1 and O_2 by the connecting-rod B. The cam whose outline is shown solid drives the follower to the right, and the follower is driven to the left by the cam whose outline is shown dotted. The displacement scales DS and D_1S_1 of the centers of the follower rollers are identical, except for numbering, and are in accordance with the displacement graph shown in Fig. 171, motion to

the right being interrupted and occupying 10 time periods while motion to the left is quite different and occupies only 4 time periods. The displacement scale DS being based on 16 equal time intervals, 16 corresponding radial lines equally spaced are drawn through the cam shaft center. As the usual expedient of rotating the follower will be employed, these radial lines are numbered to correspond to the numbering of scale DS, but in a direction counter to that of the rotation of the cam. Having numbered DS and the radial lines, it remains to number the displacement scale D_1S_1. The initial position 0 of the follower roller

SCALE 6"=12"
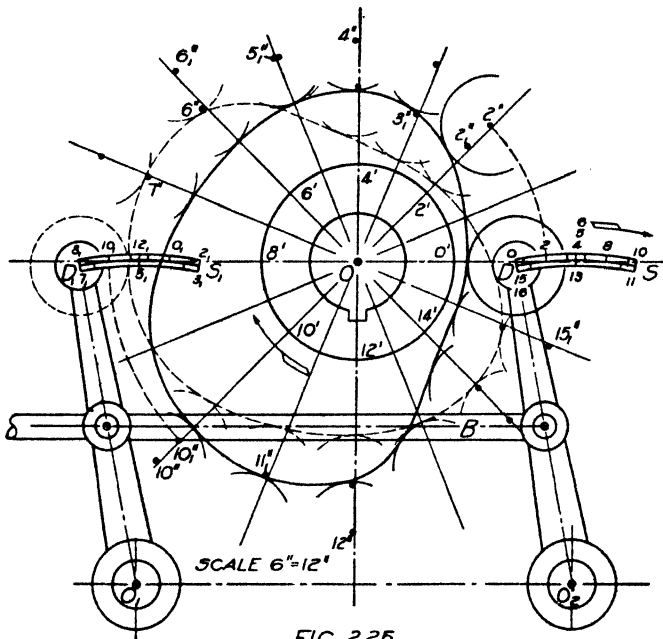
FIG 225

center D corresponds to the radial line $0'$, while the initial position of the follower roller center D_1 corresponds to the radial position $8'$ and is therefore numbered 8_1, the remaining points on the scale being numbered to agree with the radial lines $10'$, $12'$, etc. The location of a pitch point for each cam will illustrate the method employed to determine the cam outlines. Point $2''$ is the proper offset distance from radial line $2'$ on an arc of radius $O2$, and the corresponding point $10_1''$ for the return cam is on an arc of radius $O10_1$. All the other pitch points were located in a similar manner, and the cam outlines drawn tangent to arcs of the roller radius struck from these pitch points as centers.

The outlines of positive return double-disk cams to operate roller or flat-faced yoke followers are determined in a manner similar to that just described. It should be noted that where a reciprocating roller yoke follower is used, the diameter of the cam for any given maximum pressure angle may be computed by the method of Art. 95, and the diameter approximated for oscillating roller yoke followers where the paths of the roller centers do not depart too widely from a radial line of the cams.

107. Cylindrical Cams. In Fig. 226 is shown a cylindrical cam for operating a reciprocating roller follower parallel to the axis of rotation of the cam. To simplify the construction, a plain cylindrical roller was

PROJECTION   SCALE 4"=12"   DEVELOPMENT
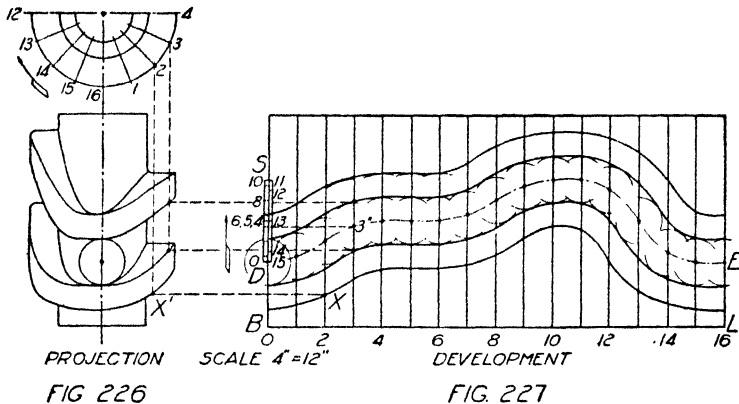
FIG 226   FIG. 227

assumed for the follower. The grooved cam is shown in plan and elevation. To the right in Fig. 227 is shown a development of the outer surface of the cylinder, the horizontal base line BL being equal in length to the outer circumference of the cam. The displacement scale for the pitch point of the follower roller is in accordance with Fig. 171. Since this is based on 16 equal time intervals and the cam is assumed to rotate at a uniform rate, 16 elements of the cylinder are drawn equally spaced and perpendicular to BL. The center of the follower roller on the pitch curve for each interval is then easily located by projection as shown by $3''$ for the third interval for position 3 of the follower. Having located the points on the pitch curve DE, the development of the sides of the roller groove are drawn tangent to arcs of the roller radius struck from centers on the pitch curve. The developments of the ridges to form the roller groove may be drawn in the same manner by striking arcs of a radius equal to the roller radius plus the desired thickness of the ridges. Having the development,

the elevation of the cam, Fig. 226, may be drawn by projecting from the plan and from Fig. 227 as shown for position *3*.

Sheet metal templates of the form shown in the development, Fig. 227, may be made, wrapped around the cam blank, and used to scribe on the surface of the blank the outlines of the sides of the roller groove. It may be noted that a translation cam, if it were possible to reciprocate it at a uniform velocity, would give the same motion to a reciprocating roller follower as the cylindrical cam of Fig. 226.

As mentioned in Art. 88, oscillating roller followers as shown in Fig. 162 may be used with grooved cylindrical cams provided the path of the center of the roller does not depart too widely from a line parallel to the axis of the cam. For cylindrical cams of comparatively small diameters, the follower roller should be in the form of the frustum of a cone whose apex is in the axis of the cam. By this expedient the bottom and top of the cam groove will tend to rotate the follower roller at the same speed. Where the roller is made of plain cylindrical form, there is more or less slipping of the roller in the groove by reason of the fact that the bottom of the groove travels at a lower velocity than the top.

It should be noted that the diameter of a cylindrical cam having a roller follower moving in a line parallel to the axis of the cam may be computed for any given maximum pressure angle by the method outlined in Art. 95, and by the same method the diameter may be approximated for a cam with an oscillating roller follower where the path of the roller center does not depart too widely from a line parallel to the axis of the cam.

108. Inverse Cam Mechanisms. As before stated, an inverse cam mechanism is one where the element corresponding to the follower of a cam mechanism is used as the driver. In such mechanisms, it is the given motion of the follower, or driver, and the prescribed motion of the cam that fix the outline or shape of the cam. Inverse cam mechanisms find a much narrower field of application than cam mechanisms.

In Fig. 228 is shown an inverse cam mechanism in which an oscillating arm fitted with a roller is to drive a translation cam up and down in accordance with the displacement scale DS. The translation cam is to move vertically upward $1\frac{1}{4}$ in. with uniformly accelerated and retarded motion during 4 and 2 time periods, respectively, and therefore vertically downward with uniformly accelerated and retarded motion during 2 and 4 time periods, respectively. By a mechanism not shown, the center of the driving roller is moved from a position on one side of the vertical center line to a position an equal distance on the other side, the horizontal travel $0'6'$ of the roller center to be simple harmonic. Since the total displacement of the cam has been based on 6 time periods,

the semicircle on $0'6'$ as a diameter is divided by radial lines into 6 equal spaces, and the positions $0'$, $2'$, $4'$, etc., of the roller center determined by projection. With the cam stationary, the position of the roller, say for position $4'$, is at $4''$, a distance M vertically below $4'$. All the other positions of the roller center were found in a similar manner. With these positions as centers, circles of a radius equal to that of the roller were drawn, and the outlines of the two sides of the slot in the translation cam drawn tangent to them. Since with the cam stationary,
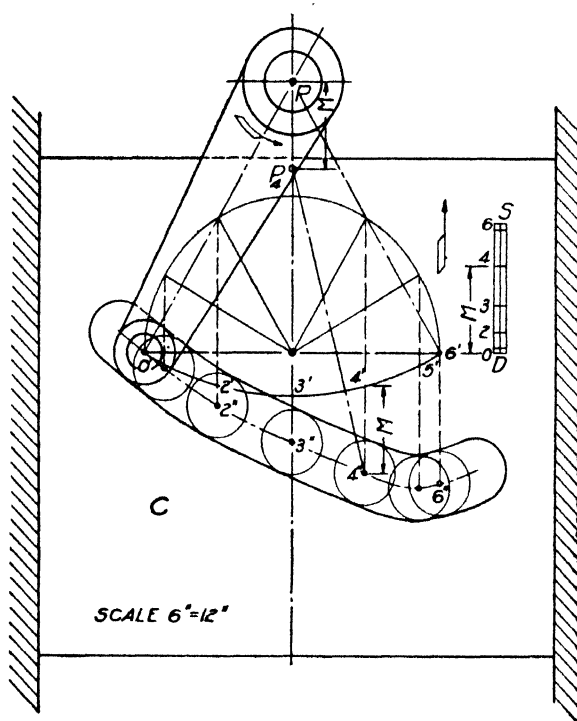
FIG. 228

the frame and center P would move downward as the roller follower swings to the right, another method is suggested for determining each position of the roller center. For position 4, the center P would be at P_4, a distance M below the position shown. Evidently the corresponding position $4''$ of the roller center is where an arc of radius PO' with P_4 as a center intersects the vertical through $4'$. As previously explained, this latter method is known as the *locus* method, and the former as the *offset* method.

CHAPTER VI

ROLLING CURVES AND FRICTION GEARING

109. Requirements for Pure Rolling. For pure rolling action between two curves in direct contact, it was shown in Art. 48 that the point of contact, and therefore the contact radii, must lie in the line joining the centers of rotation. Since the contact radii must lie in the line of centers, it follows that their sum must equal the distance between the centers and that they must make equal angles with the common tangent to the rolling curves. Also, since in pure rolling action there must be entire absence of slipping, the arcs making contact in any interval of time must be of the same length.

For pure rolling action it was also shown in Arts. 48 and 49 that the angular velocity ratio at any instant is inversely as the contact radii or inversely as the perpendiculars let fall from the centers of rotation to the common tangent or to the common normal, or inversely as the segments into which the line of centers is cut by the common normal. Hence, for constant angular velocity ratio combined with pure rolling, the point of contact must be a fixed point in the line of centers, which means that the contact radii of the curves in contact must be of constant length and therefore that the curves must be circles.

For positive driving, it was shown in Art. 50 that the normal component of the motion of the contact point of the driver must be directed toward the follower, and that the common normal must not pass through the center of rotation of either driver or follower. It is to be noted that while pairs of circular arcs satisfy the conditions for pure rolling they do not meet the requirements for positive driving. There are, however, many pairs of curves that meet the requirements for both pure rolling and positive driving. Two of these curves will be treated in detail, and a general method will be developed for deriving a curve to roll with a given curve about centers a fixed distance apart.

110. Rolling Circles. The pairs of circles shown in contact in Figs. 229 and 230 fulfill the requirements for pure rolling action in that the point of contact P remains in the line of centers as action proceeds. Since, however, the common normal passes through the centers of rotation, the requirement for positive driving is not fulfilled. Assuming

that each pair is independently turned or that the friction of contact is sufficient to give pure rolling action, the peripheral velocity V_1 of circle 1 will be the same as V_2 of circle 2. If ω_1 is the angular velocity of 1 and ω_2 is the angular velocity of 2, then

$$V_1 = \omega_1 R_1 \quad \text{and} \quad V_2 = \omega_2 R_2$$

But

$$V_1 = V_2$$

Hence

$$\omega_1 R_1 = \omega_2 R_2 \quad \text{or} \quad \frac{\omega_1}{\omega_2} = \frac{R_2}{R_1} \quad \ldots \quad (189)$$

that is, the angular velocity ratio of 1 to 2 is inversely as the contact radii R_1 and R_2. Since the contact radii do not vary in length, the
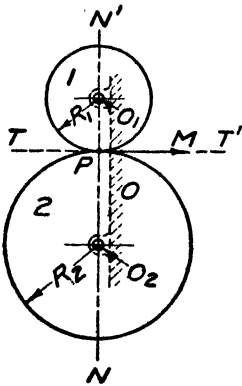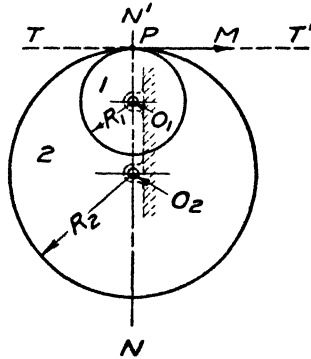
FIG. 229   FIG. 230

angular velocity ratio is constant. This is true only for rolling circles or circular arcs. For other rolling curves where the contact radii vary in length from instant to instant, the angular velocity ratio varies with the change in the length of the contact radii.

In Fig. 229, where the fixed centers O_1 and O_2 are on opposite sides of the point of contact P, clockwise or counter-clockwise rotation of one circle will cause the other circle to rotate in the reverse direction. In Fig. 230, where the fixed centers O_1 and O_2 are on the same side of the point of contact P, rotation of one of the circles in either direction causes the other to rotate in the same direction. These directional relations must be considered in solving for either R_1 or R_2 in terms of the angular velocity ratio $\dfrac{\omega_1}{\omega_2}$ and the distance O_1O_2 between centers.

For Fig. 229, the contact radius R_1 is easily found in terms of $\dfrac{\omega_1}{\omega_2}$ and O_1O_2 by adding unity to each side of equation (189). Thus,

$$\frac{\omega_1}{\omega_2} + 1 = \frac{R_2}{R_1} + 1 = \frac{R_2 + R_1}{R_1} = \frac{O_1O_2}{R_1}$$

or

$$R_1 = \frac{O_1O_2}{\dfrac{\omega_1}{\omega_2} + 1}$$

For Fig. 230, R_1 is obtained by subtracting unity from each side of equation (189). Thus,

$$\frac{\omega_1}{\omega_2} - 1 = \frac{R_2}{R_1} - 1 = \frac{R_2 - R_1}{R_1} = \frac{O_1O_2}{R_1}$$

or

$$R_1 = \frac{O_1O_2}{\dfrac{\omega_1}{\omega_2} - 1}$$

111. Rolling Ellipses. It can be shown that two equal ellipses are of such form as to roll together without slipping if the centers of rotation are taken a distance apart equal to the common major axis of the ellipses, and the center of rotation of each ellipse is located at one of its foci.

Ellipses *1* and *2* in Fig. 231 are equal, since their minor axes C_1D_1 and C_2D_2 and their major axes P_1B_1 and P_2B_2 are equal. As will be recalled, the sum of the distances from the foci of an ellipse to any point on its periphery is equal to the major axis of the ellipse. Hence the foci F_1 and O_1 of ellipse *1* are at a distance equal to one-half the major axis from the ends C_1 and D_1 of the minor axis. The foci of an ellipse are therefore located as soon as the minor and major axes are fixed. In accordance with the same property, having located the foci, the sum of the radii vectors F_1S_1 and O_1S_1 for any point S_1 on the periphery is equal to the major axis P_1B_1. Another property of the ellipse is that the tangent T_1S_1 at any point such as S_1 makes the same angle θ with the radii vectors F_1S_1 and O_1S_1. By the aid of these two properties it can be shown that the two equal ellipses *1* and *2* of Fig. 231, whose centers of rotation O_1 and O_2 are a distance apart equal to the common major axis, are of such form as to roll together without slipping. In Fig. 231 the two equal ellipses have been placed in contact

with the two major axes in the same horizontal line, making the distance between the fixed centers $O_1O_2 = O_1P_1 + O_2P_2 = P_1B_1$. Thus, for the instant, the conditions for pure rolling are satisfied. Since at the instant the common normal through the point of contact passes through the centers O_1 and O_2, the requirement for positive driving is absent and must be assumed. If the arcs P_1S_1 and P_2S_2 are taken equal, S_1 and S_2 become similar points on equal ellipses, and therefore have
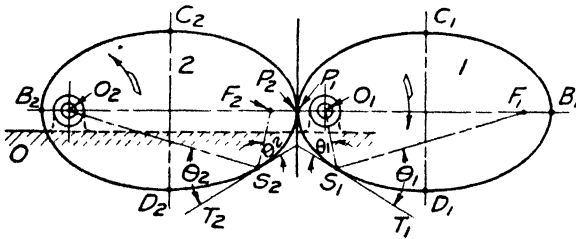
FIG. 231
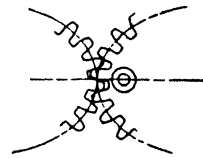
FIG. 234
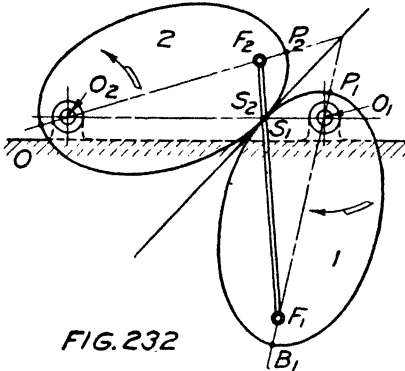
FIG. 232
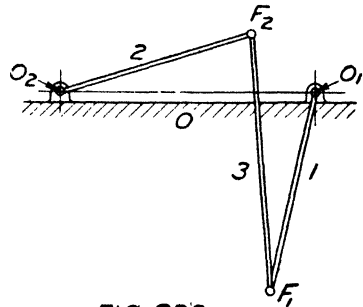
FIG. 233

equal radii vectors making the same angle with the tangents T_1S_1 and T_2S_2, or

$$O_2S_2 = F_1S_1, \quad F_2S_2 = O_1S_1, \quad \text{and} \quad \theta_2 = \theta_1$$

But

$$O_2S_2 + F_2S_2 = B_2P_2 = O_1O_2$$

Hence, by substitution,

$$O_2S_2 + O_1S_1 = O_1O_2$$

Therefore, if the driver 1 is rotated clockwise through an angle $P_1O_1S_1$ and the driven ellipse 2 is rotated counter-clockwise through an angle $P_2O_2S_2$, it follows that the radii vectors O_1S_1 and O_2S_2 will coincide with the line of centers O_1O_2 as shown in Fig. 232, that points S_1 and S_2 will become coincident points in the line of centers, and that the tangents

T_1S_1 and T_2S_2 will coincide, since they pass through coincident points at the same angle with the contact radii or the line of centers. It has therefore been shown that, for equal arcs, contact will be maintained in the line of centers with the contact radii making equal angles with the common tangent, thus proving that pure rolling action between two equal ellipses is possible.

Since for the phase shown in Fig. 231 the common normal passes through the fixed centers, one ellipse will not positively drive the other. This is also true for the phase 180 degrees removed when B_1 and B_2 are the coincident points of contact. If for these two critical phases the friction of contact is sufficient to prevent slipping, one ellipse would positively drive the other through 180 degrees when positive driving would cease. For instance, if *1* is rotated clockwise, it will positively drive *2* counter-clockwise through 180 degrees, after which the transmission of motion would cease, as the periphery of *1* would be moving away from that of *2*. Hence, if the rotation of one ellipse is to rotate the other continuously, something must be done to secure transmission through the second 180 degrees and to overcome the failure of positive driving at the two critical phases. It is to be noted by reference to Figs. 231 and 232 that the distance between the free foci F_1 and F_2 remains constant and equal to the common major axis, or to O_1O_2, the distance between centers, as one ellipse rolls on the other. Hence the equivalent mechanism shown in Fig. 233 gives the same motion to F_1 and F_2 and therefore to link *3* as the rolling ellipses give to the free foci F_1 and F_2. The rolling ellipses therefore correspond to the centrodes of such a system of links as shown in Fig. 233. If the free foci in Fig. 232 are connected by a link, rotation of one ellipse would transmit motion to the other continuously except at the two critical phases when the added link would lie in the line of centers. To insure positive driving through these two dead center positions, mating teeth of proper outline may be formed at the ends of the ellipses as shown in Fig. 234. With these additions, motion equivalent to the rolling of the ellipses could be transmitted continuously from one shaft to the other. It is to be noted, however, that the addition of the link would make it necessary to mount the ellipses on the ends of the shafts. By using full elliptical gears, as shown in Fig. 235, the link may be omitted and the gears placed anywhere along the shafts. In general such gears are used wherever it is desired to transmit motion equivalent to the rolling of equal ellipses from one shaft to another. They have been applied to the quick return mechanisms of shaping machines to secure a more uniform velocity on the cutting stroke than is possible by the use of circular gears. They have also been applied to slide-valve mechanisms to give a different

point of cut-off on one stroke than on the other. The method of properly forming the teeth on such gears will be discussed in a later chapter.

In the transmission of motion by pure rolling, as demonstrated in Art. 48, the angular velocity ratio at any instant is inversely as the contact radii. If, in Fig. 231, ω_1 represents the angular velocity of
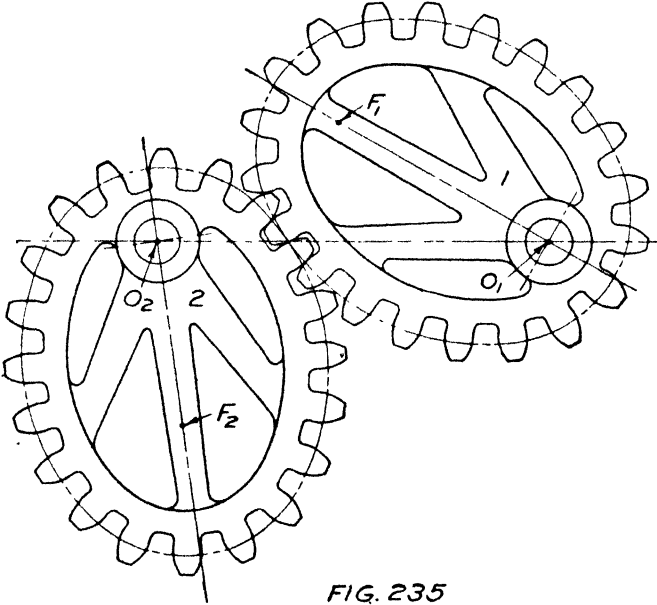
FIG. 235

ellipse *1* at the instant and ω_2 that of ellipse *2*, then the angular velocity ratio is a maximum for the phase shown, and is equal to

$$\frac{\omega_1}{\omega_2} = \frac{O_2P_2}{O_1P_1} = \frac{O_1B_1}{O_1P_1} \qquad \ldots \ldots \quad (190)$$

Obviously, as action continues this ratio grows smaller and smaller, reaching a minimum value when B_1 and B_2 become the coincident contact points, the minimum value being

$$\frac{\omega_1}{\omega_2} = \frac{O_2B_2}{O_1B_1} = \frac{O_1P_1}{O_1B_1} \qquad \ldots \ldots \quad (191)$$

The maximum and minimum values of the angular velocity ratio show that one is the reciprocal of the other. Hence, if one is fixed, the other follows. By fixing either ratio, the minor and major axes for two equal ellipses can be determined for any given distance between centers. As an example, assume that $O_1O_2 = 12$ in. and that the maximum value

of the angular velocity ratio $\dfrac{\omega_1}{\omega_2}$ is to be 2.   From (191) it follows that the minimum value of the angular velocity ratio is $\frac{1}{2}$, and from (190) it follows that

$$\frac{O_1B_1}{O_1P_1} = 2 \quad \text{or} \quad O_1B_1 = 2O_1P_1$$

Since the major axis is equal to the distance between centers,

$$P_1B_1 = O_1O_2 = O_1P_1 + O_1B_1 = O_1P_1 + 2O_1P_1 = 12 \text{ in.}$$

From which,

$$O_1P_1 = \tfrac{12}{3} = 4 \text{ in.}$$

and

$$O_1B_1 = 2O_1P_1 = 8 \text{ in.}$$

The distance between the foci is

$$O_1F_1 = O_1B_1 - O_1P_1 = 8 - 4 = 4 \text{ in.}$$

Having the major axis and the distance between the foci, the ellipses may be constructed.   Remembering that $C_1F_1 = \dfrac{P_1B_1}{2}$, the length of

FIG. 236

one-half the minor axis may be computed if desired. The computed value in this instance is 5.65 in.

Sectors of ellipses can be used as driver and follower to transmit a reciprocating or oscillating motion from one shaft to another. In this case the distance between the shaft centers and the maximum and minimum angular velocity ratios for a given angular displacement of the driver may be assumed, the angular displacement of the follower being fixed by that assumed for the driver. Since only equal ellipses can roll together the sectors will be portions of equal ellipses. As an example, assume, for Fig. 236, that the distance O_1O_2 between the centers is 3 in., that the maximum and minimum values of the angular velocity ratio $\dfrac{\omega_1}{\omega_2}$ are to be $\frac{11}{5}$ and $\frac{7}{9}$, respectively, and that α is to be
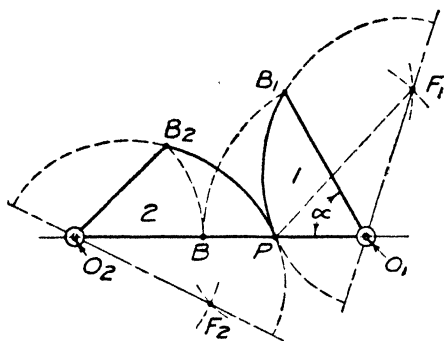
75 degrees. In accordance with the angular velocity ratios,

$$\frac{O_2P}{O_1P} = \frac{11}{5} \quad \text{and} \quad \frac{O_2B_2}{O_1B_1} = \frac{7}{9}$$

The length of O_1P is easily obtained by adding unity to each side of the first equation. Thus,

$$\frac{11}{5} + 1 = \frac{O_2P}{O_1P} + 1 \quad \text{or} \quad \frac{16}{5} = \frac{O_2P + O_1P}{O_1P} = \frac{O_1O_2}{O_1P}$$

whence,

$$O_1P = \tfrac{5}{16}(O_1O_2) = \tfrac{15}{16} \text{ in.} \quad \text{and} \quad O_2P = O_1O_2 - O_1P = 2\tfrac{1}{16} \text{ in.}$$

By a similar treatment of the second equation,

$$O_1B_1 = 1\tfrac{11}{16} \text{ in.} \quad \text{and} \quad O_2B_2 = 1\tfrac{5}{16} \text{ in.}$$

Having α and the lengths of the contact radii, it remains to locate the foci of the ellipses before the curves can be drawn. Through O_1 draw O_1B_1 making an angle $\alpha = 75°$ with O_1O_2. Since the sum of the radii vectors for any point on the periphery of the ellipses must be equal to the common major axis, or distance between centers, the focus F_1 must be a distance from P equal to PO_2 and a distance from B_1 equal to O_2B_2. Hence F_1 is the intersection of two arcs drawn with P and B_1 as centers and PO_2 and O_2B_2 as the respective radii. Since O_2F_2 is equal to O_1F_1, the focus F_2 is located as the intersection of two arcs drawn with O_2 and P as centers and O_1F_1 and PO_1 as the respective radii. Having the foci and the major axis of the equal ellipses, the curves may be constructed. Such sectors of equal ellipses may be used as rolling curves, or as the pitch curves for segmental gears to transmit the same motion.

Because of its bearing on lobed wheels, it should be noted that if the maximum and minimum angular velocity ratios are reciprocal, the rolling sectors become equal sectors of equal ellipses, and the angular displacements of driver and follower become equal.

112. Rolling Logarithmic Spirals. The polar equation of the logarithmic spiral, Fig. 237, is

$$a\theta = \log_e R \qquad \dots \dots \dots \dots \quad (192)$$

in which R is the radius vector, θ the angle of rotation of R, and a the coefficient of angularity of the spiral. It can be shown that for any given value of a the angle between the tangent and radius vector is the

same for all points on the curve.   Differentiating (192) with respect to R gives

$$a\frac{d\theta}{dR} = \frac{\log_e e}{R} = \frac{1}{R} \quad \text{or} \quad \frac{Rd\theta}{dR} = \frac{1}{a} \quad \ldots \ldots \quad (193)$$

But

$$\tan \phi = \frac{Rd\theta}{dR} \qquad \text{or} \quad \tan \phi = \frac{1}{a} \quad \ldots \ldots \quad (194)$$

Since for any regular logarithmic spiral a is constant, it follows that the angle ϕ made by the tangent with the radius vector is the same for all points on the curve. For this reason the curve is often called an equiangular spiral.
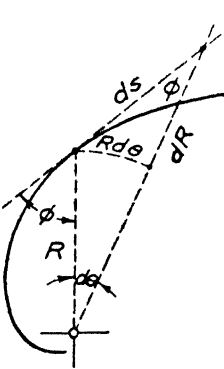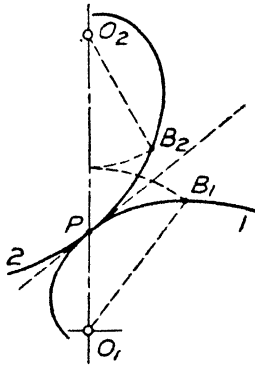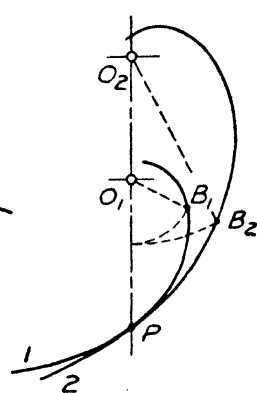
FIG. 237   FIG. 238   FIG. 239

It can also be shown that for arcs of equal length the difference in the lengths of the radii vectors is constant.   Referring again to Fig. 237, it is evident that

$$(ds)^2 = (Rd\theta)^2 + (dR)^2$$

But from (193) $Rd\theta = \dfrac{dR}{a}$, hence

$$(ds)^2 = \left(\frac{dR}{a}\right)^2 + (dR)^2 = (dR)^2\left(\frac{1}{a^2} + 1\right)$$

or

$$ds = dR\sqrt{\frac{1 + a^2}{a^2}} \quad \ldots \ldots \ldots \ldots \quad (195)$$

Integrating (195) gives

$$s = (R_1 - R_2)\sqrt{\frac{1 + a^2}{-2}} \quad \ldots \ldots \ldots \quad (196)$$

Since for similar spirals a is constant, it is therefore proved that for arcs of equal length the difference in the lengths of the radii vectors is constant. Thus, for the pairs of similar spirals shown in Figs. 238 and 239, if arcs PB_1 and PB_2 are equal, then $O_2P - O_2B_2 = O_1B_1 - O_1P$. Hence, for any pair of similar logarithmic spirals rotating about their foci as fixed centers, it follows that the point of contact will lie on the line of centers and, consequently, that the curves will roll together without slipping. In Fig. 238, where the foci or fixed centers O_1 and O_2 are on opposite sides of the point of contact P, the spirals rotate in opposite directions, while in Fig. 239, where O_1 and O_2 are on the same side of P, they rotate in the same direction.

A single pair of logarithmic spirals cannot transmit motion continuously in one direction, but a single pair of sectors of similar spirals can be used as rolling curves or as the pitch curves of segmental gears to transmit oscillating motion from one shaft to another. For a single pair of such sectors, the maximum and minimum angular velocity ratios for a given distance between centers and a given angular displacement of one of the sectors may be assumed as for sectors of equal ellipses. As an example, assume for Fig. 240 that the distance O_1O_2 between centers is to be 3 in., that 1 the driver is to rotate through 60 degrees, and that the maximum and

FIG. 240

minimum angular velocity ratios are to be $\frac{2}{1}$ and $\frac{1}{3}$, respectively. Since these ratios are inversely as the contact radii and since the center distance $O_1O_2 = 3$ in., the necessary lengths of the radii vectors are,
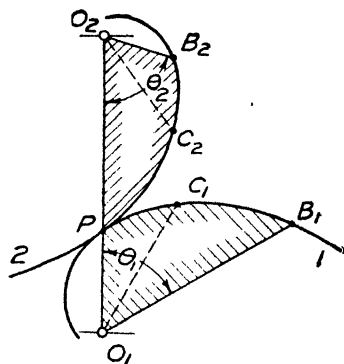
$$O_1P = 1 \text{ in.}, \quad O_2P = 2 \text{ in.}, \quad O_1B_1 = 2\tfrac{1}{4} \text{ in.}, \quad \text{and} \quad O_2B_2 = \tfrac{3}{4} \text{ in.}$$

Having the radii vectors for the extreme positions, it is necessary to determine the angular displacement θ_2 of the driven sector before the spiral curves can be constructed. Applying equation (192) to spiral 1 gives

$$a\theta'' = \log_e O_1B_1 = \log_e 2.25$$

and

$$a\theta' = \log_e O_1P = \log_e 1.0$$

Subtracting the second from the first, remembering that $\theta'' - \theta' = \theta_1$

$$= \frac{60\pi}{180} = \frac{\pi}{3} \text{ radians, gives}$$

$$a\left(\frac{\pi}{3}\right) = \log_e 2.25 - \log_e 1.0 = \log_e \frac{2.25}{1} = 0.8109$$

Hence

$$a = \frac{3 \times 0.8109}{\pi} = 0.774$$

From equation (194) $\tan \phi = \frac{1}{a} = 1.2914$. Hence ϕ, the angle between the tangents and radii vectors, is $52°\,10'\,30''$. Since the value of a is known, θ_2 may be computed by applying equation (192) to sector *2*. Hence,

$$\theta_2 = (\theta'' - \theta') = \frac{\log_e \dfrac{O_2 P}{O_2 B_2}}{a} = \frac{\log_e \dfrac{8}{3}}{0.774} = 1.2672 -$$

or

$$\theta_2 = \frac{1.2672 \times 180}{\pi} = 72.576° = 72°\,34'\,34''$$

Having θ_1 and θ_2 and the corresponding radii vectors, the curves are readily drawn. Solving equation (192) for the length R of the radius vector bisecting the angle between two known radii vectors R' and R'' in terms of each known radius vector, gives $R = \sqrt{R' \times R''}$. Hence, the length of O_1C_1 bisecting θ_1 is

$$O_1C_1 = \sqrt{(O_1P)(O_1B_1)} = \sqrt{1.0 \times 2.25} = 1.5 \text{ in.}$$

and the length O_2C_2 bisecting θ_2 is

$$O_2C_2 = \sqrt{(O_2B_2)(O_2P)} = \sqrt{\tfrac{3}{4} \times 2} = 1.2247 \text{ in.}$$

Points between P and C_1, C_2 and B_2, etc., may be found in a similar manner.

Because of its bearing on lobed wheels, it should be noted that if the maximum and minimum angular velocity ratios are reciprocal, the rolling sectors become equal sectors of similar spirals, and the angular displacements of driver and follower become equal.

113. General Case of Rolling Curves. It has been shown for rolling sectors of similar spirals and equal ellipses that, where the terminal angular velocity ratios are independently chosen for a given

distance between centers, the angular displacement of the follower is fixed by that assumed for the driver. This is true for all rolling sectors whether or not the nature of the rolling curves is known.

A general method will now be developed for constructing a curve to roll upon another curve when the distance between centers and the terminal angular velocity ratios are known for a given angular displacement of the driver. If only the terminal angular velocity ratios are known, then the curve of the driver must be given from which to construct the curve of the follower; if in addition to the terminal angular velocity ratios, intermediate ratios are known for definite divisions of the total angular displacement of the driver, then both curves may be constructed. That is to say, if for a given total angular displacement of
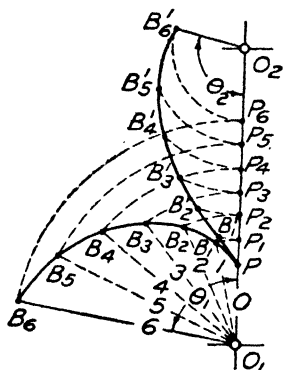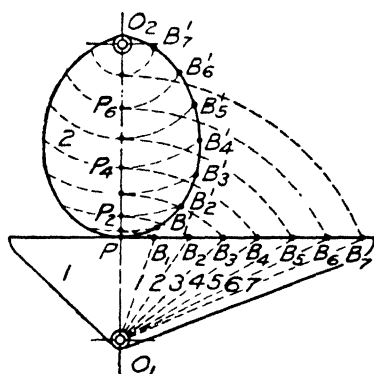
FIG. 241   FIG. 242

the driver the terminal contact radii and those for intermediate angular displacements are known, the curves of the driver and follower may be constructed. This most general case is shown in Fig. 241, where the rolling curves of the driver *1* and the follower *2* and the total displacement θ_2 of the follower are to be determined for known values of O_1O_2 and θ_1 and known locations of the contact points P, P_1, P_2, etc., on the line of centers corresponding to the divisions *0, 1, 2*, etc., of the angular displacement of the driver. Hence the given angular velocity ratios are $\dfrac{O_2P}{O_1P}$, $\dfrac{O_2P_1}{O_1P_1}$, $\dfrac{O_2P_2}{O_1P_2}$, etc.; and the radii vectors O_1P, O_1B_1, O_1B_2, etc., of the driver for positions *0, 1, 2*, etc., are respectively equal to O_1P, O_1P_1, O_1P_2, etc. The curve of the driver is obtained by passing a smooth curve through the points P, B_1, B_2, etc. The radius vector $O_2B'_1$ must be equal to O_2P_1, and the arc distance from P to B_1' must be the same as that from P to B_1. Hence the intersection of an arc

of radius PB_1 having its center at P with an arc of radius O_2P_1 having its center at O_2 determines B'_1 and the radius vector $O_2B'_1$. So also B'_2 is the intersection of two arcs of radii B_1B_2 and O_2P_2 with centers at B'_1 and O_2, respectively. Points B'_3, B'_4, etc., are successively determined in a similar manner, and the curve of the follower obtained by drawing a smooth curve through the points. How perfectly this curve will roll on the driver depends on the number of points taken on the curve of the driver and the judgment displayed in drawing a smooth curve through the corresponding points on the follower. If the intervals between the points on the curve of the driver are taken too great, errors are introduced on account of the difficulty of obtaining arcs to correspond with the chords used; while, if the intervals are taken too small, errors accumulate through the difficulty of accurately carrying out the graphical work.

In Fig. 242 a definite form of curve, a straight line, has been chosen for the driver, and the follower curve $PB'_2B'_4B'_6$ to roll with it has been determined for the angular positions $0, 1, 2, 3$, etc., of the driver in a manner similar to that outlined above. Thus B'_2 is at the intersection of two arcs of radii O_2P_2 and B_1B_2 with centers at O_2 and B'_1, respectively. As in the previous example, the accuracy of form attained for the follower will depend upon selecting points on the curve of the driver that are not too far apart or too close together.

For rolling sectors where the terminal angular velocity ratios are independently chosen, it has been shown in this and the previous articles that the angular displacement of the follower for a given angular displacement of the driver cannot be predicted. It therefore follows that the·general method just outlined for the construction of rolling curves cannot be applied where it is desired to control the angular displacement of the follower for a given angular displacement of the driver. Hence, without modification, the method cannot be applied to the construction of lobed wheels where it is required that the follower will make one revolution or some exact multiple or exact divisor of a revolution for one revolution of the driver.

Where it is desired to fix the total angular displacement of the follower for a given total displacement of the driver, the general method shown in Figs. 243 and 244 may be used. In Fig. 243, sector *1* in rotating through 180 degrees is to rotate sector *2* through 180 degrees. For convenience the driver was assumed to rotate at a uniform rate, and hence the divisions *1, 2, 3*, etc., of its total angular displacement were taken equal. A semicircle of any convenient radius O_1C is therefore divided into equal displacement arcs CE. The corresponding angular displacements desired for the follower are *1', 2', 3'*, etc. For each one

of these displacements, arcs FH, KL, etc., were drawn equal to CE. How corresponding points such as B_2 and B'_2 were found for each sector is shown in Fig. 244. Obviously the angular velocity ratio desired for the second interval is $\dfrac{\omega_1}{\omega_2} = \dfrac{\theta_2}{\theta'_2} = \dfrac{O_2G}{O_1D}$, since $(O_2G)\theta'_2$
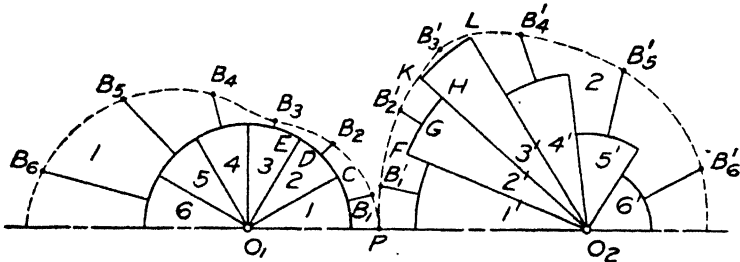
FIG. 243

$= (O_1D)\theta_2$. To obtain B_2 and B'_2, draw any two lines from O_2 and O_1 intersecting at some convenient point M. From S and N draw lines parallel, respectively, to O_2M and O_1M, intersecting at Q. A line through M and Q will divide O_1O_2 in the desired ratio since O_1RM and NRQ are similar triangles, likewise O_2RM and SRQ. Hence the intersection of an arc of radius O_1R with the mid radial line of displacement 2 determines B_2, the corresponding point B'_2 being similarly located. Applying this same process to the other corresponding divisions locates the points B_1 and B'_1, B_3 and B'_3, etc., on the driver and follower, respectively, through which smooth curves may be drawn. Since this general method of constructing rolling curves enables the corresponding angles of rotation of driver and follower to be independently chosen, it is a method suitable to the construction of irregular uni- or multi-lobed wheels. A brief treatment of regular lobed wheels, based on curves of known characteristics, will now be given.
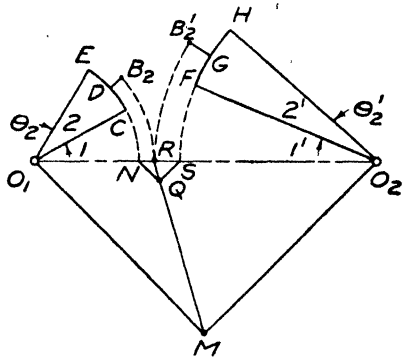
FIG. 244

114. Lobed Wheels. It has been shown that a pair of equal ellipses can rotate continuously with rolling contact, and that the angular velocity ratio passes through one maximum and one minimum value for each revolution. A pair of equal ellipses each rotating about one

of its foci, Fig. 232, is one of the most familiar examples of a pair of single-lobed wheels. Where in a complete rotation it is desired to have the value of the angular velocity ratio pass through several maxima and minima, multi-lobed wheels are used.

It was pointed out in Arts. 111 and 112 that sectors of equal ellipses and similar logarithmic spirals will each rotate through the same angle if the maximum and minimum angular velocity ratios are reciprocal. Hence each one of a pair of uni-lobed wheels may be made up of a pair of 180-degree sectors of equal ellipses or similar logarithmic spirals, and each one of a pair of bi-lobed wheels may be made up of two pairs of 90-degree sectors, and each one of a pair of tri-lobed wheels may be made up of three
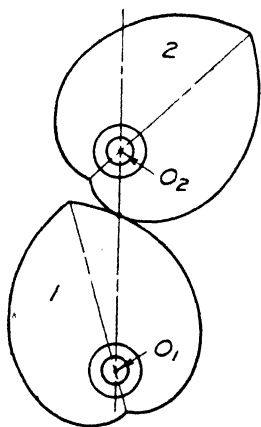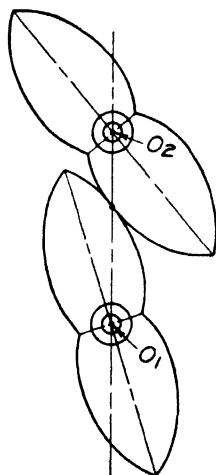
FIG. 245   FIG. 246

pairs of 60-degree sectors, etc. In Fig. 245 is shown a pair of uni-lobed wheels each composed of a pair of 180-degree sectors of similar logarithmic spirals; in Fig. 246 a pair of bi-lobed wheels each composed of two pairs of 90-degree sectors of equal ellipses, and in Fig. 247 a pair of tri-lobed wheels each composed of three pairs of 60-degree sectors of similar spirals. Such wheels will, of course, work together only in similar and equal pairs, the follower making one revolution to one of the driver. The number of changes in the angular velocity ratio from a maximum to a minimum or from a minimum to a maximum will correspond during each revolution to the number of lobes on each wheel.

The lobes of lobed wheels are not necessarily symmetrical. In Fig. 248 is shown a pair of unsymmetrical, uni-lobed wheels each composed of a 270-degree and a 90-degree sector of similar logarithmic

spirals. This is possible, since the maximum and minimum angular velocity ratios are reciprocal. For the same reason unsymmetrical
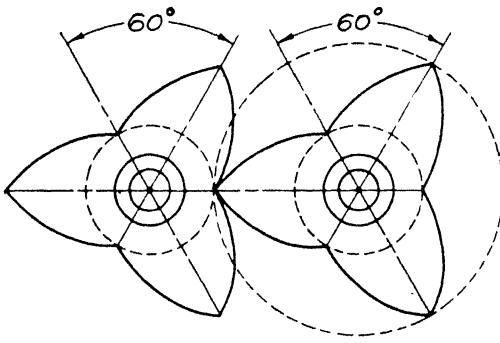
FIG. 247
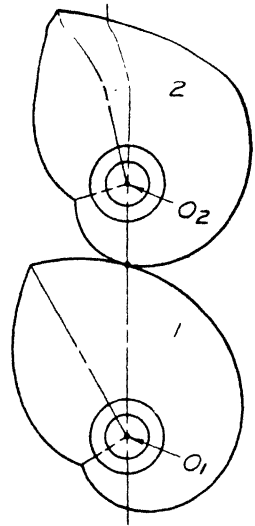
FIG. 248

multi-lobed wheels composed of sectors of similar spirals may be easily constructed.

It was shown for similar logarithmic spirals in Art. 112 that arcs between any two radii having the same difference in length are equal.
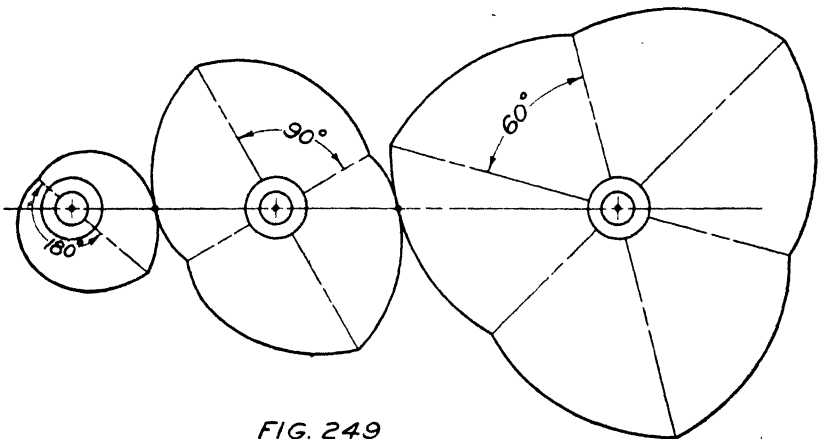
FIG. 249

Hence lobed wheels having an unequal number of lobes may be easily derived from similar spirals. A series of three such wheels that will

act together continuously are shown in Fig. 249, the difference between the maximum and minimum radius of each wheel being the same.

Lobed wheels having an unequal number of lobes may be derived from equal rolling ellipses. This is effected by a process known as the *contraction* or *expansion* of angles, illustrated in Figs. 250 and 251. The pair of rolling sectors shown in Fig. 250 make contact on the line of centers at P. As sector *1* rotates through the angle PO_1C_1, sector *2* rotates through the angle PO_2C_2, points C_1 and C_2 becoming coincident points in the line of centers at C. If in Fig. 251 the corresponding
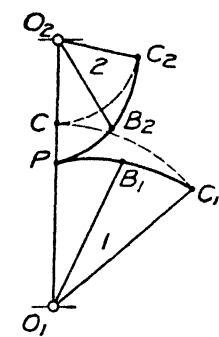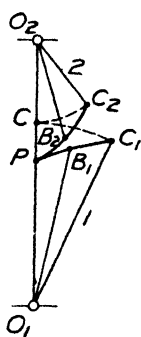
FIG. 250   FIG. 251
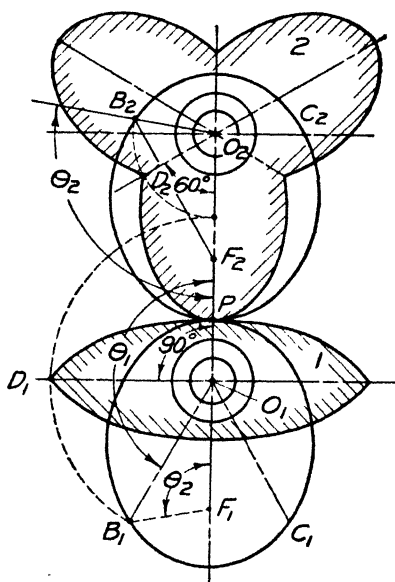
FIG. 252

radiants are made the same in length as those of Fig. 250, but each corresponding angle is reduced in exactly the same proportion, the sectors formed by passing smooth curves PB_1C_1 and PB_2C_2 through the points thus located will roll together with the same angular velocity ratio as the sectors in Fig. 250. This process has been applied in Fig. 252 in constructing the two- and three-lobed wheels to roll together during continuous rotation. The wheels are based on the two equal ellipses PB_1C_1 and PB_2C_2. The bi-lobe is to make $\frac{1}{2}$ of a revolution while the tri-lobe makes $\frac{1}{3}$ of a revolution. Hence for the two equal arcs PB_1 and PB_2, the angles PO_1B_1 and PO_2B_2 must be in the ratio of 3 to 2. If angle PO_1B_1 is taken 150 degrees and angle PO_2B_2, or

$O_1F_1B_1$, is taken 100 degrees and the distance between centers O_1O_2 is taken $2\frac{1}{2}$ in., then

$$O_1F_1 = \frac{(O_1O_2)\sin(\theta_1 - \theta_2)}{\sin\theta_1 + \sin\theta_2} \quad \ldots \ldots \quad (197)$$

$$= \frac{2.5 \times 0.76604}{0.50 + 0.98481} = 1.29 \text{ in.}$$

Since the ellipses are equal, focus F_2 is likewise located, and the ellipses can be constructed and points B_1 and B_2 located. In deriving sector PO_1D_1 of the bi-lobe from sector PO_1B_1 and in deriving sector PO_2D_2 of the tri-lobe from sector PO_2B_2 after the manner described above, angles PO_1B_1 and PO_2B_2 are reduced, respectively, to 90 and 60 degrees. For the center distance and angles chosen, the maximum and minimum angular velocity ratios are

$$\frac{O_2P}{O_1P} = 3.13 \quad \text{and} \quad \frac{O_2D_2}{O_1D_1} = 0.507$$

To secure positive driving in all possible phases of action, lobed wheels are fitted with teeth, the rolling curves of the lobed wheels serving as pitch curves in forming the teeth. If the teeth are of proper form, the resulting toothed gears will act together to transmit motion exactly equivalent to the rolling of the curves of the lobed wheels.

115. Rolling Surfaces. In the first chapter it was shown that the plane motion of any body may be completely represented by the motion of a plane figure. Hence a plane figure rotating about a fixed center may represent a body rotating about an axis through the fixed center perpendicular to the plane of motion. It therefore follows that rolling curves in the same plane may represent rolling surfaces rotating about parallel axes. Any of the rolling curves treated in the preceding articles of this chapter may be and were regarded as rolling surfaces rotating about parallel axes, the surfaces being generated by a straight line moving along the rolling curves while remaining parallel to the axes of rotation. Hence rolling circles become rolling right circular cylinders, and rolling curves become rolling cylinders in a general sense. As before indicated, such rolling surfaces rotating about parallel axes are used as the pitch surfaces of a class of toothed gears known as spur gears.

When the axes of rotation, instead of being parallel, are at an angle and intersect, the rolling surfaces become tangent cones or frusta having a common contact element and a common apex at the intersection of the axes. While rolling cones are not necessarily right circular cones, the use of other than right circular cones is so rare that only such cones

will be treated.   Toothed gears whose pitch surfaces are rolling cones are known as bevel gears.

Axes of rotation that are at an angle but do not intersect may be connected by two members which will roll together making contact along a common rectilinear element.   The rolling surfaces are hyperboloids of revolution generated by a straight line rotating about each axis, the generating line neither intersecting nor paralleling either axis. While in a sense the action between such surfaces departs somewhat from pure rolling, the manner of departure is not such as to affect the angular velocity ratio or to prevent the surfaces from being used as the pitch surfaces of toothed gears. Such gears are known as *hyperboloidal gears.* If the pitch surfaces are right circular cylinders with their axes at an angle but not intersecting, the gears are known as *helical gears.*

116. Rolling Cylinders. It has just been shown that the motion of rolling surfaces having parallel axes may be completely represented by rolling curves rotating in the same plane about fixed centers. As for the corresponding rolling curves, the angular velocity ratio at any instant for such rolling surfaces is inversely as the contact radii. Since for rolling right circular cylinders the lengths of the contact radii do not change as action proceeds, the angular velocity ratio is constant, as explained in Art. 110, for rolling circles. Hence for rolling right circular cylinders with axes parallel,

$$\frac{\omega_1}{\omega_2} = \frac{R_2}{R_1}$$

The angular velocities may be expressed in degrees, radians, or revolutions per minute or per second. For any given distance between centers, R_1 may easily be obtained for any given angular velocity ratio by adding or subtracting unity from each member of the above equation and solving for R_1 in terms of the distance between centers and the angular velocity ratio. If the axes of rotation are on opposite sides of the line of contact, unity is added, and is subtracted if the axes are on the same side of the line of contact. As explained in Arts. 110 and 112, the location of the axes with respect to the line of contact will depend upon whether the rolling cylinders are to rotate in the opposite or in the same direction.

117. Rolling Cones. Since, as before explained, rolling cones other than right circular cones are so rarely used, only right circular cones will be treated. In Fig. 253 are shown two cylinders combined with two cones. Since each cylinder has a base in common with the base of its cone, each cylinder and cone have a common axis. If the cones are to roll together they must be of the same slant height. As shown

in Fig. 253, the cylinders are in contact with their axes parallel. Hence the bases of the cones have a common tangent through P perpendicular to the plane of the axes. If the axes BB and CC are rotated in their common plane about the tangent through P, the apices of the cones, since the slant heights are the same, will meet at F, as shown in Fig. 254; the cones will be tangent along the line FP, and the base circles of the cones will still have a common tangent through P perpendicular to the plane of the axes. Hence the coincident contact points of the base circles at P will have equal velocities along the common tangent, and the base circles in the new positions can roll together as in the original positions. Since this is true of other transverse sections an equal
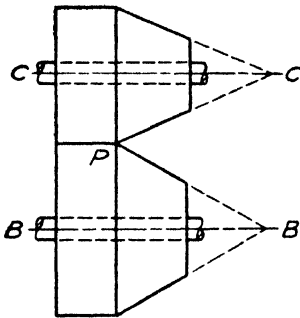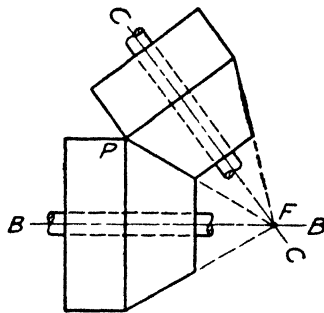
FIG. 253   FIG. 254

distance from F along the contact element FP, the cones can have the same rolling action as the original cylinders.

Rolling cones may be used to connect two shafts which intersect at a given angle and which are to rotate at a given angular velocity ratio. Assume the shafts BB and CC in Fig. 255 to intersect at any given angle α, the angular velocity ratio $\dfrac{\omega_B}{\omega_C}$ to be of any given value. Lay off on BB and CC from D the respective lengths DE and DF in the ratio of $\dfrac{\omega_B}{\omega_C}$. Through F and E draw lines respectively parallel to BB and CC intersecting at G. It can be proved that a line through D and G is the contact element of the desired cones. From G drop perpendiculars GH and GK on the axes BB and CC. Since FK is parallel to EG and EH is parallel to FG, the two right triangles FKG and EHG are similar. Hence,

$$\frac{KG}{HG} = \frac{FG}{EG} = \frac{DE}{DF} = \frac{\omega_B}{\omega_C}$$

that is, the contact radii of the cones are inversely as the desired angular velocities, and therefore the cones in rolling together will transmit motion from one shaft to the other at the desired ratio. Any point such as P in DG may be chosen as the contact point of the base circles of the frusta to be used.
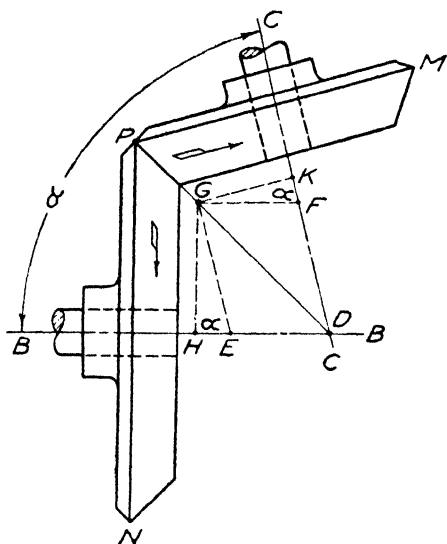
In Fig. 255 a pair of cones was determined for a given angular velocity ratio between shafts BB and CC by making the construction in the acute angle α. This construction for the same conditions as in Fig. 255 is repeated in (1) in Fig. 256. In (2), to the right of (1), the same construction has been applied to the obtuse angle $(180° - \alpha)$, giving cones whose base circles are in the same ratio as those in (1) but whose angles of slope are different. Only for $\alpha = 90°$ would the slope angles and the pairs of cones be alike. It is to be noted that the pairs of cones in (1) and (2) drive shafts CC in opposite directions. Whether
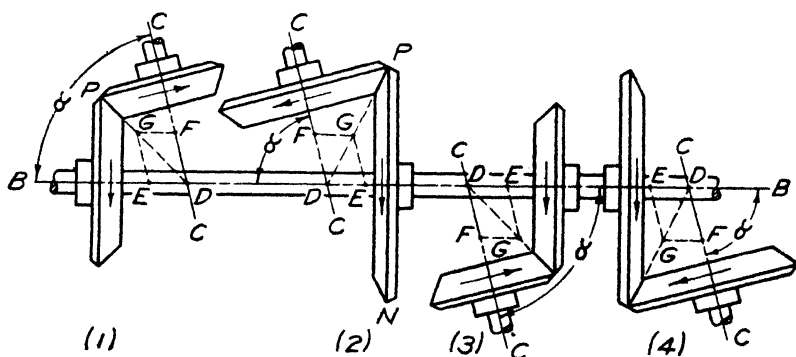
FIG. 255

FIG. 256

the construction is to be applied to the acute or to the obtuse angle depends upon the relation of the shafts and the direction of rotation desired. The two pairs of cones in (3) and (4) for shafts CC below shaft

BB are respectively identical with those in (1) and (2), and each corresponding pair rotates its shaft *CC* in the same direction.

Where the two cones are of about the same size and the direction of rotation and relation of the shafts are such as to make it necessary to use the obtuse angle, one frustum may cut through the shaft of the other as shown in Fig. 257, making it necessary to mount the frusta on the ends of the shafts. Again the angular velocity ratio and relation of the shafts may be such as to cause the contact element of the pair of frusta to be perpendicular to one of the shafts, as shown in Fig. 258, in which case one of the cones becomes a flat disk. Or, again, the
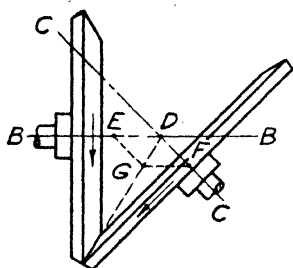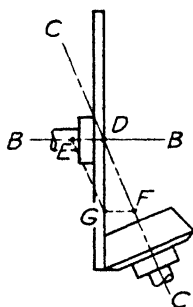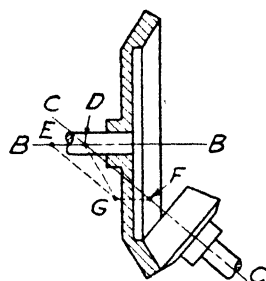
FIG. 257   FIG. 258   FIG. 259

angular velocity ratio and relation of the shafts may be such as to cause the contact element to fall inside one of the frusta, as in Fig. 259.

In the great majority of applications of rolling cones, the axes intersect at right angles. As before mentioned, the principal application of rolling conical surfaces is as the pitch surfaces of bevel gears.

118. Rolling Hyperboloids. A straight line rotating about an axis which it does not intersect, and to which it is not parallel, generates a surface called an hyperboloid of revolution. The rotating straight line is the generatrix of the warped surface generated. It being a surface of revolution, the sections normal to the axis of the figure are circles, the smallest being known as the *gorge* circle. The intersection of an axial plane with the surface is an hyperbola, which hyperbola, if rotated about the axis, would generate the same surface as the revolving straight line. Because of this fact the figure is known as an hyperboloid of revolution. In Fig. 260 is shown a pair of these surfaces which are tangent to each other along the common element *EE*. If the axes *BB* and *CC* are fixed in the positions corresponding to this tangency, it is evident that the two surfaces will remain tangent as the two hyperboloids rotate about their axes.

It can be shown that two hyperboloids of revolution can be tangent along an element only when the radii of the gorge circles are proportional to the tangents of the angles between the contact element and the respective axes, or only when $\dfrac{R_1}{R_2} = \dfrac{\tan \phi_1}{\tan \phi_2}$. In Fig. 261 let BB and CC represent the fixed axes of rotation and EE the common element between two tangent hyperboloids. A plane through BB parallel to
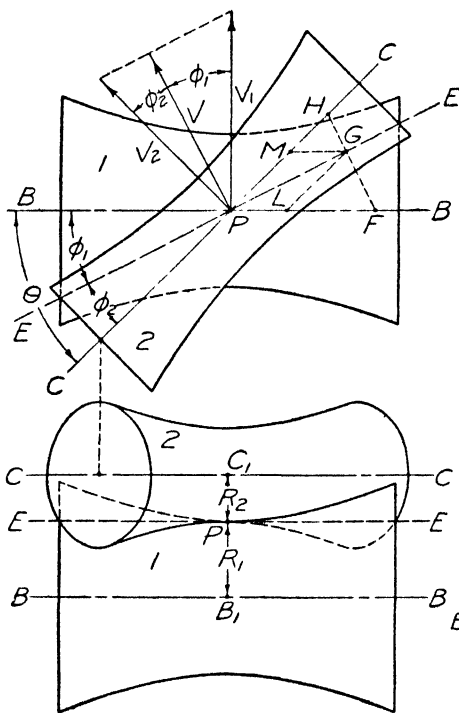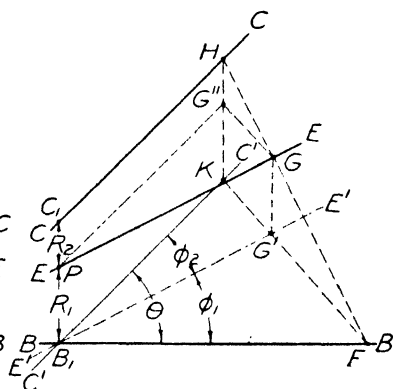
FIG. 260   FIG. 261

CC and a plane through CC perpendicular to the first plane will intersect along $C'C'$, giving the angle θ between the axes BB and CC. Line B_1C_1, being the intersection of planes through BB and CC perpendicular to FB_1K, is perpendicular to both axes and equal to the sum of the gorge circle radii. Since radii from B_1 and C_1 are perpendicular to the hyperboloids, the point of contact P is in C_1B_1, and the gorge circle radii B_1P and C_1P are perpendicular to the common tangent EE. Line $E'E'$ is the projection of EE on the base plane FB_1K, and gives the angles ϕ_1 and ϕ_2 made by the common element with the axes. A

plane through any point G normal to EE gives the lines FG and HG, which are normal to the hyperboloids at point G on the line of contact. Evidently,

$$\frac{\tan \phi_1}{\tan \phi_2} = \frac{G'F}{G'K} = \frac{GF}{GH} = \frac{G'G}{G''H} = \frac{PB_1}{PC_1} = \frac{R_1}{R_2} \quad . \quad . \quad (198)$$

Any point in hyperboloid 1 must move in a plane perpendicular to its axis BB, and any point in hyperboloid 2 must move in a plane perpendicular to its axis CC. Hence, in Fig. 260, the velocity of point P as a point in 1 is V_1, and as a point in 2 is V_2. For proper action at all points along the contact element, the components of these velocities perpendicular to the contact element must be equal, that is, $V_1 \cos \phi_1$ must equal $V_2 \cos \phi_2$. It therefore follows that the rate of sliding along the contact element will be the algebraic difference between $V_1 \sin \phi_1$ and $V_2 \sin \phi_2$. Since these components can affect neither the motion nor the normal components, the angular velocity ratio of the rolling hyperboloids is not affected. It is this characteristic of the rolling action between the hyperboloids that was referred to in Art. 115. Hence the angular velocity ratio between hyperboloids 1 and 2 is

$$\frac{\omega_1}{\omega_2} = \frac{\dfrac{V_1}{R_1}}{\dfrac{V_2}{R_2}} = \frac{V_1 R_2}{V_2 R_1} = \frac{\left(\dfrac{V}{\cos \phi_1}\right) R_2}{\left(\dfrac{V}{\cos \phi_2}\right) R_1} = \frac{R_2 \cos \phi_2}{R_1 \cos \phi_1} \quad . \quad . \quad (199)$$

Substituting the value of $\dfrac{R_2}{R_1}$ from (198) gives

$$\frac{\omega_1}{\omega_2} = \frac{\tan \phi_2 \cos \phi_2}{\tan \phi_1 \cos \phi_1} = \frac{\sin \phi_2}{\sin \phi_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

The method of constructing a pair of rolling hyperboloids to transmit motion between two shafts in parallel planes a fixed distance $B_1 C_1$ apart, and at an angle θ for a given angular velocity ratio $\dfrac{\omega_1}{\omega_2}$, is shown in Fig. 260. Line CC of the plan view in Fig. 260, and $C'C'$ in Fig. 261, represent the projection of shaft CC on a plane through BB parallel to CC, the shafts being at an angle θ and the angular velocity ratio between hyperboloids 1 and 2 being $\dfrac{\omega_1}{\omega_2}$. Lay off from P on BB and CC in Fig. 260 the respective lengths PL and PM in the ratio of ω_1 to ω_2. Lines through L and M parallel, respectively, to CC and BB

will intersect at G. A line through G and P determines the projection EE of the contact element. Through G draw a line perpendicular to EE intersecting BB and CC at F and H. Then the gorge circle radii of the hyperboloids are in the ratio of GF to GH, since $PFGH$ of Fig. 260 is similar to $B_1FG'K$ of Fig. 261, for which it was shown that $\dfrac{R_1}{R_2}$ must equal $\dfrac{G'F}{G'K}$ for proper action. Hence, as $\dfrac{GF}{GH}$ of Fig. 260 is equal to $\dfrac{G'F}{G'K}$ of Fig. 261, the respective gorge circle radii R_1 and R_2 of the required hyperboloids are obtained by dividing the distance B_1C_1 between shafts in the ratio of GF to GH.

119. Friction Gearing. Pairs of surfaces that can rotate continuously in rolling contact have been treated in the preceding articles. Such surfaces divide themselves into two classes: those that roll together at a constant angular velocity ratio, and those that roll together at a varying angular velocity ratio. For this latter class of surfaces it has been pointed out that while one will positively drive the other between certain phases during a revolution, there are certain critical phases between which there is no positive driving and between which they could not drive by friction, since the driver tends to leave contact with the follower. Hence, such surfaces cannot be used in friction gearing where it is required that the surfaces remain in contact under pressure as action proceeds. The former class of surfaces which can roll together at a constant angular velocity ratio include all the surfaces of revolution. Since for such surfaces the common normals pass through the axes of rotation, there can be no positive driving. This very fact makes them suitable for friction gearing since pressure between them can be maintained even though slip occurs while they drive. Hence, rolling cylinders, cones, hyperboloids, and other surfaces of revolution may be used in friction gearing. Of these, cylinders and cones are most used, cylinders being used more frequently than cones. By the use of cylindrical friction wheels power may be transmitted between parallel shafts at practically a constant angular velocity ratio. For shafts that intersect at an angle, rolling cones are used. For shafts at an angle that do not intersect, an auxiliary shaft and a combination of cylinders and cones would in general be used in preference to rolling hyperboloids.

In Fig. 262 are shown two cylinders on parallel shafts which are supported by bearings B symmetrically placed. Let cylinder 1 be the driver and cylinder 2 the follower. If the cylinders are pressed

together, the total pressure normal to their surfaces along the line of contact and the pressure on each pair of bearings will be the same. Since all materials are more or less elastic, the cylinders will be somewhat deformed by the pressure between them, and instead of line there will be surface contact. If the cylinders are rolled together while under pressure, the deformation and readjustment that takes place constitutes a resistance to motion known as rolling friction. A turning moment would therefore have to be applied to the driver to turn the follower even though there was no resisting torque to be overcome at the shaft and bearings of the follower. In transmitting power from one shaft to another, it therefore follows that the friction at contact acting in the plane of motion must be sufficient to overcome the rolling friction at contact, the friction at the follower bearings, and the resisting torque at the follower shaft. The friction between two surfaces in contact under pressure is the force required to cause relative sliding between them. Within limits, this force, or friction, varies directly with the normal pressure between the two surfaces in contact. Hence if the total normal pressure between cylinders *1* and *2* is N, the friction F is

$$F = \mu N$$

where μ is the coefficient of friction. The value of μ for dry, unlubricated surfaces at low velocities depends upon the kind of materials in contact and the finish of the surfaces. For a slip between driver and follower of not more than 2 to 3 per cent, 0.20 would be a fair value of μ for cast iron in contact with cast iron, and 0.30 for cast iron in contact with leather. Since the coefficients of friction for certain fibrous materials in contact with metals are higher than for metals in contact with metals, the driver is generally surfaced with some fibrous material such as leather, wood, or asbestos, while the follower surface is made of metal. This practice of making the follower surface of the harder material is followed to avoid having a spot worn in the follower should driving fail and the rotating driver be in contact with a stationary follower, as might occur if an excessive overload were thrown on the follower. This would also be the case if the driver were put in motion before being pressed against the follower. The pressure per inch of length of contact is generally limited to about one-fifth the crushing strength of the fibrous material used, which would be about 150 pounds per inch for leather.

To illustrate the general method of designing friction wheels, suppose, in Fig. 262, that the shafts are 15 in. apart, and that a leather-covered driver rotating at 210 R.P.M. is to transmit 3.6 horse-power to a cast-iron follower which is to rotate at 140 R.P.M. Neglecting slip, the

diameters of driver and follower must obviously be 12 in. and 18 in., respectively, and the peripheral velocity of each would be

$$V = \pi D(\text{R.P.M.}) = \pi(1\tfrac{1}{2})210 = 660 \text{ ft./min.}$$

But

$$\text{H.P.} = \frac{FV}{33000}$$

or

$$F = \frac{\text{H.P.}33000}{V} = \frac{3.6 \times 33000}{660} = 180 \text{ pounds}$$

FIG. 262   FIG. 263   FIG. 264
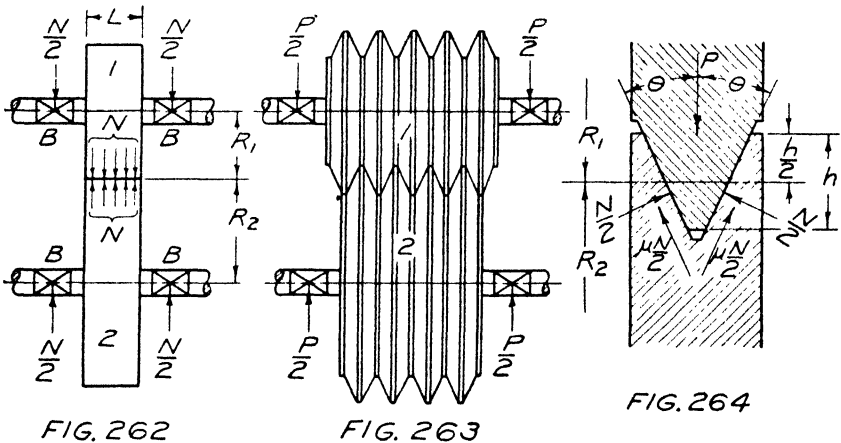
Hence, the wheels must be pressed together with a force of

$$N = \frac{F}{\mu} = \frac{180}{0.30} = 600 \text{ pounds}$$

Allowing, as mentioned above, a pressure of 150 pounds per inch of length of contact, the width of the wheels must be

$$L = \frac{N}{150} = \frac{600}{150} = 4 \text{ in.}$$

As before pointed out, rolling cylinders, cones, and hyperboloids may be used as elements in friction gearing, and may be used also as the pitch surfaces of toothed gears. Friction wheels and toothed gears are therefore both suitable for about the same distances between shafts, but differ in their capacity to transmit power and in their manner of transmitting it. Where the power to be transmitted from one shaft to another is relatively small and a small amount of slip is not objec-

tionable, friction gears may possibly be used to advantage over toothed gears. It should be remembered, however, that while friction gears are simpler in form and more easily produced, they are larger for the same power than toothed gears, may wear out more rapidly, and may in the end be more expensive. Under sudden changes of load, friction gears have the advantage of absorbing a certain amount of the shock in slipping. Where the power to be transmitted is large, toothed gears are to be preferred to friction gears, and are preferred, whether the power to be transmitted is small or large, where the angular velocity ratio between the shafts is to remain constant from instant to instant. It is this requirement that makes the use of toothed gears positively necessary in so many situations. Thus friction wheels would not do as timing gears for automobile engines, nor as part of the mechanism to drive the tool carriage of a lathe in cutting a thread. For long distances between shafts, where a slip of from 2 to 3 per cent is not objectionable, belt and rope drives are used. Powers of considerable magnitude may be so transmitted. For positive driving over shorter distances, transmission chains are used.

120. Grooved Friction Wheels. For two plain cylinders on parallel shafts, it was shown in the problem of the preceding article that the total pressure on each pair of bearings was equal to the total normal pressure between the two wheels. By using a pair of grooved wheels as shown in Fig. 263, the load on the bearings for a given total pressure normal to the contact surfaces of the wheels may be greatly reduced, thereby reducing the wear and frictional loss at the bearings.

In Fig. 264 is shown a section at contact of a single wedge and groove of two wheels on parallel shafts. The forces represented are in the plane of the axes of the parallel shafts. The wedge of one wheel is forced into the groove of the other by drawing the two shafts together with a force P acting radially in the plane of the axes of the shafts. The force P is resisted by two forces $\dfrac{N}{2}$ normal to the sides of the groove, and two forces $\mu\left(\dfrac{N}{2}\right)$ acting along the line of contact. For equilibrium the sum of the radial components of these forces must be equal to P. Hence,

$$P = 2\left(\frac{N}{2}\right)\sin\theta + 2\left(\mu\frac{N}{2}\right)\cos\theta = N(\sin\theta + \mu\cos\theta) \quad . \quad . \quad (201)$$

For the resisting forces $\mu\left(\dfrac{N}{2}\right)$, it is assumed that, due to the elasticity

of the materials, there is impending slip after the wedge and groove have been brought together and are held by the force P. It is doubtful if this component is fully effective even for wheels that are stationary. Experiments on wheels in motion seem to show that this component should be about one-fourth that given above. Equation (201) may therefore be written

$$P = N\left(\sin\theta + \frac{\mu}{4}\cos\theta\right) \quad \ldots \ldots \quad (202)$$

or

$$N = \frac{P}{\sin\theta + \dfrac{\mu}{4}\cos\theta} \quad \ldots \ldots \quad (203)$$

which gives the normal pressure of contact N for a radial force P acting in the plane of the shafts. The power that can be transmitted is dependent upon the frictional resistance to slip in the tangent plane at contact. Hence,

$$F = \mu N = \frac{\mu P}{\sin\theta + \dfrac{\mu}{4}\cos\theta} \quad \ldots \ldots \quad (204)$$

Assuming for the problem of the preceding article that the friction wheels have 60-degree grooves $(2\theta = 60°)$ the required force P at each pair of bearings for $F = 180$ pounds becomes

$$P = \frac{F}{\mu}\left(\sin\theta + \frac{\mu}{4}\cos\theta\right) = \frac{180}{0.30}\left(0.50 + \frac{0.30}{4} \times 0.866\right) = 345 \text{ pounds}$$

This is seen to be over 40 per cent less than the force at the bearings for plain friction wheels. The advantage of the grooved wheels is therefore apparent.

In Fig. 264, h is the working depth of the wedge and groove. To be certain that the contact will be only on the sides of the wedge and groove, the total depth of the groove and the height of the wedge must be made greater than the working depth. This difference is made sufficient to allow for a reasonable amount of wear. If, at the mid line of the working depth, the mating wheels have the same velocity, it follows that on either side of this line they will have different velocities. Hence, the greater the working depth of the groove relative to the mean radii of the wheels the greater will be the relative sliding or wearing action. If, in Fig. 264, R_1 is the mean radius of the driver and R_2 that of the

follower, and it is assumed that the two wheels have the same velocity V at these radii, then

$$V = 2\pi R_1 N_1 = 2\pi R_2 N_2 \quad \text{or} \quad R_1 N_1 = R_2 N_2$$

where N_1 and N_2 represent the respective revolutions per minute of driver and follower. Remembering that $R_1 N_1 = R_2 N_2$, the velocity of sliding at the base of the wedge of the driver is

$$V_s = 2\pi\left(R_2 + \frac{h}{2}\right)N_2 - 2\pi\left(R_1 - \frac{h}{2}\right)N_1 = \pi h(N_1 + N_2) \quad (205)$$

In a similar manner it can be shown that the velocity of sliding at the bottom of the groove in the follower is the same. It is to be noted that this relative sliding which results in wear is independent of any slipping that may occur at the mean depth of the grooves and therefore represents an action that is not present in plain cylindrical friction wheels. Since this secondary sliding is directly proportional to h, the working depth
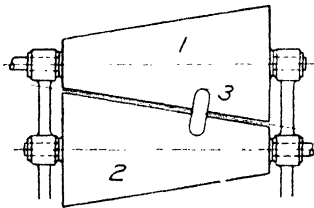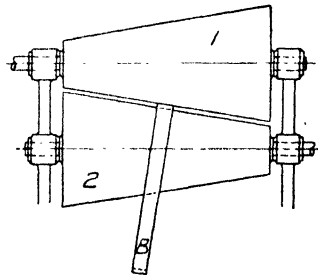
FIG. 265   FIG. 266

of the grooves should be made as small as practicable. Several grooves of small depth are therefore better than one deep groove.

Friction wheels of all kinds are generally so mounted that they can be engaged and disengaged at will, and the pressure between the wheels regulated to suit the power to be transmitted.

121. Friction Speed Variators. As pointed out in the preceding articles, pairs of friction wheels may be used where it is desired to transmit small and moderate amounts of power from one shaft to another at practically a fixed angular velocity ratio. If a transmission is desired where the angular velocity ratio can be changed at will over quite a wide range with little or no change in the amount of slip, a combination of rolling surfaces of revolution is used. Such combinations are known as speed variators. Secondary sliding is usually present, which reduces somewhat the efficiency of such gearing. Three variators for parallel shafts are shown in Figs. 265, 266, and 267.

In Fig. 265 are shown two equal frusta of cones mounted on parallel shafts, the distance between the shafts being such that there is a safe clearance between the cones. In a plane parallel to the cone shafts is an auxiliary shaft carrying a friction wheel *3*. The angle which this shaft makes with the cone shafts will depend upon the diameter of *3*. By having the operating mechanism such that *3* may be raised, moved to any desired position on its shaft, and then pressed against the
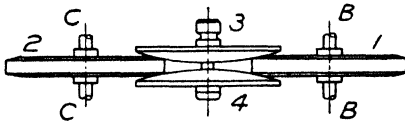
FIG. 267
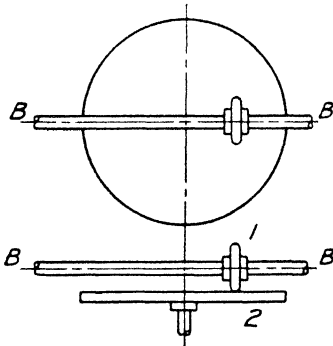
FIG. 269
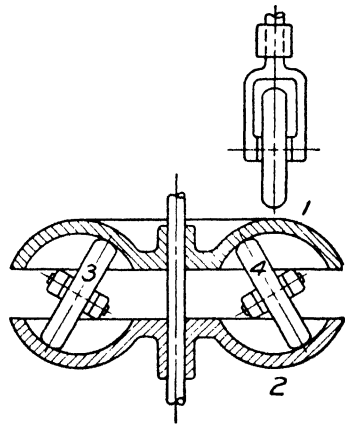
FIG. 268

two cones, the angular velocity ratio of the two cone shafts may be varied over quite a wide range. If the diameters of the ends of the frusta are in the ratio of 2 to 1, then the combination admits of the angular velocity ratio of the driver and driven shafts being varied from 1/2 to 2/1. It is to be noted that the cones will rotate in the same direction. In Fig. 266 is shown a better combination for the same purpose in which one cone drives the other in the opposite direction. This speed variator is known as the Evans friction cone pulleys. In this use of two equal frusta, an endless loop of belt *B* is pinched between the cones. The angular velocity ratio of the drive is varied by shifting the loop of belt to different positions between the cones.

In Fig. 267 is shown a combination first applied to a machine tool feed mechanism, and known as Sellers feed disks. Two equal disks *1* and *2* are mounted in the same plane on parallel shafts a fixed distance apart.

Between them are mounted two convex disks *3* and *4*. ₊These disks are pushed together by helical springs pressing against washers with spherical seats at the centers of the disks. The disks may therefore be spread apart farther on one side than on the other. With the axis of *3* and *4* midway between shafts *B* and *C*, the angular velocity ratio between shafts *B* and *C* will be unity. By swinging *3* and *4* towards shaft *B*, the angular velocity ratio of *B* to *C* is lowered, and is raised as *3* and *4* are moved away from shaft *B*.

In Fig. 268 is shown a combination made by the Power and Speed Controller Company for varying the speed ratio between two disks *1* and *2* whose axes coincide. Between the two disks are two equal friction wheels *3* and *4*, each wheel being mounted in a forked spindle as shown separately in detail. The operating mechanism is such that the forked spindles and the two wheels can be turned in opposite directions through equal angles. Disks *1* and *2* are surfaces of revolution having torus-shaped channels of the same radius as wheels *3* and *4*. One of these disks, say *1*, is keyed to the drive shaft, while the other, being free to rotate relative to the shaft, is pushed toward *1* by the action of a helical spring. When the axes of *3* and *4* stand perpendicular to the drive shaft, the angular velocity ratio of *1* to *2* is unity. With *3* and *4* turned as shown, the angular velocity ratio is more than unity, and it would be less than unity for both tilted the other way.

In Fig. 269 is shown a speed variator for shafts at right angles. The friction element *2* is a flat circular disk in contact with the friction wheel *1* which is mounted on a splined shaft *BB* parallel to the surface of *2*. The operating mechanism is such that *1* may be moved and held in any position along shaft *B*. When *1* is pressed against *2* on the extreme right, the angular velocity ratio $\dfrac{\omega_1}{\omega_2}$ is a maximum, the ratio decreasing as *1* is moved along *BB* to the left toward the center of *2*. If *1* is moved to the left beyond the center, the rotation of *2* is reversed. This device has been used as a speed variator for sensitive drill presses, and was used for a time in certain makes of automobiles in place of the usual clutch and transmission gears.

CHAPTER VII

STRAIGHT AND HELICAL SPUR GEARS

122. Classification of Gears. In the preceding chapter rolling surfaces were classified as those that roll together at a constant angular velocity ratio and those that roll together at a varying angular velocity ratio. The treatment of the latter class of rolling surfaces was confined to sectors and to regular and irregular uni- and multi-lobed wheels having plane motion about parallel axes. It was pointed out that by using such surfaces as the pitch surfaces of toothed gears, motion equivalent to the pure rolling of the surfaces could be transmitted from one shaft to another. Apart from the occasional use of rolling sectors, equal ellipses, and now and then a pair of irregular lobed wheels, such surfaces are seldom fitted with teeth and used as gears. Practically all of the gears used in machinery are for the purpose of transmitting motion from one shaft to another at a constant angular velocity ratio, and are, therefore, based on the first class of rolling surfaces. In classifying gears, the intersection of the working surface of a tooth with the pitch surface of the gear will be referred to as the *pitch element* of the tooth.

As previously pointed out, gears that are to transmit motion between parallel shafts are called *spur gears*. If they are to transmit motion at a constant angular velocity ratio, their pitch surfaces must be rolling right circular cylinders. Where the pitch elements of the teeth are straight the gears are known as *straight* spur gears, and as *curved* spur gears if the pitch elements of the teeth are curved. Where the pitch elements are helices, the gears are known as *helical* spur gears. A very unusual combination of gears is shown in Fig. 270, which illustrates a part of a horizontal milling machine. A mating pair of straight spur gears is shown at *1* and *2*, and a pair of mating helical spur gears at *3* and *4*.

Gears to transmit motion between shafts that intersect are called bevel gears. If they are to transmit motion at a constant angular velocity ratio, their pitch surfaces must be rolling right circular cones. If the pitch elements of the teeth are straight lines passing through the apices of the pitch cones, the gears are known as *straight* bevel gears. In *spiral* bevel gears the teeth are curved and oblique, while in *skew* bevel gears they are straight and oblique with respect to the elements of the pitch cones. A mating pair of straight bevel gears is shown at *5* and *6*

in Fig. 270. Where the axes intersect at right angles and the bevel gears of the mating pair are of the same size, as shown at *7* and *8* in Fig. 270, the gears are called *miter* gears.

Motion between shafts that are at an angle but do not intersect is transmitted positively and continuously at a constant angular velocity ratio either by *helical* gears, erroneously called *spiral* gears, or by
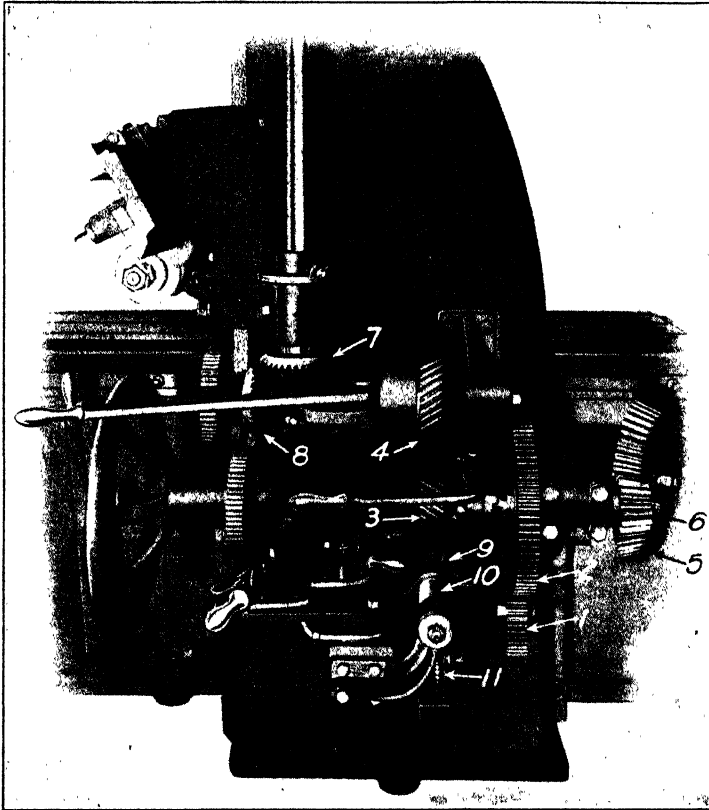
Fig. 270.

hyperboloidal gears. If the pitch surfaces are right circular cylinders, the gears are called *helical* gears; if the pitch surfaces are portions of hyperboloids of revolution, the gears are called *hyperboloidal* gears. A mating pair of helical gears is shown at *3* and *9* in Fig. 270, also at *10* and *11*. It is to be noted that *3* works as a helical spur gear with *4* and as a helical gear with *9*. Helical gear *10* is a single thread worm mating with a worm wheel *11*. The worm is a regular helical gear, while the worm wheel is not. The worm, like all helical gears, is formed by cutting

helical tooth spaces in a right circular cylinder, while the worm wheel is formed by cutting tooth spaces in a cylinder whose face has been hollowed out to conform to the worm. Because of the difficulty of producing them, hyperboloidal gears, until recently, have been very little used. It is the only general class of gears not shown in Fig. 270.

Gears may, therefore, be classified, as given below in Table 4, in accordance with the relation of the axes, the kind of pitch surfaces, the character of the pitch elements of the teeth, the kind of teeth, and the nature of the contact between mating teeth.

TABLE 4

CLASSIFICATION OF GEARS

Name of Gear	Relation of Axes	Pitch Surfaces	Pitch Elements of Teeth	Kind of Teeth	Tooth Contact
Straight spur.... . Helical spur......	Parallel Parallel	Cylinders Cylinders	Straight lines Helices*	Straight Helical	Straight line Oblique straight line †
Straight bevel.... Spiral bevel...... Skew bevel.......	Intersecting Intersecting Intersecting	Cones Cones Cones	Straight lines Curved lines Curved lines	Straight Curved oblique Straight oblique	Straight line Curved line Oblique straight line
Helical......... Worm and worm- wheel	At any angle not intersecting Usually at 90° not intersecting	Cylinders Cylinder for worm	Helices Helices for worm	Helical Helical for worm	Point Curved line
Hyperboloidal ...	At any angle not intersecting	Hyperboloids of revolution	Straight or curved lines	Oblique, straight or curved	Straight or curved line

* Spur gears may have curved, oblique teeth other than helical, but no such gears have ever come into general use.
† For involute teeth.

Besides being employed much more frequently, spur gears are simpler than the gears of any other class, and an understanding of them is of very considerable help in the study of bevel, helical, and hyperboloidal gears. This chapter will therefore be devoted to the theory of spur gears.

123. History.* A few historical remarks are here introduced in the hope of arousing the student's interest in gears and of awakening a proper appreciation of these highly important and indispensable machine elements. Although toothed gears have been used for over 2200 years, it was only about 250 years ago that the theory of correct tooth shapes

* Principally from ."Historical Notes on Gear Teeth" by L. D. Burlingame, *Machinery* (N. Y.), March, 1924.

was first enunciated, and only about 75 years ago that a means of accurately cutting gear teeth was first made commercially available.

The works of Aristotle indicate that toothed gears were used as early as 350 B.C. As early as 150 B.C., Ctesibius employed spur and bevel gears in his water-clocks. That gears were employed by the Romans in the early Christian era is proven by the sculptured gears on the Column of Trajan in Rome. Leonardo da Vinci, in his work, Codice Atlantico, published about 1492, shows sketches of spur, bevel, and worm gears applied to widely different purposes. It is not apparent that he or any of his predecessors had solved the problem of theoretically correct tooth shapes.

So far as known, the celebrated Danish astronomer, Olaf Roemer, was, in 1674, the first to deduce the theory of correct tooth shapes for gears. He proposed the use of cycloidal outlines. In 1695 Philippe de Lahaire, a Frenchman, advocated involute outlines as equally suitable. In 1760 the involute was further elucidated by Leonard Euler, a Swiss mathematician. He is generally credited as the first to seriously suggest involute outlines. In an essay, prepared in 1752, Charles E. L. Camus, a French genius, who at the age of twelve gave lectures in science in Paris, gave such a comprehensive treatment of cycloidal tooth outlines as to enable mechanics for the first time to make practical application of the theory. He pointed out the requirements for positive, continuous driving, and showed that action was smoother in recess than in approach. He also conceived the idea of interchangeability. Camus favored cycloidal over involute outlines as did Hawkins, an Englishman, who translated Camus' work. However, when, in 1837, Hawkins brought out the second edition of his translation, he devoted a lengthy appendix to the merits and superiority of involute outlines over cycloidal. In a paper before the Institution of Civil Engineers in 1837, Robert Willis set forth the requirements for an interchangeable system of gears and described his odontograph for laying out involute gear teeth. This paper was republished in his "Principles of Mechanism" in 1841. The angle of obliquity and tooth proportions adopted later by the Brown and Sharpe Manufacturing Company are essentially the same as those proposed by Willis. He perhaps deserves more credit than any one else for improving practice and for the present almost universal recognition of the superiority of involute over cycloidal outlines for gear teeth. When the contest between these two systems was at its height in this country, George B. Grant was probably the outstanding champion of the involute system.

The first gear-cutting machines used reciprocating or rotating tools shaped to correspond to the tooth spaces to be cut. Apparently the

first rotating gear-cutter was made before 1782 by a famous French mechanic named Jacques de Vaucanson. The end of a bar was first turned to conform to the space to be cut, and the fine teeth were then formed by the use of hammer and chisel before the cutter was hardened. The first gear-cutting machine appeared about 1800. It was small and hand operated. The first power-driven, precision gear-cutting machine to be built in this country was built in 1855 by Joseph R. Brown of the Brown and Sharpe Manufacturing Company. In 1864 Brown invented the formed milling cutter, a cutter that could be sharpened by grinding without changing the shape of its cutting face. Previously, in 1861, he had invented the universal milling machine. With this machine and with a set of formed cutters provided for each pitch, a shop was equipped to cut spur, bevel, and helical gears. The method is inherently inaccurate for bevel gears, and accurate for spur gears only when the
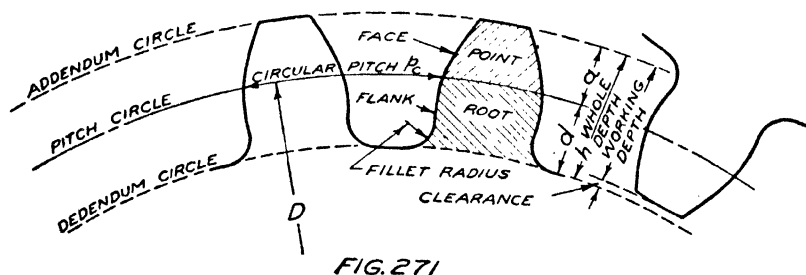
FIG. 271

shape of the cutter corresponds to the number of teeth in the gear to be cut. Since the number of cutters in a set is necessarily limited by the consideration of cost, the method can be only closely approximate for spur gears. The generating method as patented by Hugo Bilgram in 1884 constituted the next and most important advance in the art of cutting gear teeth. Improvement in the accuracy and the rapidity with which gears can be cut is due to the application of the principle invented by Bilgram. Many machines have been developed for generating gear teeth. Some employ a straight-sided cutter, some a pinion-shaped cutter, while others use a hob. A brief treatment of the methods now employed will be found in the next three chapters.

124. Terminology. Time and space will be conserved if certain terms to be used are defined and illustrated before considering in detail the profiles and the action of mating gear teeth. Most of the following terms are illustrated in Fig. 271.

The *pitch diameter*, D, is the diameter of the pitch circle.

The *pitch point*, P, of a pair of mating gears is the point of tangency of their pitch circles.

The *circular pitch*, p_c, of a gear is the distance, measured on the pitch circle, from a point on one tooth to the corresponding point on the next tooth. It is therefore equal to the circumference of the pitch circle divided by the number of teeth.

The *diametral pitch*, p'_c, is the ratio of the number of teeth in a gear to its pitch diameter expressed in inches. By expressing the circular pitch and the diametral pitch in terms of the number of teeth N and the pitch diameter D in inches, the relation between them is easily obtained.

$$p_c = \frac{\pi D}{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

$$p'_c = \frac{N}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (207)$$

The product of (206) and (207) is

$$p_c p'_c = \pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

The *module*, m, is the reciprocal of the diametral pitch, and therefore represents the number of inches of diameter per tooth.

The *addendum*, a, is the radial distance from the pitch circle to the ends of the teeth, or to the addendum circle.

The *dedendum*, d, is the radial distance from the pitch circle to the bottom of the tooth spaces, or to the dedendum circle.

The *clearance*, c, is the radial distance from the end of a tooth on one gear to the bottom of the tooth space of the mating gear.

The *whole depth*, h, or height of tooth, is the sum of the addendum and dedendum.

The *working depth* is the depth to which the teeth of one gear extend into the spaces of the mating gear. It is the sum of the addenda of the mating gears. It is also equal to the whole depth minus the clearance.

The *point* of a tooth is that portion included between the pitch and addendum circles.

The *root* of a tooth is the portion included between the pitch and dedendum circles.

The *face* of a tooth is that portion of the tooth outline, or profile, which extends from the pitch to the addendum circle.

The *flank* of a tooth is that portion of the tooth profile which extends from the pitch circle to the bottom of the tooth space, or to the dedendum circle.

The *backlash* is the width of a tooth space of one gear, measured along its pitch circle, minus the thickness of a tooth of the mating gear, measured along its pitch circle. For cut industrial spur gears, the Amer-

ican Gear Manufacturers' Association recommend for the backlash a minimum of 0.03 in., an average of 0.04 in., and a maximum of 0.05 in. divided by the diametral pitch.

The term *pinion* is used to designate the smaller gear of a mating pair.

A *rack* is a portion of a spur gear whose pitch radius is infinitely long. Its pitch surface is therefore a plane and its pitch curve a straight line. A rack may have either straight or oblique teeth.

125. Fundamental Law of Gear Tooth Profiles. Mating pairs of gear teeth may be regarded as direct contact members or cams fastened to mating pitch surfaces to secure positive driving. In order that the driving may be continuous as well as positive, the cams, or teeth, on each gear must be sufficient in number to admit of one pair coming into action before the preceding pair goes out of action. If the teeth are not of proper shape, the driving will not be equivalent to the pure rolling of the pitch surfaces or curves, and the angular velocities from instant to instant will not be inversely as the contact radii of the pitch curves. To secure transmission equivalent to the rolling of the pitch curves, *the common normal through the point of contact of the mating teeth must pass through the point in the line of centers at which the pitch curves are tangent.* This follows from what has previously been demonstrated: that rolling curves make contact in their line of centers; that their angular velocities are inversely as their contact radii; and that the angular velocities of direct contact members are, as demonstrated in Art. 43, inversely as the segments into which the line of centers is cut by the common normal through the point of contact of the members. Hence, at any instant, the common normal to mating tooth profiles must pass through the point of tangency of the pitch curves in the line of centers if driving is to be equivalent to the rolling of the pitch curves. If the pitch curves are circles, the point of tangency, or pitch point, is a fixed point in the line of centers; if the pitch, or rolling, curves are not circles, the pitch point will be a moving point in the line of centers.

Within limits, the profile of a tooth on one gear may be assumed, and the profile of the tooth on the mating gear may then be derived by applying the fundamental law stated above. A tooth so derived to mate with a given tooth is known as a *conjugate* tooth. It can also be shown that tooth profiles described by a point on a curve which rolls upon each of a pair of pitch curves, will fulfill the requirement above stated.

126. Conjugate Spur Gear Teeth. A portion of a gear having 36 teeth is shown in Fig. 272. For convenience, straight line profiles have arbitrarily been chosen for the teeth, the included angle of each tooth being 60°. The path of the point of contact, the conjugate rack tooth,

and a conjugate gear tooth for the given tooth are shown in Figs. 274, 276, and 278.

In Fig. 273, five equally spaced positions of the given profile are shown. According to Art. 125, the points of contact must be where
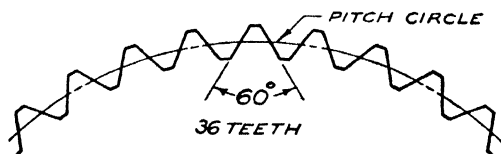
FIG. 272
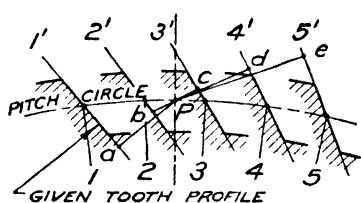
FIG. 273
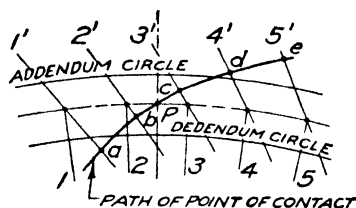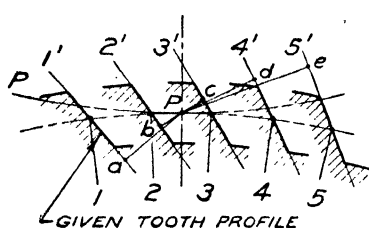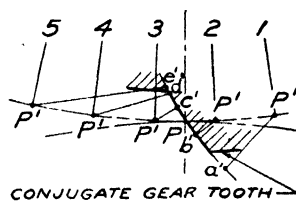
FIG. 274
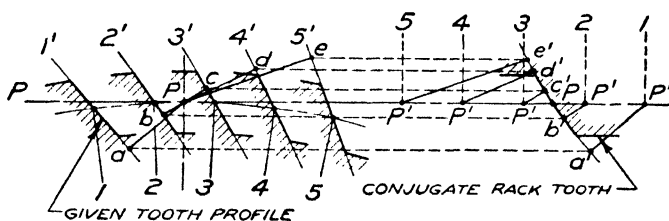
FIG. 275

FIG. 276

FIG. 277

FIG. 278

lines through the pitch point P, perpendicular to the different positions of the given profile, intersect the profile lines *1'*, *2'*, *3'*, etc. Since the given profile is a straight line, the normals and therefore the points of contact a, b, c, d, and e are easily determined. A smooth curve through these points gives the path of the point of contact. To draw a smooth curve through the points a, b, etc., in Fig. 273 would obscure the per-

perdiculars *Pa*, *Pb*, etc. The positions and points of Fig. 273 were therefore repeated in Fig. 274, and the path of the point of contact *abPcde* drawn as shown.

Fig. 275 is a repetition of Fig. 273 with the addition of the pitch line *PP* of the rack. Since, in driving the rack, the flank of the given gear tooth will come in contact with the face of the rack tooth, the position lines *1, 2, 3, 4,* and *5* for the rack in Fig. 276 are in the reverse order to those of Fig. 275. These lines are perpendicular to the pitch line of the rack and are spaced in accordance with the lengths of the corresponding pitch arcs between the five different positions of the given gear tooth profile, which, for convenience, were taken equal. Point *a'*, as a point on the conjugate rack tooth profile, is where an arc of radius *P'a'*, equal to the perpendicular *Pa* of Fig. 275, intersects a line through *a* parallel to the pitch line *PP*. Points *b'*, *c'*, etc., were determined in a similar manner. A smooth curve through the points so determined gives the profile of the conjugate rack tooth.

The tooth profile shown in Fig. 278 is conjugate for the given profile and is for a gear of the same number of teeth, 36, as the given gear. Fig. 277 is a repetition of Fig. 273 with the addition of the pitch curve *PP* of the conjugate gear. For the same reason as for the rack, the radial position lines of Fig. 278 are in the reverse order to those in Fig. 277, the displacement angles in each case corresponding to equal arcs. Point *a'*, as a point on the conjugate gear tooth profile, is where an arc of radius *P'a'*, equal to the perpendicular *Pa* of Fig. 277, intersects an arc having its center at the center of the conjugate gear and a radius equal to the distance from this center to the point *a* in Fig. 277. The remaining points were found in a similar manner.

It is to be noted that while a 36-tooth gear having tooth profiles as shown in Fig. 278 will work properly with a 36-tooth gear having tooth profiles as shown in Fig. 277, two gears similar to either one will not work properly together. Also, the given gear will work with the conjugate rack, but, for the conjugate gear of Fig. 278 to work with the rack, the shading of the profile in Fig. 276 must be on the other side. In other words, the gears are not interchangeable. For interchangeability, as will be shown, the path of the point of contact must be symmetrical with respect to the line of centers.

It is evident that the above constructions can be carried out for any given profile.* To construct a conjugate gear tooth profile from

an arbitrary rack tooth profile is the reverse of the problem illustrated in Figs. 275 and 276. It would require perpendiculars to be drawn through the pitch point P to different and convenient positions of the given rack tooth profile. While for every possible path of the point of contact there is a conjugate rack tooth profile, there is no simple and general graphical construction for making the determination.

The conjugate profile of Fig. 278 could be determined from the arbitrary profile of Fig. 277 by the method illustrated in Fig. 279. That a conjugate so determined would fulfill the fundamental law stated in Art. 125 is by no means as obvious as for the method of determination just described above. In Fig. 279, *1* and *2* are two pitch curves, P being the pitch point on the line of centers O_1O_2. M is a thin metal band fastened to *1* and *2* to enable one to be rolled on the other without slipping. T is a thin metal template of the arbitrary tooth profile which is fastened to *1* so that the sheet of paper S, which is fastened to *2*, may slide under the template as *1* and *2* are rolled together. If

FIG. 279

for each of several relative positions of *1* and *2*, the curve of the given profile is traced on the paper and a smooth curve is then drawn tangent to the several tracings on S, the resulting smooth curve C will be the conjugate of the given profile.
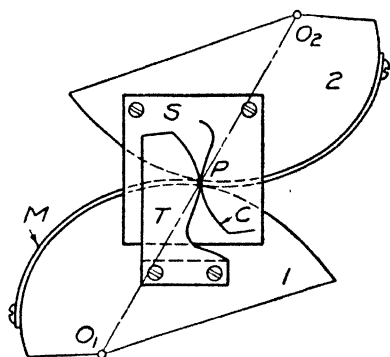
The method illustrated by Fig. 279 suggests how a tooth acting as a cutter could produce its conjugate profile. If *1* and *2* were rotated about O_1 and O_2, respectively, by small angular displacements corresponding to the rolling of the pitch curves, and T were reciprocated parallel to O_2 after each small angular displacement, T would cut its conjugate in *2*. From this point of view, Fig. 279 may be regarded as an illustration, though incomplete, of the generating process of cutting gear teeth.

While the concepts of this article and the fundamental law of gear tooth profiles are of greater practical importance in the cutting of gear teeth, the describing method of generating gear tooth profiles to be given in the next and some of the succeeding articles is quite necessary to an understanding of gear tooth action and to an appreciation of what constitutes a suitable generating profile.

127. General Method of Describing Gear Tooth Profiles. The general method of describing gear tooth profiles by means of an auxiliary rolling curve, or generator, is illustrated in Fig. 280. Suppose the outlines of the two plane figures *1* and *2* to be two curves capable of
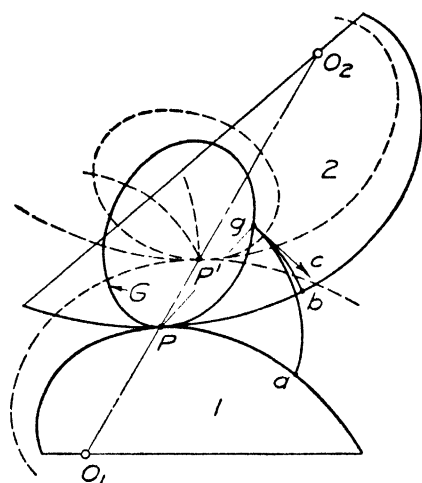

FIG. 280

rolling together about the fixed centers O_1 and O_2. Since they are rolling curves, their point of contact, or tangency, will lie on the line of centers, and their angular velocity ratio at any instant will be inversely as their contact radii. For a pair of teeth to give a relative motion to *1* and *2* equivalent to the rolling of the curves, the common normal to the teeth in contact must pass through the point *P* on the line of centers at which the curves are tangent. The profiles of a pair of teeth to fulfill this requirement may be described by a point on any curve such as

G, whose maximum radius of curvature will permit it to roll on either pitch curve.

Starting with point *g* of *G* in contact with *1* at *a*, the point *g* will describe the curve *ag* on *1* as *G* is rolled backward on *1* to the position shown. Since *G* has been rolled on *1*, the arcs *Pa* and *Pg* are equal. If the arc *Pb* is made equal to *Pa* and *G* is rolled on the concave side of *2* from a position with *g* of *G* in contact at *b* to the position shown, the point *g* will describe the curve *bg* on *2*. Having rolled *G* on *1* and then on *2*, let it now be assumed that they roll together simultaneously beginning with the positions shown by the full lines as initial. A counterclockwise rotation of *1* will cause *2* and *G* to rotate clockwise. Since the arcs *Pa*, *Pb*, and *Pg* are equal, point *g* in *G* will retrace the curve *ga* on *1* and the curve *gb* on *2*, and the points *a*, *b*, and *g* will come together at *P'* with *1*, *2*, and *G* in the positions shown dotted. Evidently as *1*, *2*, and *G* roll together simultaneously, the instant center of rotation of *G* relative to *1* and *2* is always the point of contact in the line of centers, and the point of contact between the two curves that are traced is always the describing point *g* in *G*. For the initial position, this instant center of rotation is *P*. Hence, the describing point *g* is, at the instant, moving in the direction *gc* normal to its rotational radius *Pg*. Since at the instant *g* is likewise describing the curves *ga* and *gb*, the common

tangent to these curves coincides with *gc*, and *Pg* is, therefore, the common normal to *ga* and *gb*. It having been proved that the common normal to curves *ga* and *gb* passes through the point on the line of centers at which the pitch curves are tangent, it follows that, with *gb* as a tooth profile on *2* and with *ga* as a tooth profile on *1*, clockwise rotation of *2* will positively drive *1* counter-clockwise with a relative motion between *2* and *1* equivalent to the rolling of the pitch arcs *Pb* and *Pa*.

In choosing the pitch curves for Fig. 280, circles were purposely avoided. Since the pitch surfaces of toothed gears are almost invariably surfaces of revolution, circles or circular arcs will be used as pitch lines in all but one of the remaining articles of this chapter.

128. Forms of Spur Gear Teeth. It is evident from the preceding article that there are many curves that could be used as generators to describe gear tooth profiles. However, on account of the practical difficulties of generation and cutting, the circle is the only curve that has been used to generate tooth profiles by rolling on the pitch circles. As a generator of tooth profiles the straight line is now used almost to the exclusion of the circle, but, as will be shown, it is not rolled on the pitch but on a base circle.

The curve traced by a point on a circle as it rolls on the convex side of another circle is called an *epicycloid;* the curve traced by a point on a circle as it rolls on the concave side of another and larger circle is called a *hypocycloid;* and the

FIG. 281
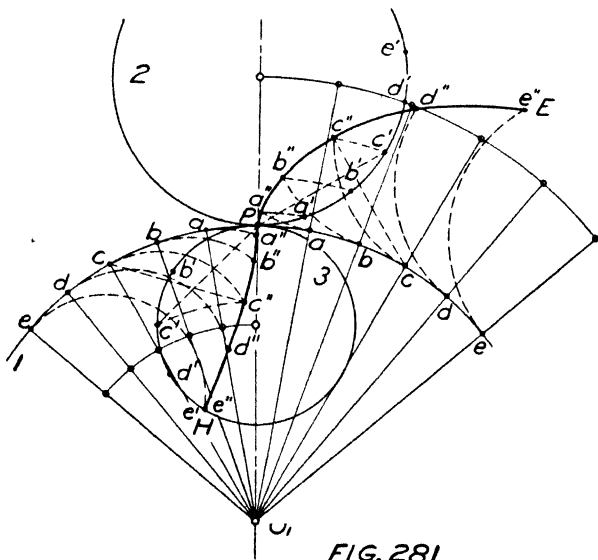
curve traced by a point on a circle as it rolls on a straight line is called a *cycloid*. It will be found that gear teeth whose profiles have thus been traced are referred to either as *epicycloidal teeth* or as *cycloidal teeth*. The latter term will be used in this text.

In Fig. 281 the describing circles *2* and *3* are shown tangent to circle *1* at *P*. If *2* is rolled to the right on *1*, the point on *2* coincident with

P will trace the epicycloid E. To carry out the construction, the circumference of circle 1 to the right of P is divided into equal arcs of convenient length, Pa, ab, bc, etc.; and the circumference of circle 2 to the right of P is divided into equal arcs of the same length, Pa', $a'b'$, $b'c'$, etc. Hence, when 2 is rolled to the right on 1, the point a' will come into coincidence with a, b' with b, c' with c, etc., and the describing point on 2 will have moved to a'', b'', c'', etc., corresponding to the dotted positions of the describing circle. If the describing circle 3 is rolled to the left on the inside of 1, the point on 3 coincident with P will describe the hypocycloid H. The points a'', b'', c'', etc., on this curve were located by the same method as employed in plotting the epicycloid. If the diameter of 3 had been taken equal to the radius of 1, the hypocycloid H would have coincided with the radial line PO_1. It is obvious from the method employed in plotting the epicycloid and the hypocycloid, that points a'' and a', b'' and b', c'' and c', etc., must lie on concentric arcs from O_1 as a center. This suggests a simpler method than that employed for plotting the curves whereby the drawing of the different positions of the describing circles may be avoided. Thus, as for each of the other positions, cc'' is the revolved position of chord $c'P$. Hence c'' is the intersection of an arc through c' from O_1 as a center with an arc of radius $c'P$ from c as center. This construction is shown more clearly in Fig. 282, points a'', b'', d'', etc., being located in the same manner as c''.
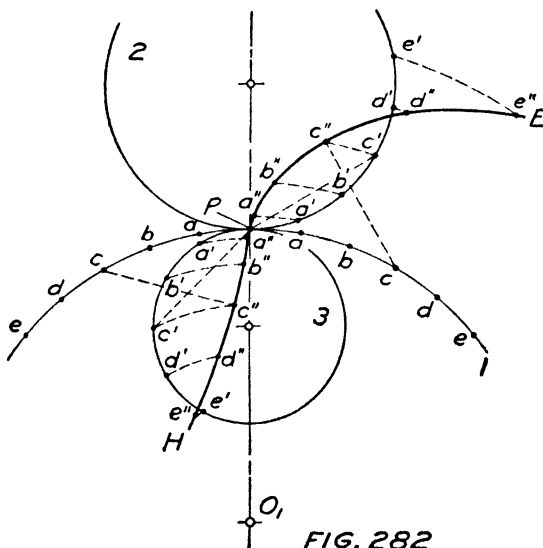
FIG. 282

When a right line rolls upon a circle, called the *base* circle, any point in the line traces a curve called the *involute* of a circle. It is a property of this curve that the normal at any point is tangent to the base circle. Gear teeth whose profiles have thus been traced are referred to as *involute teeth*.

In Fig. 283 the straight line 2 is shown tangent at P to the base circle 1 upon which it is to roll. To carry out the construction, the

circumferential arc of *1* to the left of *P* is divided into equal arcs of convenient length, *Pa*, *ab*, *bc*, etc.; and the straight line *2* to the left of *P* is divided into equal parts, *Pa'*, *a'b'*, *b'c'*, etc., of the same length. Hence, when *2* is rolled on *1*, the point *a'* will come into coincidence with *a*, *b'* with *b*, *c'* with *c*, etc., and the describing point on *2* will have moved to *a''*, *b''*, *c''*, etc.

corresponding to the dotted positions of the generating line *2*. It is obvious from the method employed in plotting the involute curve *N* that points *a'* and *a''*, *b'* and *b''*, *c'* and *c''*, etc., must be on concentric arcs from O_1 as a center. This suggests a simpler method than that employed for plotting the curve whereby the dotted positions of the generating line may be eliminated. Obviously, if *c''* is on an arc through *c'* from center O_1, it must be a distance
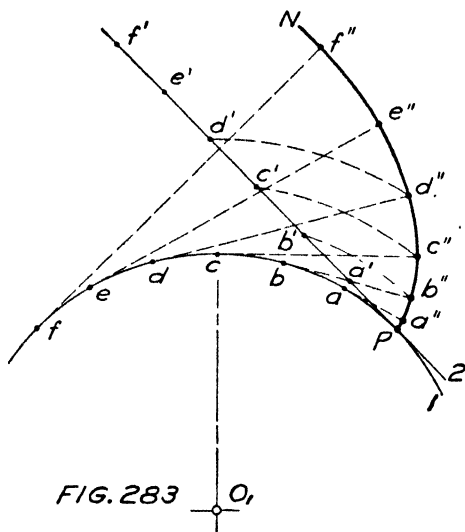
FIG. 283

from *c* equal to *c'P*. Hence *c''* is where an arc through *c'* from O_1 as a center intersects an arc of radius *c'P* from *c* as a center, points *a''*, *b''*, *d''*, etc., being located in a similar manner. For the polar equation of the involute see equation (235) Art. 147.

It is to be noted that while *P* in *2* is describing the involute shown, every other point in the generating line is describing an identical involute. Thus the involute described by *c'* after making contact with *1* at *c* would be the same as the involute *N* and at every point would be at a normal distance from
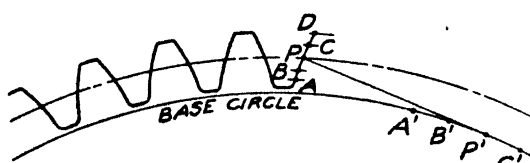
FIG. 284

N equal to the arc *Pc*. If *N* and the new involute were corresponding profiles of adjacent teeth, the arc *Pc* becomes the *base pitch*, p_b.

Where, for illustrative purposes, it is necessary to draw the teeth on a gear, an easy approximate method of drawing the involute profiles is a convenience. Such a method is shown in Fig. 284, which shows a line, drawn through a point *P* on the pitch circle tangent to the base

circle at P'. An arc through P from P' as a center will be found to be in fair agreement with an involute through P. The profiles of the teeth shown were drawn in this manner. Where a closer approximation is desired, use may be made of other centers on the base circle. Thus let B and C be determined as the intersections of an arc through P from P' as a center with arcs concentric to the pitch circle. Then determine the tangency points B' and C' on the base circle by drawing tangents through B and C. With B' and C' as centers, draw the arcs BA and CD, and finish the flank by drawing an arc from A with the point of tangency A' as a center. The radii $A'A$, $B'B$, $P'P$, etc., may be used for additional tooth profiles if desired.

As before pointed out, the cycloidal tooth form was the first to be adopted, and for many years was used exclusively. Now the involute is used almost to the exclusion of the cycloidal form. Practically the only use now made of cycloidal curves is in conjunction with the involute in a composite form of gear tooth. The involute and composite are the principal forms of gear teeth used to-day. However, a brief treatment of the cycloidal form is justified, and will be found to contribute to a better understanding of the involute and composite forms.

129. Cycloidal Spur Gear Teeth. In Fig. 285, the two pitch circles *1* and *2*, with centers at O_1 and O_2, respectively, are tangent at point P. The generator, or describing circle, *3* has its center O_3 on the line of centers O_1O_2. In the figure arcs Pa, Pb, and Pg are equal. If *1*, *2*, and *3* roll together simultaneously as they rotate about their respective centers O_1, O_2, and O_3, point g on *3* will trace the hypocyloid ga on *1* and the epicycloid gb on *2* as *1*, *2*, and *3* rotate through the equal arcs Pa, Pb, and Pg, points a, b, and g becoming coincident at P. During this rotation the instant center of rotation of *3* relative to *1* and *2* evidently remains at the pitch point P. Since g in *3* is describing both curves, the curves at any instant are in contact at g, and therefore their common tangent must coincide with the direction of motion of g which is perpendicular to Pg; hence, Pg is the common normal to the described curves. Since, during action, the instant center remains at P, the common normal will always cut the line of centers at a fixed point. The hypocycloid of *1* acting on the epicycloid of *2* will therefore transmit motion from *1* to *2* at a constant angular velocity ratio corresponding to the rolling of the pitch circles. Hence the described curves are suitable as tooth profiles for the gears.

If *1* were fitted with a hypocycloidal flank ag and *2* with an epicycloidal face bg, counter-clockwise rotation of *1'* through an arc aP would drive *2* clockwise through an arc bP. To continue the driving beyond P, with the same pair of teeth, it is necessary to add an epicy-

cloidal face to *1*, such as $a'g'$, and a hypocycloidal flank to *2*, such as $b'g'$. These may be described by another generator such as *4* rolling on the outside of *1* and the inside of *2*. A tooth having such a profile composed of a face and a flank is shown shaded at *c* for *1* and shaded at *e* for *2*. These two profiles acting together would drive from one
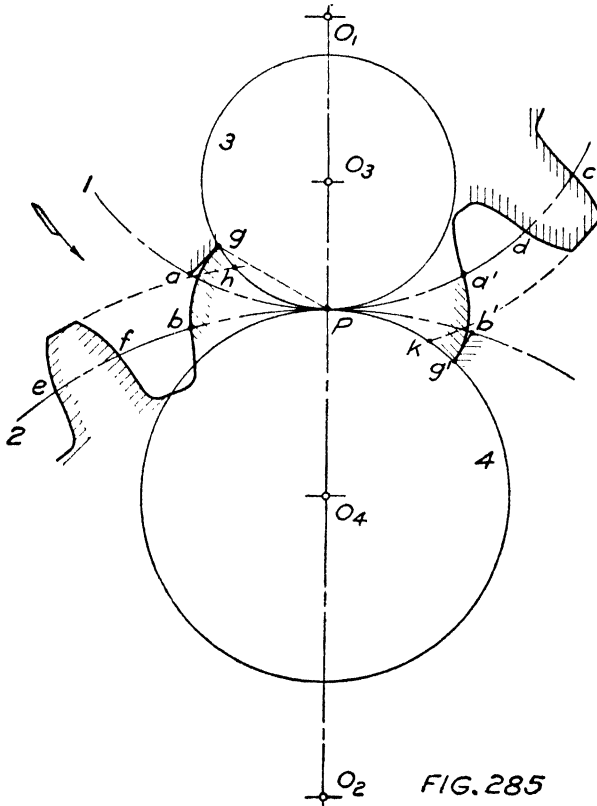

FIG. 285

side of *P* to the other. Similar but reversed profiles *d* and *f* are added to enable either gear to drive the other in either direction. With single profiles, one gear could drive the other only in one direction.

It is to be noted that the face of profile *c* has not been made as long as $a'g'$, nor has the face of profile *e* been made as long as *bg*. Hence, if *1* and *2* were fitted with teeth as shown at *cd* and *ef*, contact would begin at *h* and cease at *k* for counter-clockwise rotation of *1* acting as the driver. Since at the beginning of contact the flank of *1* engages with the face of *2*, the point at which contact begins is determined by the intersection of the addendum circle of *2* with the describing circle *3*,

and the point at which contact ends is determined by the intersection of the addendum circle of *1* with the describing circle *4*. The path of the point of contact between a pair of teeth during action is therefore the reverse curve *hPk*.

To meet essential requirements, a pair of mating gear wheels would require that the common circular pitch be an exact divisor of the circumferences of the pitch circles *1* and *2*, that the addenda of the mating teeth be sufficiently long to insure one pair of teeth coming into action before the preceding pair goes out of action, and that the depth of the tooth spaces be greater than the sum of the addenda of the mating teeth to provide clearance in action.

130. Action of Cycloidal Spur Gear Teeth. (*a*) *Path of the Point of Contact.* In Fig. 286, *1* is the driver and *2* the driven gear. The
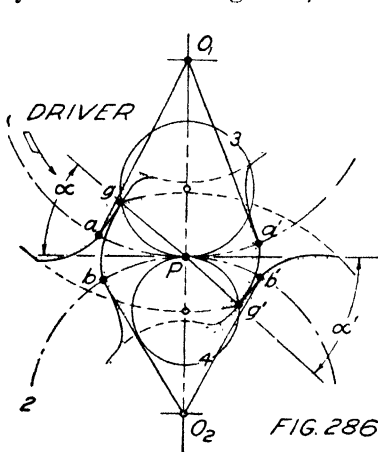
FIG. 286

pair of teeth to the left of the line of centers is just coming into contact, while to the right a pair is shown just quitting contact. As was shown in the preceding article, the reverse curve *gPg'* is the path of the point of contact during action between the pair of mating teeth. It was also shown that arcs *Pa*, *Pb*, and *Pg* are equal, likewise arcs *Pa'*, *Pb'*, and *Pg'*. Hence *gPg' = aPa' = bPb'*, or the point of contact of a pair of mating gear teeth traverses the path *gPg'* while the mating gears turn through the equal arcs *aPa'* and *bPb'*.

(*b*) *Arc and Angle of Action.* The distance on the pitch circle between the positions of a tooth from the beginning to the end of contact, is called the *arc of action*. Thus *aPa'* is the arc of action of *1*, and *bPb'* that of *2*. Since the action of mating gears is equivalent to the rolling of the pitch circles, it follows that the arcs of action of mating gears are equal. The angle subtended by the arc of action is called the *angle of action*. This angle is, therefore, the angle turned through by a gear from the beginning to the end of tooth contact. Thus aO_1a' is the angle of action of *1*, and bO_2b' that of *2*. The angle turned through by a gear from the beginning of tooth contact to contact at the pitch point *P*, is called the *angle of approach*. Thus aO_1P is the angle of approach of *1* and bO_2P that of *2*, *aP* and *bP* being the respective arcs of approach. The *angle of recess* is the angle turned through by a gear from tooth contact at the pitch point to the end of contact. Thus

PO_1a' is the angle of recess of *1* and PO_2b' that of *2*, Pa' and Pb' being the respective arcs of recess.

(c) *Continuous Driving and Length of Teeth.* To secure continuous driving by having one pair of teeth come into action before the preceding pair goes out of action, the length of the arc of action or path of contact must be greater than the circular pitch. The ratio of the arc of action to the circular pitch is called the *contact ratio.* For durability and quiet running in the transmission of any appreciable amount of power, it has been found from experience that the contact ratio should, if possible, be 1.40 or more. The nearer the ratio used approaches unity, the higher the degree of accuracy of profile shapes required to secure quiet running. When it is necessary to use a ratio of less than 1.40, 1.20 should be regarded as the minimum. Since g and g' mark the intersections of the addendum circles of *2* and *1* with the generating circles *3* and *4*, the length of the path of contact gPg', and therefore the arc of action, is determined by the addenda of the mating gears. The greater the addenda the greater the length of the path of contact or arc of action. The maximum addendum possible is reached when the teeth become pointed, which fixes the maximum length of teeth and the maximum arc of action possible for any given pitch and pair of pitch and describing circles. If the addenda and pitch circles are fixed, the length of the arc of action will depend upon the size of the describing circles. The larger they are the greater will be the arc of action but the weaker will be the teeth. For this reason a diameter of describing circle larger than $\frac{5}{8}$ the pitch diameter of the gear is seldom used.

(d) *Angle of Obliquity and Pressure Angle.* The angle between the common normal to the teeth at their point of contact and the common tangent to the pitch circles is called the *angle of obliquity,* or commercially, the *pressure angle.* Thus α is the maximum angle of obliquity during approach and α' the maximum during recess. It is to be noted that during approach the angle of obliquity decreases from α to zero, and, during recess, increases from zero to α'. Since the obliquity increases the pressure between the teeth and at the bearings, the angle of obliquity should be no greater than that required for satisfactory and continuous driving. For the transmission of any appreciable amount of power, the maximum obliquity should not exceed 30 degrees. For more than one pair of teeth continuously in action, this would give an average obliquity of less than 15 degrees. For light work and extreme cases, the maximum obliquity may be as high as 36 degrees. Neglecting the friction of sliding, the pressure between the teeth would act at an angle equal to the angle of obliquity. Hence the pressure normal to the teeth will be equal to that corresponding to the torque

to be transmitted divided by the cosine of the angle of obliquity. The per cent increase of the normal over the tangential pressure will therefore be $100\left(\dfrac{1}{\cos\alpha} - 1\right)$, which equals slightly over 15 per cent for an angle of 30 degrees.

The effect of the friction of sliding between the teeth is to make the pressure angle greater than the angle of obliquity during approach and to make it less during recess. The energy lost in friction at the teeth and at the bearings is therefore less during recess than during approach. This, to some degree, accounts for the action being smoother during recess than during approach. The principal reason, however, for the smoother action is the difference in the direction of sliding between the teeth during approach and recess. During approach the teeth, while sliding on each other, are pushing into mesh, while during recess they are drawing out of mesh. Gears would therefore be more efficient and durable if the action were confined to recess. However, continuous driving and other requirements will often make such a limitation of action, at least for straight spur gears, either impossible or undesirable. As will be evident later, the best tooth proportions for any given conditions of operation are generally a compromise between requirements and possibilities.
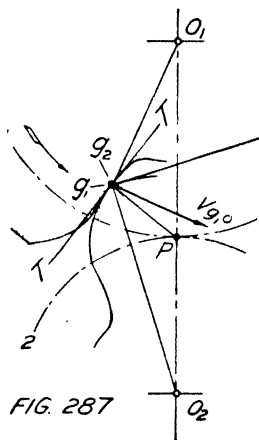
FIG. 288

FIG. 287

(e) *Velocity of Sliding between Teeth.* The determination of the velocity of sliding between teeth at the beginning of contact is shown in Figs. 287 and 288. Fig. 287 is a repetition of part of Fig. 286, g_1 being the point in 1 in contact with point g_2 in 2. The velocities of these respective points must be perpendicular to their respective radii g_1O_1 and g_2O_2. With velocity V_{g_1o} of point g_1 of the driver known, the velocities V_{g_2o} and $V_{g_1g_2}$ may be found by the method of Art. 47 or by the velocity vector diagram method of Chapter IV. In Fig. 288, vector og_1 is drawn perpendicular to radius g_1O_1 and of a length to represent the known velocity V_{g_1o}. It is known that the velocity of g_2 is perpendicular to g_2O_2, and that the velocity of g_2 relative to g_1 must be along the common tangent TT, or perpendicular to the common

normal g_2P. Hence the intersection of lines through o and g_1, perpendicular, respectively, to g_2O_2 and g_2P, determines g_2 and the velocity vectors og_2 and g_1g_2 representing the velocity V_{g_2o} and the velocity of sliding $V_{g_2g_1}$ of g_2 relative to g_1. It is to be noted that the farther the coincident points of contact g_1 and g_2 are from the pitch point P, the greater the angle between vectors og_1 and og_2 and the greater the vector g_1g_2 representing the velocity of sliding between the teeth. As the coincident points of contact approach the pitch point P, vectors og_1 and og_2 are brought together, the relative sliding being reduced to zero for contact at P.

131. Interchangeable Cycloidal Spur Gears. It has been shown that the two gears *1* and *2* in Fig. 286 will work together satisfactorily because the respective engaging faces and flanks were described by the same generating circle. If the tooth profiles of a third gear are described by the same generating circles, *3* and *4*, to work with *1*, it will not work with gear *2*. The flanks of the third gear, described by generator *4*, will engage with the faces of gear *2* which were described by generator *3*; and the faces of the third gear, described by generator *3*, will engage with the flanks of gear *2* which were described by generator *4*. The third gear will therefore not work satisfactorily with gear *2* because the respective engaging faces and flanks were not described by the same generator, or describing circle. Obviously, to make the three gears interchangeable, the describing circles *3* and *4* must be of the same diameter. *It can therefore be stated that for a set of gears of the same pitch to be interchangeable, the describing circles used must be of the same diameter.*

The diameter of the describing circle for an interchangeable set of gears must obviously be smaller than the pitch diameter of the smallest pinion that would be considered desirable. It must also be of such a diameter as to give suitable flanks. The smaller the describing circle the greater the angle of obliquity for a given addendum and the shorter the arc of action. The larger the diameter of the describing circle the greater will be the arc of action for a given addendum or the smaller the addendum for a given arc of action. These considerations would point to as large a describing circle as possible. If, however, the diameter of the describing circle is made larger than the pitch radius of the pinion, the flanks of the teeth will be inside of a radial line from the center of the gear to the point where the face meets the pitch circle. If the describing circle diameter is made smaller than the pitch radius of the pinion, the flank will fall outside of the radial line, giving a stronger form of tooth. If the diameter of the describing circle is made equal to the pitch radius of the pinion, the flanks described will be radial.

This is the compromise usually made. With equal describing circles, the path of contact becomes symmetrical with respect to the line of centers, the condition for interchangeability mentioned in Art. 126. Other obvious conditions for interchangeability are that all gears of the same pitch must have the same addendum and a tooth thickness equal to one-half the circular pitch.

132. Proportions of Cycloidal Spur Gear Teeth. It has just been shown that for a set of gears of the same pitch to be interchangeable, the describing circles must be equal in diameter. The Brown & Sharpe Company adopted for their system of cycloidal gears a diameter of describing circle equal to the pitch radius of the 15-tooth pinion of the set. This makes the flanks of the adjacent teeth of a 12-tooth pinion practically parallel but sufficiently convergent to permit the use of a rotating cutter. The William Sellers Company adopted a describing circle diameter equal to the pitch radius of the 12-tooth pinion, thus securing somewhat stronger tooth forms. In addition to having equal describing circles, all gears of the same pitch must have similarly proportioned teeth. The gears of a set must have the same addendum, and the depth of the tooth spaces must be twice the addendum plus a suitable clearance. For this reason interchangeable gears are referred to as *equal* addendum gears to distinguish them from *unequal* addendum gears.

Proportions of the teeth of cut and cast gears are given below in terms of the diametral pitch. The generally accepted values are given for cut gears, and the proportions for cast gears are quite representative of average practice. It is to be noted that for cast gears the clearance is increased and the thickness of the teeth is made less than the width of the tooth spaces to allow for inaccuracies of size, form, and spacing.

		CUT GEARS	CAST GEARS
Addendum	$= a$	$\dfrac{1.0}{p'_c}$	$\dfrac{1.0}{p'_c}$
Clearance	$= c$	$\dfrac{0.157}{p'_c}$	$\dfrac{0.25}{p'_c}$
Dedendum	$= d = a + c =$	$\dfrac{1.157}{p'_c}$	$\dfrac{1.25}{p'_c}$
Thickness of tooth $= t =$		$\dfrac{1.5708}{p'_c}$	$\dfrac{1.50}{p'_c}$
Backlash	$= b =$	zero	$\dfrac{0.1416}{p'_c}$

Using the addendum given above and the describing circle diameter of the Sellers system, it can easily be shown that two mating 12-tooth pinions would have a maximum obliquity of 20.304 degrees, and a contact ratio, or ratio of the arc of action to the circular pitch, of 1.3536. The angle of obliquity and contact ratio increase slightly as a 12-tooth pinion is mated with gears having a greater number of teeth than itself. If mated with a rack, the maximum obliquity becomes 20.405 degrees, and the contact ratio becomes 1.3603. Using the same addendum and the same diameter for the describing circles, it can be shown that the contact ratio and maximum obliquity for two 10-tooth pinions would be 1.319 and 19.785 degrees. In this case the diameter of the describing circles is 0.60 that of the pitch diameter of the pinions. A value for this ratio of 0.625 is usually regarded as a maximum. Where possible a value of 0.50 is to be preferred, since it gives teeth having radial flanks and of stronger form. Using a ratio of 0.50 and an addendum equal to the reciprocal of the diametral pitch, it can be shown that two 6-tooth pinions would have a

FIG. 289

contact ratio of 1.02 and a maximum obliquity of 30.6 degrees. Two 4-tooth pinions having pointed teeth and a diameter of the describing circles equal to 0.625 the pitch diameter of the pinions, would have a contact ratio slightly greater than unity and a maximum obliquity of 36 degrees.
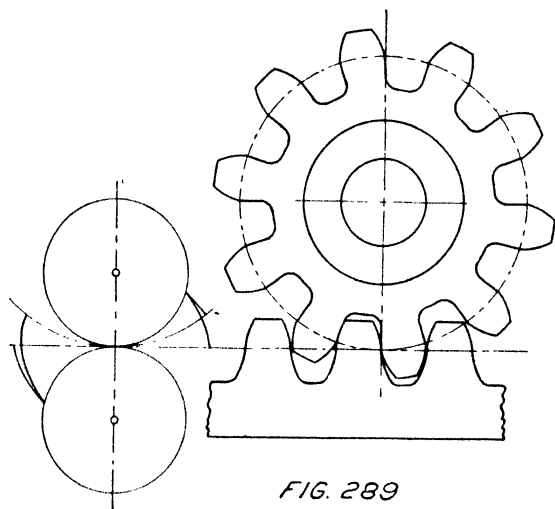
133. Cycloidal Rack and Pinion. If one of a pair of mating spur gears has a pitch radius of infinite length, its pitch circle becomes a straight line. A gear of infinite radius is called a *rack*, and its mate of finite radius is called a pinion, the combination being known as a rack and pinion. Such a combination is shown to the right in Fig. 289. The method of describing the tooth profiles is precisely the same as for a pair of gears of finite pitch radii. Since the pitch curve of the rack is a straight line, the faces and flanks are both *cycloids* and will be alike, if, as for interchangeable gears, only one size of describing circle is used.

In this event, any size of gear of the same pitch will mate properly with the rack. The method of describing the respective faces and flanks of the rack and pinion teeth are shown to the left in Fig. 289, the complete rack and pinion being shown to the right. Since a rack must be of limited length, the motion transmitted must be reciprocating.

134. Internal Cycloidal Gears. In mating spur gears so far considered, the parallel axes of the pitch cylinders or pitch circles were

FIG 290
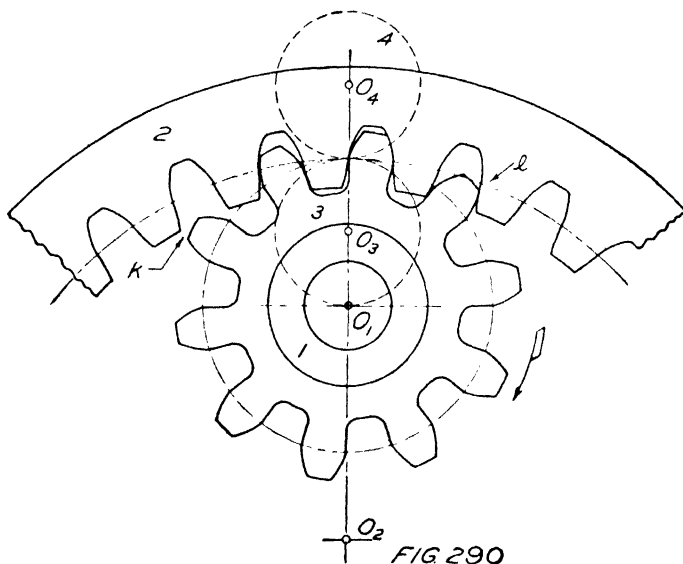
located on opposite sides of the pitch point, the pitch circles making contact externally. Such gears are referred to as *external gears*. Where the centers of the pitch circles are on the same side of the pitch point, the pitch circles make contact internally, and the gears are called *internal gears*. Such a pair of mating gears is shown in Fig. 290; the enveloping gear, only a part of which is shown, is sometimes referred to as an *annular gear*.

As shown in Fig. 290, the tooth profiles are described in the same manner as those for external gears. The result, however, is somewhat different. In external gears, epicycloidal faces mate with hypocycloidal flanks, while in internal gears, the mating faces and flanks are either both epicycloids or both hypocycloids. If the diameters of the pitch and describing circles are not properly related, the faces of the teeth of the pinion and gear may interfere, or foul, as the pinion teeth enter and leave the tooth spaces of the gear at k and l.

Reference must be made to the double generation of the epicycloid and the hypocycloid to determine whether or not the active and inactive faces of internal cycloidal gears will interfere. The faces of the pinion teeth of Fig. 290 were generated by rolling the describing circle 4 on the outside of the pitch circle of 1. The face ac of the pinion tooth of Fig. 291 was generated in a similar manner by rolling 4 from a to the left on 1. The face ac may also be generated by rolling the enveloping circle $4'$ to the left from a to P on 1, provided R_4', the radius of $4'$, is equal to the sum of the radii R_1 and R_4. Since $4'$ lies between the pitch circles 1 and 2, it is called the *intermediate describing circle of the pinion.*

The face bd of the tooth on the gear was generated by rolling the describing circle 3 from b to the right on the inside of 2, arc Pb having been taken equal to Pa. The face bd may also be generated by rolling the circle $3'$ to the right from b on the inside of 2 provided R_3', the radius of $3'$, is equal to the difference between the radii R_2 and R_3. Since $3'$ lies between the pitch circles 1 and 2, it is called the *intermediate describing circle of the gear.* If, as in Figs. 290 and 291, the intermediate describing circle of the pinion is smaller than the intermediate describing circle of the gear, the faces of the pinion teeth will not touch or foul the faces of the teeth of the gear as the pinion teeth enter and leave the tooth spaces of the gear. If, as in Fig. 292, the intermediate describing circles $4'$ and $3'$ are of the same radius, the faces of the pinion and gear teeth will touch but will not foul. Only in this case is the path of contact extended beyond that given by the describing circles 3 and 4. The active and inactive faces of the teeth of the pinion and gear will, however, foul, if, as in Fig. 293, the radius R_4' of the intermediate describing circle of the pinion is greater than R_3', the radius of the intermediate describing circle of the gear. Hence, if the active and inactive faces are not to touch or foul,
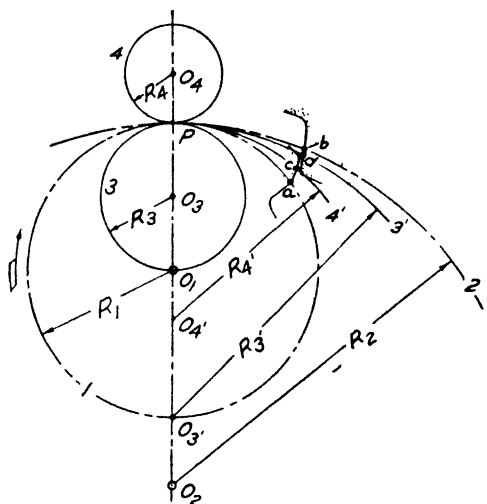
FIG. 291

$$(R_1 + R_4) \text{ must be less than } (R_2 - R_3) \quad . \quad . \quad . \quad (209)$$

If the describing circles *3* and *4* are of the same size, then

$$(R_3 = R_4) \text{ must be less than } \tfrac{1}{2}(R_2 - R_1) \quad . \quad . \quad (210)$$

135. Pin Gears. In Fig. 294 a generating circle having a diameter equal to the pitch diameter of *2* is used for pitch circles *1* and *2*. A point on this generator will describe an epicycloid if rolled on the outside of *1*, and the hypocycloid described on *2* by the same generator will be a point. Obviously, if the epicycloid could act on a mere point, the motion transmitted would be equivalent to the rolling of the pitch circles.

For clockwise rotation of *1*, action will cease at *c* where the addendum circle of *1* intersects the generating circle, which is also the pitch circle, of *2*. Hence, arc *Pc* is the path of contact, also the arc of action. Since *Pc* is greater than the pitch arc, action will be continuous. If the circular pitch and the relation of the pitch circles were such that point *d* would fall on the generating circle, the ratio of the arc of action to the circular pitch would be unity, the theoretical limit for continuous driving. So long as *d* falls within the generating circle, continuous driving is assured. It is to be noted that with *1* as the driver the entire action is in recess, and with *2* as the driver it is in approach. Since action is smoother during recess than approach, it is always more satisfactory to have the pin gear the follower.
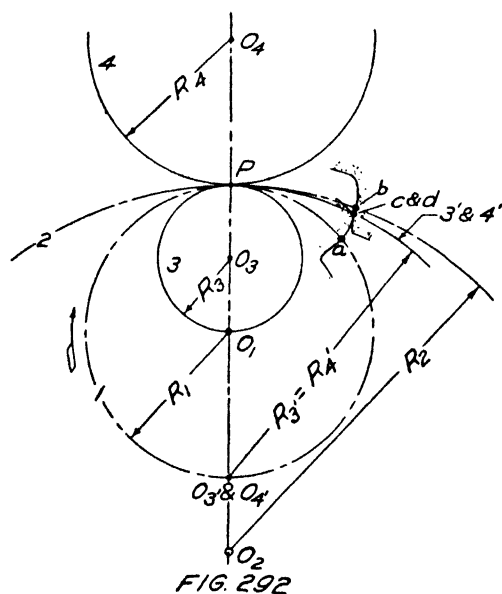
FIG. 292

To get actual driving it is necessary to fit gear *2* with pins of sensible diameter, as indicated in Fig. 294. The tooth profiles of *1* then become curves parallel to the epicycloids, the curves being tangent to a series of arcs having a radius equal to the radius of the pins and drawn from centers in the epicycloids. In the figure these curves are shown dotted. The common normal *Pe* intersects the tooth profile at *f*. If the dotted tooth profiles are topped off by an arc through *f* from center O_1, the pin, whose center is at *e*, would be just quitting action. If it can be assumed that the pin whose center is at *P* is just coming into action, then the
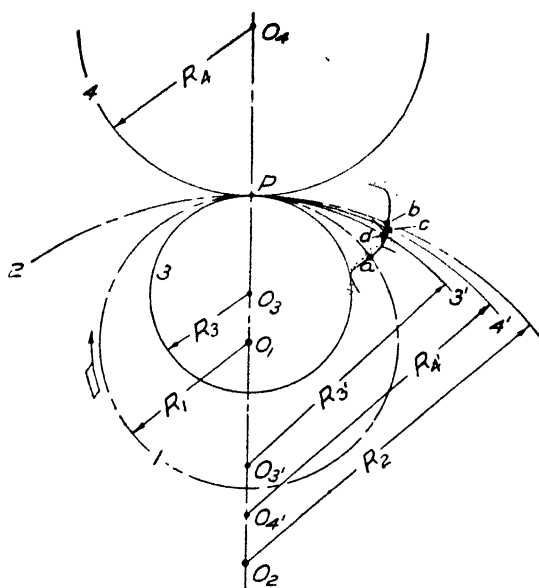
FIG. 293
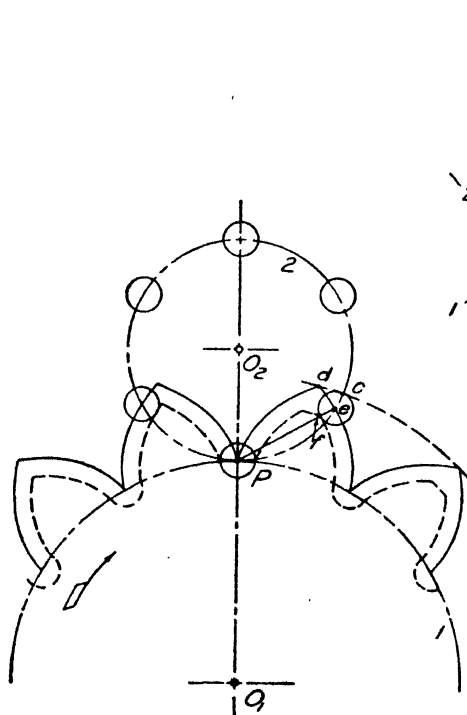
FIG. 295
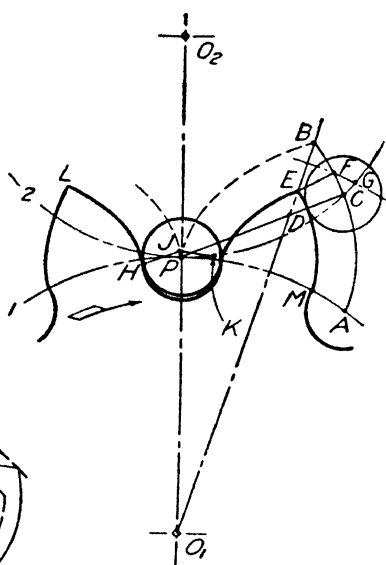
FIG 294

arc of action would be equal to the circular pitch and continuous driving theoretically possible. This condition would not, however, obtain, since the dotted profile parallel to the epicycloid would not make contact with the pin until its center had passed the pitch point, P.

The points on the profiles are at the ends of normals to the epicycloids, the length of the normals being equal to the radius of the pins. If one end of a line equal to the radius of the pins is moved along while remaining normal to the epicycloid, the other end will trace the profile curve desired. Since this moving radius will first be tangent to the pitch circle at P, the first point on the curve will lie outside of the pitch circle on this tangent. As the radius is moved outward from P along the epicycloid, the tracing end of the moving radius will at first move inward and then outward forming a cusp inside the pitch circle. As only one branch, the outward branch, of the traced curve can be used as the tooth profile, contact with the pin cannot begin with the pin center at P. The angle of approach, which is small in any event, is thereby reduced, also the arc of action. Whether or not practical conditions for continuous driving exist can only be determined graphically.

In Fig. 295 is shown a construction for determining the size of pin and conditions for continuous driving. The point B of the epicycloidal tooth must fall within the generating and pitch circle 2 as shown, and the radius of the pin must be made less than CD, D being the intersection of the normal CP with the radial line O_1B. If the radius of the roller were taken equal to CD, the tooth point would be at D and therefore just quitting contact with the pin. Draw from E a line EF normal to the epicycloid AB. An arc through F with O_1 as a center determines G, the position of the pin center when the tooth profile is quitting contact with the pin. Point H is where the outward branch of the parallel curve, traced as explained above, cuts the circumference of the pin whose center is at P. Line HJ is the moving radius normal to the epicycloid at J. An arc through J with O_1 as a center intersects the generating circle 2 at K. When J has moved to K the pin will be in contact with the profile HL at H. If KG is less than PC, the radius of the pin is too large; if it is equal to or greater than PC, the radius of the roller is satisfactory. When KG is greater than PC the tooth ME may be topped off.

Pin gears may be two mating gears as shown in Fig. 294, or they may be in the form of a rack and pinion, or they may constitute a pair of internal gears. In any event, as before pointed out, the follower and not the driver should be fitted with pins for smoothest action. Such gears are very little used except for instruments and clockwork.

136. Involute Spur Gear Teeth. In Fig. 296 a line *ab* is shown tangent to the two base circles, *1* and *2*, having their centers at O_1 and O_2, respectively. If the tangent line were pulled to the right with pure rolling between it and the base circles, the base circles would be rotated through equal arcs of a length equal to the distance moved by
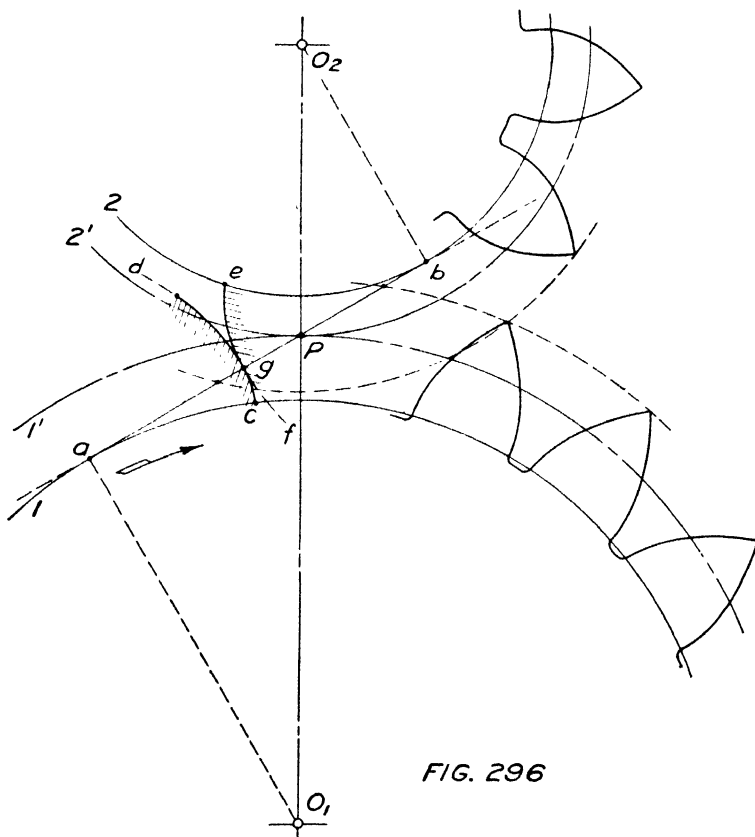

FIG. 296

the tangent line, the motion of the base circles being the same as though they rolled together in contact. If a thin band ran from the periphery of *2* to the periphery of *1*, counter-clockwise rotation of *2* would cause the same relative motion of *1* and *2*. During this motion any point such as *g* on the moving straight line would trace an involute curve *cd* on *1* and an involute *ef* on *2*. This can be shown by parting the line *ab* at any point such as *g* and considering the relative motion between a straight line and a circle. Rolling a straight line on a stationary circle gives the same relative motion between the line and the circle as rotating the circle and having the straight line move in a fixed straight path

as though the pitch line of a rack. Hence, with *1* stationary, the rolling of *ga* on it gives the same relative motion between the line and circle as before. Obviously the free end of the rolling straight line *ga* will trace the involute *cd* on *1*, and the free end of *gb* will trace the involute *ef* if rolled on *2*. At the instant shown, *a* is the instant center of *ga* and *b* the instant center of *gb*. The direction of motion of the free end of each segment is therefore perpendicular to *ab*; the involutes being described are tangent at *g*; and the line *ab* is their common normal. Hence, with *cd* and *ef* as the respective tooth profiles of *1* and *2*, the common normal will, at any instant, be tangent to the base circles and will cut the line of centers at the fixed point *P*. The involute curves therefore fulfill the fundamental requirement for tooth profiles and will transmit motion from one wheel to the other equivalent to the pure rolling of the base circles. The constant angular velocity ratio of the motion transmitted is not only inversely as the radii of the base circles but also inversely as the segments into which the line of centers is cut by the common normal. Hence the motion transmitted is also equivalent to the rolling together of the two nominal pitch circles *1'* and *2'*.

Since the common normal through the point of contact of involute profiles is always tangent to the base circles, it follows that the point of contact must follow this tangent line. The beginning of contact is where the addendum circle of the driven gear intersects the tangent to the base circles, and the end of contact is where the addendum circle of the driver intersects this tangent. The path of the point of contact is therefore that portion of the tangent line between these two points.

137. Action of Involute Spur Gear Teeth. (a) *Path of the Point of Contact.* It was shown in the preceding article that the point of contact of involute teeth travels along a line tangent to the base circles. It is therefore evident, as shown in Fig. 297, that *ab* is the maximum length possible for the path of contact. For contact to begin at *a* and end at *b*, the addendum radii of *2* and *1* would have to be O_2a and O_1b, respectively. For the pitch and the distance between centers and the base circles chosen, this maximum length cannot be realized. Assuming that *1* acts as the driver, that *2* and *1* are to have 12 and 20 teeth, respectively, and that the teeth are carried to a point, contact would begin at *m* and cease at *n*; as for any kind of teeth, contact begins where the addendum circle of the driven wheel intersects the path of contact, and contact ceases where the addendum circle of the driver intersects the path of contact.

(b) *Arc and Angle of Action.* The profiles of a pair of mating teeth in contact at *m* at the beginning and at *n* at the end of action are shown dotted. The dotted profiles *cmd*, *c'P*, and *c''d''n* are traced by points

m, P, and n as ab is rolled on 1, and the dotted profiles $e''nf''$, $e'P$, and efm as ba is rolled on 2. Evidently dd'' and ff'' are the respective pitch arcs turned through by gears $1'$ and $2'$ during action between a pair of mating teeth, and are, therefore, the arcs of action. Accordingly dP and Pd'' are the respective arcs of approach and recess of $1'$, and fP and Pf'' the respective arcs of approach and recess of $2'$. Since the action
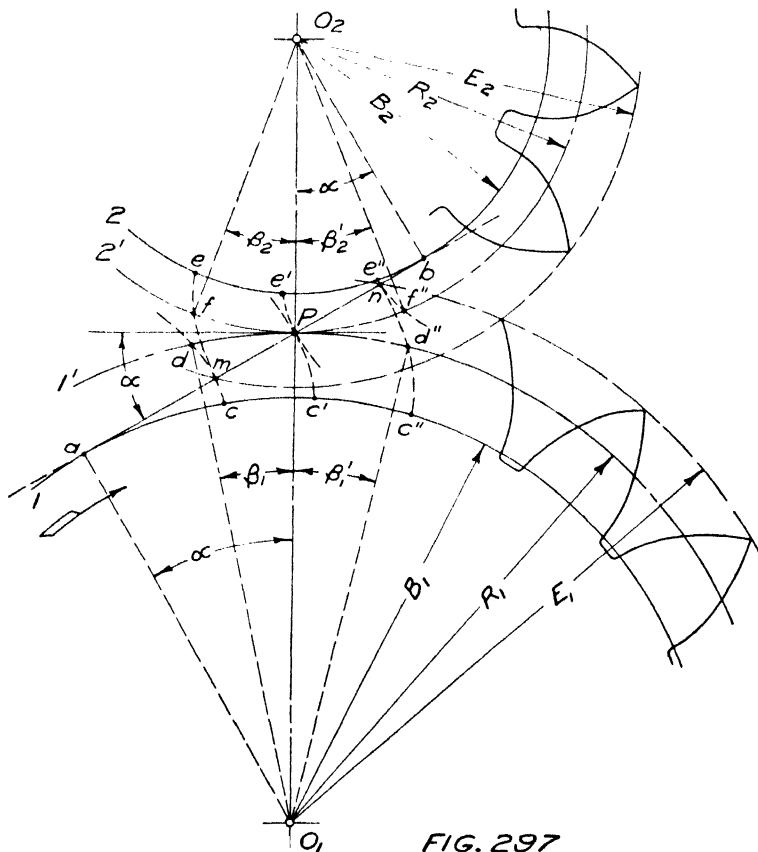

FIG. 297

of mating gears is equivalent to the rolling of the base or pitch circles, it follows that the respective arcs of action, approach, and recess are equal. Hence $dd'' = ff''$, $dP = fP$, and $Pd'' = Pf''$. The angles of action, approach, and recess of either the driver or the follower are the angles corresponding to the arcs of action, approach, and recess. The respective angles of approach and recess are β_1 and β_1' for $1'$, and β_2 and β_2' for $2'$.

(c) *Angle of Obliquity and Pressure Angle.* The angle of obliquity α is the angle between the common normal and a line through the pitch

point perpendicular to the line of centers. This angle for involutes is constant since the path of contact follows the tangent to the base circles. Since the obliquity increases the pressure between the teeth and at the bearings, the angle of obliquity should be no greater than that required for satisfactory and continuous driving. For involute teeth this angle is constant during action, and about 30 degrees has come to be regarded as the permissible maximum. Neglecting the friction of sliding, the 'pressure between the teeth will act along a line tangent to the base circles. Hence the pressure will be equal to that corresponding to the torque to be transmitted divided by the cosine of the angle of obliquity. The per cent increase in the pressure due to the obliquity will therefore be $100\left(\dfrac{1}{\cos \alpha} - 1\right)$, which equals slightly less than 15.5 per cent for an angle of 30 degrees.

As before pointed out, the effect of the friction of sliding between the teeth is to make the pressure angle greater than the angle of obliquity during approach and less during recess. This, to some degree, accounts for the action being smoother during recess than during approach. The principal reason for the smoother action, as before mentioned, is the difference in the direction of sliding between the teeth during approach and recess. During approach, the teeth, while sliding on each other, are pushing into mesh, while during recess they are drawing out of mesh. Gears would therefore be more efficient and durable if the action were confined to recess. However, continuous driving and other requirements will often make such a limitation of action, at least for straight spur gears, either impossible or undesirable. As before stated, the best tooth proportions for any given conditions of operation are generally a compromise between requirements and possibilities.

(*d*) *Base Pitch.* The base pitch, p_b, is the distance, measured on the base circle, from a point on one tooth to the corresponding point on the next tooth. In order to run together, it is a fundamental requirement of involute gears that they have the same base pitch. As pointed out in Art. 128, it is the constant normal distance between adjacent involute profiles. In terms of the number of teeth N and the radius of the base circle B, the base pitch is,

$$p_b = \frac{2\pi B}{N} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

But B in terms of the pitch circle radius R and the angle of obliquity α is,

$$B = R \cos \alpha$$

Substituting in (211) and solving gives p_b in terms of the circular pitch p_c and the angle of obliquity α:

$$p_b = \frac{2\pi R \cos \alpha}{N} = p_c \cos \alpha \quad \dots \quad (212)$$

(c) *Continuous Driving and Length of Teeth.* For continuous driving one pair of teeth must come into action before the preceding pair goes out of action. This means that the ratio of the arc of action to the circular pitch or the ratio of the length of the path of contact to the base pitch should be greater than unity. ·As before stated, this ratio, called the contact ratio, should be, if possible, 1.40 or more for durability and quiet running in the transmission of any appreciable amount of power. The nearer this ratio approaches unity, the higher the degree of accuracy required in the tooth profiles to secure quiet running. For this reason 1.20 should be regarded as a minimum value for this ratio. For the wheels shown in Fig. 297, where the angle of obliquity is 30 degrees, this ratio is almost 1.45.

It is evident by inspection of the figure that the longer the teeth the greater will be the length of the path of contact and the greater the number of pairs of teeth in action. Since, for involutes, contact cannot extend within the base circles, the longest teeth possible are fixed by the points a and b where the line of action is tangent to the base circles. The largest possible size of addendum circles would therefore pass through these points, and the teeth of each gear could be made of corresponding length, provided, for the number of teeth desired, the teeth did not come to points within these circles.

138. Distance between Centers of Involute Spur Gears. One of the most valuable properties of involute profiles for gear teeth is that the distance between centers of mating gears may be varied without changing the value of the angular velocity ratio or its constancy. This property, which is peculiar to involute gears, is very valuable in the design and building of machines. It provides tolerance in building, and makes it possible to have change gears function properly without exact adjustment of the center distances. It also permits, where required, a change in the center distance while in operation. For a pair of cycloidal gears, there is only one theoretically correct distance for which they will transmit motion at a constant angular velocity ratio.

In Fig. 298 the involutes for the base circles *1* and *2* are shown in contact at *g* in the common tangent or path of contact *ab*. If the center O_1 of base circle *1* is moved to O'_1, the base circle moving to *1'*, the

common tangent or path of contact becomes $a'b'$. Since the diameters of the base circles have not been changed, the involutes have not been changed. The new point of contact of the involutes is, therefore, at g' where the involute of 2 intersects the path of contact $a'b'$. The common normal for the first position of the base circles cuts the line of centers at P, and for the second position at P'. It is evident by inspection that triangles PO_1a and PO_2b are similar, likewise $P'O'_1a'$ and $P'O_2b'$. Hence,

$$\frac{O_2P}{O_1P} = \frac{O_2b}{O_1a}$$

and

$$\frac{O_2P'}{O'_1P'} = \frac{O_2b'}{O'_1a'}$$

But $O_2b' = O_2b = B_2$ and $O'_1a' = O_1a = B_1$; whence,

$$\frac{O_2P}{O_1P} = \frac{B_2}{B_1}$$

and

$$\frac{O_2P'}{O'_1P'} = \frac{B_2}{B_1}$$

(213) and (214)

FIG. 298

Since the angular velocity ratio of two mating gears is inversely as the segments into which the line of centers is cut by the common normal, it follows from (213) and (214) that the angular velocity ratio of the involute gears in Fig. 298 is the same in the second as in the first position. Hence,

$$\frac{\omega_1}{\omega_2} = \frac{O_2P}{O_1P} = \frac{O_2P'}{O'_1P'} = \frac{B_2}{B_1} \qquad \cdots \cdots \quad (215)$$

The angular velocity ratio of mating involute gears is therefore deter-

mined by the diameters of the base circles and not by the distance between centers.
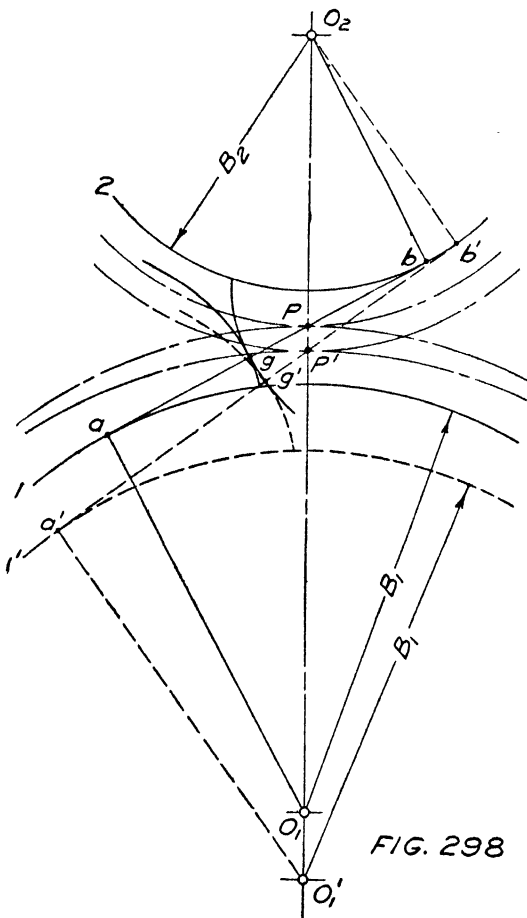
It is to be noted that as gear 1 is drawn away from gear 2, the angle of obliquity, the diameters of the pitch circles, the circular pitch, and the backlash are increased. Since the length of the path of contact, and therefore the arc of action, are determined by the intersection of the addendum circles with the common tangent, it is also to be noted that the path of contact is reduced as the gears are drawn apart. The proportions of the teeth and the conditions for positive driving therefore determine how far apart a pair of mating involute gears may be drawn. Theoretically they may be drawn apart until the ratio of the path of contact to the base pitch is reduced to unity. The minimum distance between centers is that which will give zero backlash.

139. Interchangeable Involute Spur Gears. It is evident from the preceding discussion that, for any given angular velocity ratio and distance between centers, the diameters of the base circles are fixed by the angle of obliquity desired. Only when this angle is fixed is there a definite relation between the pitch and base circles. ·For a given angle of obliquity, any two gears of a set of gears of the same pitch will mate properly, provided all of the gears of the set have the same addendum and a tooth thickness equal to one-half the circular pitch. *Hence, for interchangeability, involute gears of the same pitch must have the same angle of obliquity, the same addendum, and a tooth thickness equal to one-half the circular pitch.*

In fixing the proportions of interchangeable involute gears, the angle of obliquity and the addendum must be so chosen that the smallest gear of the set will mesh with a rack without interference, and that a pair of the smallest gears of the set will mesh and drive continuously. As will be shown, a pinion that will mesh with a rack without interference will mesh without interference with any other gear of the set having an equal or larger number of teeth than itself. It will also be shown that if a pair of the smallest gears of a set will drive continuously, any other pair of gears of the same pitch will drive continuously.

140. Involute Rack and Pinion. The involute described by a straight line rolling on a circle more and more nearly approaches a straight line as the radius of the circle upon which it rolls is increased, the involute becoming a straight line for a circle of infinite radius. Hence, the involute profiles of the teeth of a rack, since a rack is a portion of a gear of infinite radius, are straight lines. Since, for involute teeth, the common normals during action coincide with a line through the pitch point tangent to the base circles, the straight-sided rack tooth

profiles must be perpendicular to this common normal. As the angle between the common normal and the pitch line of the rack is the angle of obliquity α, the straight-sided rack tooth profiles will be at an angle of $(90 - \alpha)$ with the pitch line of the rack. The rack and pinion shown in Fig. 299 were drawn for an angle of obliquity of $22\frac{1}{2}$ degrees.

141. Interference. Since the involute curve begins at the base circle, no involute action can take place within the base circle. Hence,
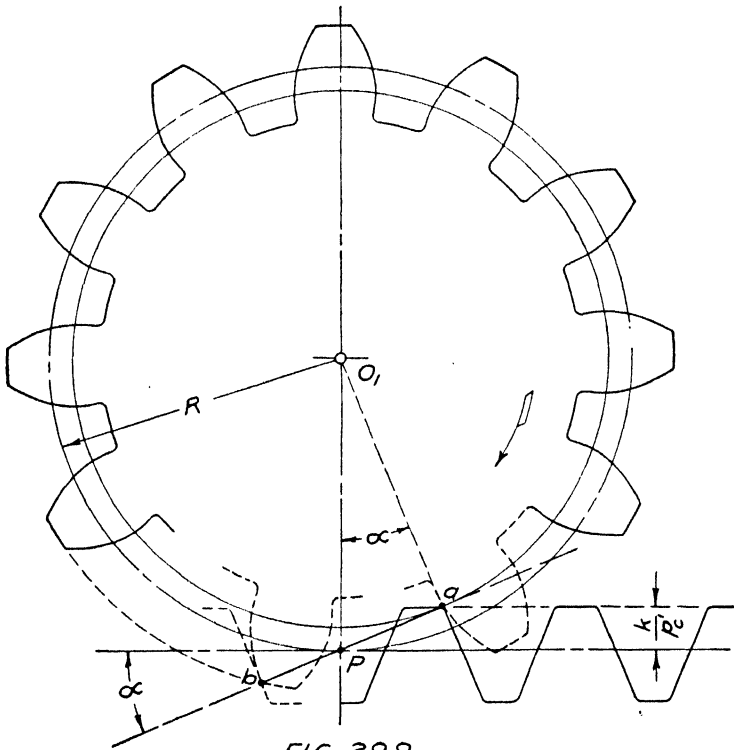
FIG. 299

for two pitch circles, the points of tangency of the line of obliquity with the base circles determine the maximum addendum for each of the pair to work together without interference. Thus, for the rack and pinion in Fig. 299, point a, for a given angle of obliquity α, fixes the maximum addendum for the rack. The addendum assigned to the pinion fixes point b. For interchangeable gears, the addendum of the pinion is made the same as that of the rack.

From the base circle inward the flanks of the pinion teeth in Fig. 299 were made radial. If the addendum of the rack teeth in Fig. 299 were made greater than shown, there would be interference, and action

between the rack and pinion would be impossible without undercutting the pinion teeth as shown in Fig. 300. The path of the corner of the rack tooth which determines the outline of the undercutting of the pinion teeth may be obtained by rolling the pitch line of the rack on the pitch circle of the pinion. It is to be noted that the extended rack tooth would cut away the flank for a certain distance outside as well as inside of the base circle. Hence, besides weakening the teeth, undercutting reduces the arc of action. For these reasons excessive undercutting is to be avoided.

It is evident by reference to Fig. 299, that if a gear of finite radius
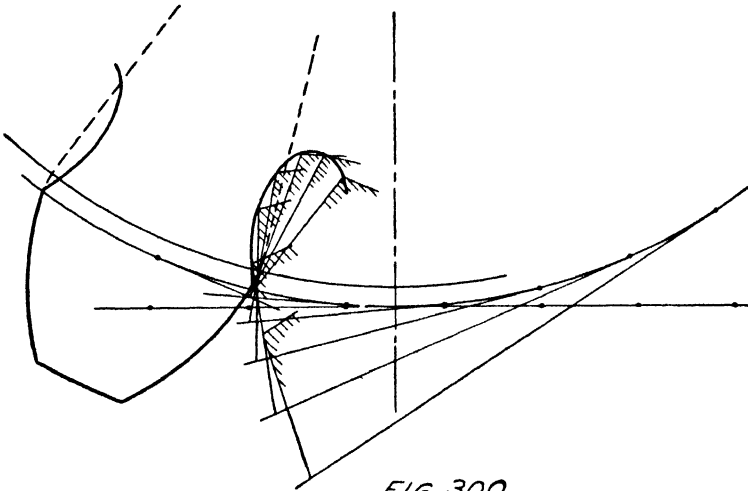
FIG. 300

had the same addendum as the rack and were to mesh with the pinion, action would begin at a point on the line of obliquity nearer P than a. Hence, if the number of teeth on the pinion is such that it will mesh with a rack without interference, it will mesh without interference with any other gear having the same or a greater number of teeth than itself. It is also evident, for a given rack tooth addendum, that the number of teeth must be such that action will begin at a or between P and a if interference is to be avoided. Referring to Fig. 299,

$$\sin \alpha = \frac{(Pa)}{R} \quad \text{and} \quad \sin \alpha = \frac{\dfrac{k}{p'_c}}{(Pa)}$$

The product of these two equations gives

$$\sin^2 \alpha = \frac{k}{Rp'_c} \qquad \cdots \cdots \quad (216)$$

But p'_c in terms of the number of teeth N and the pitch radius R of the pinion is

$$p'_c = \frac{N}{2R}$$

Substituting this value of p'_c in (216) and solving for N gives

$$N = \frac{2k}{\sin^2 \alpha} \qquad \cdots \cdots (217)$$
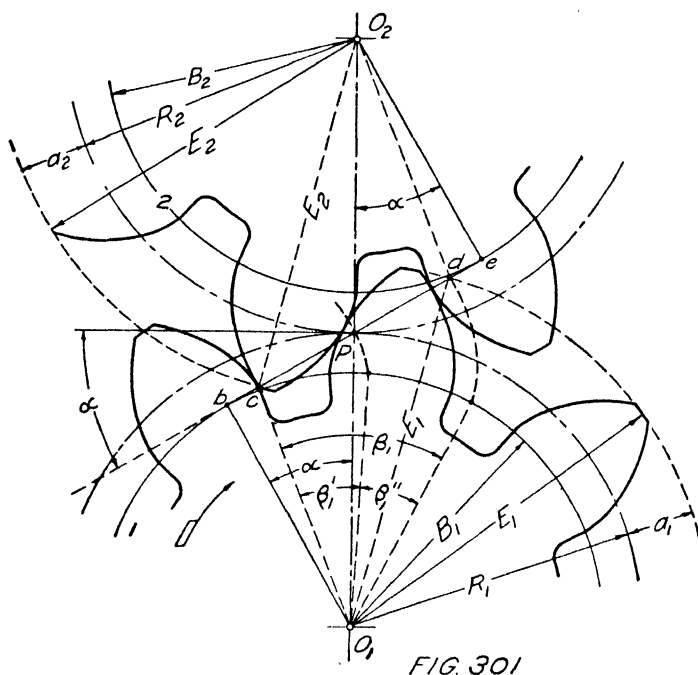
FIG. 301

By the use of equation (217) the smallest number of teeth for a pinion that will mesh with a rack without interference can be determined for any given addendum and angle of obliquity. Solving (217) for k gives

$$k = \frac{N \sin^2 \alpha}{2} \qquad \cdots \cdots (218)$$

By use of equation (218), the maximum addendum for non-interference can be determined for any given angle of obliquity and number of teeth on the pinion. For Fig. 299, where the pinion has 12 teeth and the angle of obliquity is $22\frac{1}{2}$ degrees, k equals 0.878.

142. Duration of Contact. Not only must involute gears be free from interference, but the duration of contact must be sufficient to insure continuous driving; that is, one pair of teeth must come into contact before the preceding pair goes out of contact. This means that the ratio of the base arc of action, or length of the path of contact, to the base pitch must be greater than unity. In Fig. 301, where *1* is the driver and *2* the driven gear, let

B_1 and B_2 = the respective radii of the base circles of *1* and *2*;

R_1 and R_2 = the respective pitch radii of *1* and *2*;

E_1 and E_2 = the respective outside, or external, radii of *1* and *2*;

a_1 and a_2 = the respective addenda of *1* and *2* $= \dfrac{k_1}{p'_c}$ and $\dfrac{k_2}{p'_c}$

α = the angle of obliquity;

β_1 = the angle of action of *1*;

β'_1 = the angle of approach of *1*;

β''_1 = the angle of recess of *1*;

N_1 and N_2 = the respective numbers of teeth of *1* and *2*;

p_b = the base pitch.

It has already been shown that the length of the path of contact cd is equal to the product of the angle of action and the radius of the base circle, or, $cd = \beta_1 B_1$. Hence the ratio of cd to the base pitch is the contact ratio n, or average number of pairs of teeth in action. Hence,

$$n = \frac{cd}{p_b}$$

Referring to Fig. 301,

$$cP = ce - Pe = \sqrt{E_2^2 - B_2^2} - R_2 \sin \alpha$$

and

$$Pd = bd - bP = \sqrt{E_1^2 - B_1^2} - R_1 \sin \alpha$$

Whence,

$$cd = cP + Pd = \sqrt{E_2^2 - B_2^2} + \sqrt{E_1^2 - B_1^2} - (R_1 + R_2) \sin \alpha$$

Hence,

$$n = \frac{cd}{p_b} = \frac{1}{p_b}[\sqrt{E_2^2 - B_2^2} + \sqrt{E_1^2 - B_1^2} - (R_1 + R_2) \sin \alpha] \quad (219)$$

But from equation (212)

$$p_b = p_c \cos \alpha = \frac{\pi \cos \alpha}{p'_c}$$

also,

$$R_1 = \frac{N_1}{2p'_c} \quad \text{and} \quad B_1 = R_1 \cos \alpha = \frac{N_1 \cos \alpha}{2p'_c}$$

$$R_2 = \frac{N_2}{2p'_c} \quad \text{and} \quad B_2 = R_2 \cos \alpha = \frac{N_2 \cos \alpha}{2p'_c}$$

$$E_1 = R_1 + a_1 = \frac{N_1}{2p'_c} + \frac{k_1}{p'_c} = \frac{N_1 + 2k_1}{2p'_c}$$

$$E_2 = R_2 + a_2 = \frac{N_2}{2p'_c} + \frac{k_2}{p'_c} = \frac{N_2 + 2k_2}{2p'_c}$$

Substituting the above values of p_b, B_1 and B_2, R_1 and R_2, and E_1 and E_2 in (219), and solving, gives

$$n = \frac{1}{2\pi}\left[\sqrt{\left(\frac{N_2+2k_2}{\cos \alpha}\right)^2 - N_2{}^2} + \sqrt{\left(\frac{N_1+2k_1}{\cos \alpha}\right)^2 - N_1{}^2} - (N_1+N_2)\tan \alpha\right]$$

$$(220)$$

where $k_1 = a_1 p'_c$ and $k_2 = a_2 p'_c$. If $N_2 = N_1 = N$, and $k_2 = k_1 = k$, equation (220) reduces to

$$n = \frac{1}{\pi}\left[\sqrt{\left(\frac{N + 2k}{\cos \alpha}\right)^2 - N^2} - N \tan \alpha\right] \qquad \ldots \ldots \ldots (221)$$

The above equations hold where each gear of the pair has a finite number of teeth and where $(ce - Pe)$ is not greater than bP nor $(bd - bP)$ greater than Pe; or where $(\sqrt{E_2{}^2 - B_2{}^2} - R_2 \sin \alpha)$ is not greater than $R_1 \sin \alpha$ nor $(\sqrt{E_1{}^2 - B_1{}^2} - R_1 \sin \alpha)$ greater than $R_2 \sin \alpha$.

It has been shown that k must be less for a gear to mesh with a rack without interference than to mesh with a gear having an equal or greater number of teeth than itself. Hence the maximum addendum to be used in the above equations is determined by equation (218) if one gear is to mesh with an equal or larger gear than itself without interference and without undercutting.

It is to be noted that if two equal gears will drive continuously, either one will give continuous driving when mated with any other gear larger than itself. Hence, if two of the smallest gears of an interchangeable system will give continuous driving, any pair of gears of the system will drive continuously.

143. Standard Interchangeable Involute Gear Teeth. Previous to 1921 a committee of the A.S.M.E., appointed for the purpose, recommended a basic involute rack tooth form for general adoption having an angle of obliquity of $22\frac{1}{2}$ degrees. The proportions recommended are shown in Fig. 302. Since any two gears that will mesh with a rack without interference will mesh with each other without interference, the rack tooth proportions shown may be used as a basis for an interchangeable system of involute gears. The first column of Table 5 gives the limits of such a system. It is to be noted that a pinion having 12 teeth is the smallest that will mesh with the rack without interference and without undercutting the flanks of the pinion teeth. A pair of such pinions will drive continuously and satisfactorily since the contact ratio would be 1.218. Under this system, pinions having a smaller number of teeth than 12 would be undercut after the manner shown in Fig. 300, thus weakening the teeth and somewhat reducing the duration of contact. Two 10-tooth pinions are, the smallest that will give continuous driving, the contact ratio being 1.175.
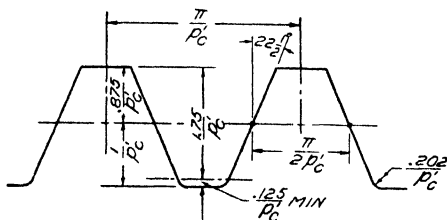
FIG. 302

In January, 1927, the American Standards Association approved two spur gear tooth forms as Tentative American Standards. This work was sponsored by the A.S.M.E. and the American Gear Manufacturers' Association. The proportions approved for the basic involute rack are shown in Fig. 303, and the proportions approved for the basic rack of the composite system are shown in Figs. 304 and 305.
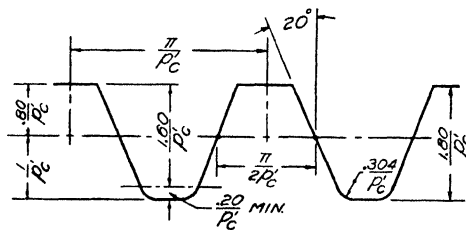
FIG 303

The limits of the A.S.A. 20-degree stub involute tooth are given in the second column of Table 5. It is to be noted that a pinion having 14 teeth is the smallest that will mesh with a rack without interference and without undercutting the flanks of the pinion teeth. A pair of such pinions will drive continuously and satisfactorily since the contact ratio would be 1.217. Under this system, pinions having a smaller number of teeth than 14 would be undercut after the manner shown in

TABLE 5

COMPARISON OF INTERCHANGEABLE INVOLUTE SYSTEMS

		A.S.M.E. $22\frac{1}{2}°$ Tooth	A.S.A. 20° Stub Tooth	Full Depth $14\frac{1}{2}°$ Tooth	Full Depth A.S.A. Composite Tooth
1	Angle of obliquity $= \alpha =$	$22\frac{1}{2}$	20	$14\frac{1}{2}$	$14\frac{1}{2}$
2	Addendum $= a = \dfrac{k}{p'_c} =$	$\dfrac{0.875}{p'_c}$	$\dfrac{0.80}{p'_c}$	$\dfrac{1.0}{p'_c}$	$\dfrac{1.0}{p'_c}$
3	Working depth $= 2a = \dfrac{2k}{p'_c} =$	$\dfrac{1.75}{p'_c}$	$\dfrac{1.60}{p'_c}$	$\dfrac{2.0}{p'_c}$	$\dfrac{2.0}{p'_c}$
4	Minimum whole depth $=$ $h = 2a + c =$	$\dfrac{1.875}{p'_c}$	$\dfrac{1.80}{p'_c}$	$\dfrac{2.157}{p'_c}$	$\dfrac{2.157}{p'_c}$
5	Minimum clearance $=$ $c = h - 2a =$	$\dfrac{0.125}{p'_c}$	$\dfrac{0.20}{p'_c}$	$\dfrac{0.157}{p'_c}$	$\dfrac{0.157}{p'_c}$
6	Number of teeth in smallest pinion to mesh with rack without interference (Equation 217)........	12	14	32	12*
7	Maximum addendum for a 12-tooth pinion to mesh with rack without interference (Equation 218).....	$\dfrac{0.878}{p'_c}$	$\dfrac{0.7018}{p'_c}$	$\dfrac{0.376}{p'_c}$	$\dfrac{1.0}{p'_c}$*
8	Number of pairs of teeth in action for a pair of 12-tooth pinions having the addendum of line 7 (Equation 221). Should be greater than unity for continuous driving	1.223	1.061	0.7392	1.375*
9	Number of pairs of teeth in action for a pair of equal pinions having the number of teeth of line 6 and the addendum of line 2 (Equation 221).........................	1.218	1.217	1.972	1.375*

* Equations 217, 218, and 221 do not apply to $14\frac{1}{2}$-degree composite system.

Fig. 300, thus weakening the teeth and somewhat reducing the duration of contact. Two 12-tooth pinions are the smallest that would give continuous driving, the contact ratio being 1.049. By shortening the

addendum to $0.7018/p'_c$, undercutting is eliminated and the duration of contact slightly increased, the contact ratio being increased from 1.049 to 1.061.

In the third column of Table 5 are shown the limits for a full depth $14\frac{1}{2}$-degree involute system. For such a system a pinion having 32 teeth is the smallest that will mesh with a rack without interference and without undercutting the flanks of the pinion teeth. Under this system, pinions having a smaller number of teeth than 32 would be undercut after the manner of Fig. 300, thus weakening the teeth and reducing the duration of contact. Two 20-tooth pinions are the smallest that would give continuous driving, the contact ratio being $n = 1.08$. For n to be greater than unity for two mating gears, the sum of the numbers of teeth on the pinion and gear must be 39 or greater with the pinion having not less than 15 teeth. For n to be equal to or greater than 1.40, the sum of the numbers of teeth must be 45 or greater with the pinion having not less than 20 teeth. By reducing the addendum to $0.53/p'_c$, the number of teeth in the smallest pair of pinions to give continuous driving may be reduced from 20 to 17, for which n would equal 1.044. Hence, a strictly $14\frac{1}{2}$-degree involute cannot be used as a basis for an interchangeable set of gears if it is desired to have a 12-tooth pinion as the smallest gear of the system. In general the full depth $14\frac{1}{2}$-degree involute system is best adapted to large gears having 40 teeth or more, while the 20-degree stub tooth system is best adapted to gears having a small number of teeth.

In the $14\frac{1}{2}$-degree composite system, which adopts the 12-tooth pinion as the smallest gear of the set, the tooth profile of the basic rack is a combination of involute and cycloidal curves. The last column of Table 5 gives the limits of the A.S.A. $14\frac{1}{2}$-degree composite system. In Fig. 304 the line bPc is drawn at an angle of $14\frac{1}{2}$ degrees with the pitch line 2 of the rack, circle 1 being the pitch circle of a 12-tooth pinion with its center at O_1. The corresponding base circle 3 is tangent to line bPc at c. Since there can be no involute action beyond c, the distance ce is the maximum addendum of a $14\frac{1}{2}$-degree involute rack tooth, the computed value for which, as before mentioned, is $0.37614/p'_c$. Since PcO_1 is evidently a right triangle, point c lies on a circle, 4, having a diameter equal to the radius of the 12-tooth pinion. Hence, if kl is described by rolling 4 on pitch line 2 and fg is described by rolling 4 on the inside of 1, the cycloidal face kl of the rack tooth will mesh properly with the hypocycloidal flank fg of the pinion tooth. Beyond c, the path of contact, cd, would therefore follow the describing circle 4. The flank mn of the rack tooth will mate properly with the face bh of the pinion if the flank mn is made a cycloid described by rolling 5 on pitch line 2,

and the face bh is made an epicycloid by rolling 5 on the outside of 1. The rack tooth profile is therefore cycloidal from n to m, involute from m to k, and cycloidal from k to l, while the pinion tooth profile is hypo-
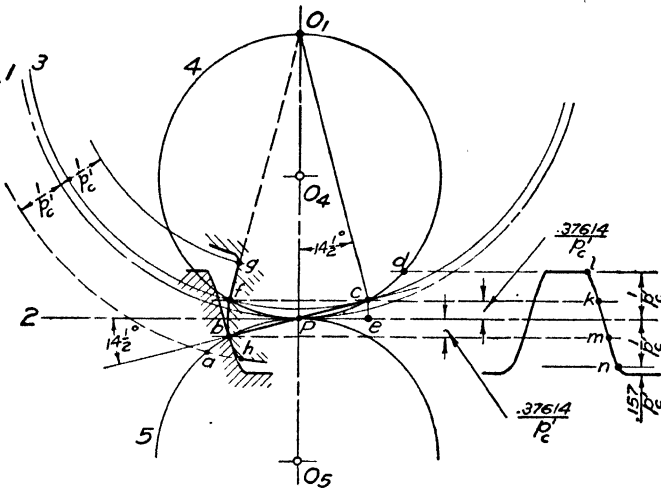
FIG. 304

cycloidal from g to f, involute from f to b, and epicycloidal from b to h. With the diameter of 4 equal to the radius of 1, the hypocycloid fg becomes a radial line. If the rack tooth is to be basic for an inter-
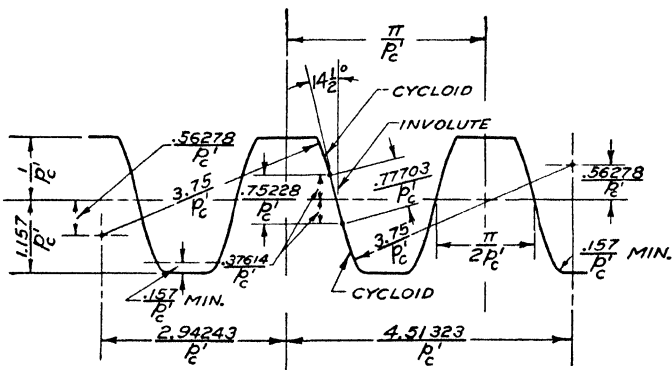
FIG. 305

changeable system of gears, the generating circle 5 must be made equal to 4, in which case the path of the point of contact $abPcd$ becomes symmetrical with respect to the center line O_1P. It is in the above

manner that the basic rack of the $14\frac{1}{2}$-degree composite system as shown in Fig. 305 was derived. In the figure the radii and the location of the centers are given that would very accurately approximate the true cycloidal faces and flanks of the rack.

It is to be noted that the rack as shown in Fig. 305 is basic for an interchangeable system of gears having 12 or more teeth. It is not basic for gears having less than 12 teeth, since for such gears the generating circle *4* would intersect the $14\frac{1}{2}$-degree line beyond the point of tangency with the base circle of the pinion. It is also to be noted that with this system the distance between centers must be accurately maintained. If the gears are drawn apart, the duration of contact for the transmission of motion at a constant angular velocity ratio is shortened to that corresponding to the involute portions of the profiles.

Besides the systems above mentioned there is the 20-degree standard involute stub-tooth system of the Fellows Gear Shaper Company, in which the addendum and clearance for the teeth of a gear are determined by the use of a larger diametral pitch than that used to determine the circular pitch and number of teeth. Their standard pitch designations are as follows: $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{7}$, $\frac{6}{8}$, $\frac{7}{9}$, $\frac{8}{10}$, $\frac{9}{11}$, $\frac{10}{12}$, and $\frac{12}{14}$. For each designation, the numerator is to be used to determine the circular pitch and number of teeth, and the denominator used to determine the addendum and clearance of the teeth. Thus a tooth of 4 diametral pitch would have an addendum of unity divided by 5 and a clearance of 0.25 divided by 5. In a similar manner an 8 diametral pitch tooth would have an addendum of $\frac{1}{10}$ or 0.10 in. and a clearance of 0.25/10 or 0.025 in.

144. Specific Sliding and Velocity of Sliding.   Two equal involutes *1'* and *2'* are shown in Fig. 306, *e* and *e'* being the coincident points of contact.   Points *a*, *b*, *c*, etc., on *1'* and *a'*, *b'*, *c'*, etc., on *2'* correspond to equal divisions of the equal base circles *1* and *2*.   It is evident from the process of generation that the radius of curvature at any point on an involute is equal to the length of the generating line from the point on the involute to the point of tangency on the base circle.   It is to be noted that the radius of curvature, which for either involute in Fig. 306 is zero at the origin *a* or *g'*, increases at a less and less rapid rate as the generating line is rolled on its base circle.   The ratio of the radius of curvature at *b* to that at *a* is infinite; at *c* it is twice that at *b*; at *d* it is 1.5 times that at *c*; etc.   In other words, the farther the portion of an involute is from the origin the less sensitive it is; and the nearer the portion is to the origin the more sensitive it is as to form and the more difficult it is to produce with acceptable accuracy.   The active portions of gear tooth profiles should therefore be no nearer to the base circle than is absolutely necessary.

As curve *2'* acts on *1'*, the portion *e'd'* will roll and slide on portion *ed*, *d'c'* on *dc*, *c'b'* on *cb*, and *b'a'* on *ba*. As these corresponding portions must act together, the amount of sliding between any two will be their difference in length. Hence there is more sliding between *b'a'* and *ba* than between *b'c'* and *bc*, and more between *b'c'* and *bc* than between *c'd'* and *cd*. The distance that *b'a'* slides on *ba* or that *ba* slides on *b'a'*,
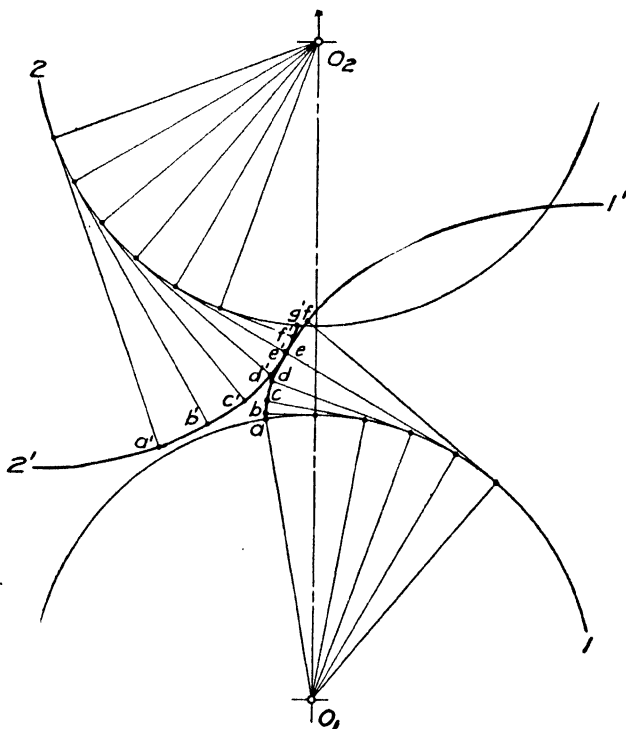
FIG. 306

is the difference in the lengths of the two portions of the curves. Since *ba* is much shorter than *b'a'*, each element of *ba* is exposed to much more sliding and wear than *b'a'*, or, contrariwise, each element of *b'a'* is exposed to much less sliding and wear than *ba*. The measure of this action is called the *specific sliding*. Thus the specific sliding of *2'* on *1'* for the portions *b'a'* and *ba* is $\dfrac{b'a' - ba}{ba}$, and the specific sliding of *1'* on *2'* is $\dfrac{ba - b'a'}{b'a'}$.

In Fig. 307 points c_1 and c_2 are the coincident points of contact of the involute tooth profiles of a pinion and gear having their centers at O_1 and O_2, respectively.   In Fig. 307, let
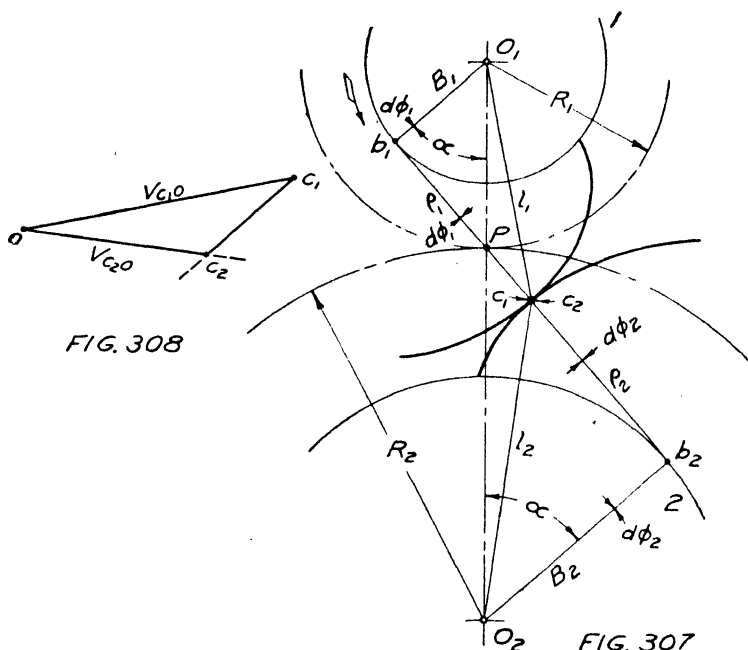
FIG. 308

FIG. 307

N_1 = the number of teeth on the pinion;

N_2 = the number of teeth on the gear;

R_1 = the pitch radius of the pinion;

R_2 = the pitch radius of the gear;

B_1 = the radius of the base circle of the pinion;

B_2 = the radius of the base circle of the gear;

α = the angle of obliquity;

$\rho_1 = b_1c_1$ = the radius of curvature of the involute of 1 at c_1;

$\rho_2 = b_2c_2$ = the radius of curvature of the involute of 2 at c_2;

S_1 = the specific sliding on the pinion tooth;

S_2 = the specific sliding on the gear tooth;

V_1 = the velocity of sliding on the pinion tooth;

V_2 = the velocity of sliding on the gear tooth;

If the pinion is displaced an infinitesimal angle $d\phi_1$, the angular displacement of the radius of curvature ρ_1 will be the same. The corresponding displacement of the gear and ρ_2 will be $d\phi_2$. Hence,

$$arc_1 = \rho_1 d\phi_1 \quad \text{and} \quad arc_2 = \rho_2 d\phi_2$$

But, since the action of the pinion and gear is equivalent to the rolling of the base circles,

$$d\phi_2 B_2 = d\phi_1 B_1 \quad \text{and} \quad \frac{B_1}{B_2} = \frac{N_1}{N_2}$$

Hence,

$$d\phi_2 = d\phi_1\left(\frac{B_1}{B_2}\right) = d\phi_1\left(\frac{N_1}{N_2}\right)$$

The specific sliding S_1 on the pinion tooth is, therefore,

$$S_1 = \frac{arc_1 - arc_2}{arc_1} = \frac{\rho_1 d\phi_1 - \rho_2 d\phi_2}{\rho_1 d\phi_1}$$

$$= \frac{\rho_1 d\phi_1 - \rho_2 d\phi_1\left(\dfrac{N_1}{N_2}\right)}{\rho_1 d\phi_1} = \frac{\rho_1 N_2 - \rho_2 N_1}{\rho_1 N_2} \qquad (222)$$

and the specific sliding S_2 on the gear tooth is

$$S_2 = \frac{arc_2 - arc_1}{arc_2} = \frac{\rho_2 N_1 - \rho_1 N_2}{\rho_2 N_1} \qquad . \quad . \quad . \quad (223)$$

It is to be noted that for contact at b_1, $\rho_1 = 0$. Substituting this value of ρ_1 in (222) gives the specific sliding on the pinion as $-\infty$, and substituting in (223) gives the specific sliding on the gear as unity. For contact at b_2, $\rho_2 = 0$, which, when substituted in (222), gives the specific sliding on the pinion as unity, and a specific sliding on the gear of $-\infty$ when substituted in (223). For contact at the pitch point P,

$\rho_1 = B_1 \tan \alpha$ and $\rho_2 = B_2 \tan \alpha = B_1\left(\dfrac{N_2}{N_1}\right) \tan \alpha$, which, when substituted in (222) and (223), gives a specific sliding of zero on the pinion and on the gear. For contact at P, there is, therefore, pure rolling between the tooth profiles.

Since rack tooth profiles are straight, equal angular displacements of the pinion mean equal portions of the rack tooth profiles. Also, since the specific sliding for contact at the pitch point is zero, the profile arcs of the teeth of the pinion and rack are equal for such contact. Each portion of the rack tooth profile may therefore be taken equal to the portion of the pinion tooth profile for contact at the pitch point, or

$$arc_2 = \rho_1 d\phi_1 = B_1 \tan \alpha d\phi_1$$

Substituting this value of arc_2 in (222) and (223) will give the expressions

for the specific sliding on the pinion and on the rack. Thus, for a *rack and pinion*.

$$S_1 = \frac{\rho_1 d\phi_1 - B_1 \tan \alpha d\phi_1}{\rho_1 d\phi_1} = \frac{\rho_1 - B_1 \tan \alpha}{\rho_1} \qquad . \quad (224)$$

and

$$S_2 = \frac{B_1 \tan \alpha \, d\phi_1 - \rho_1 d\phi_1}{B_1 \tan \alpha \, d\phi_1} = \frac{B_1 \tan \alpha - \rho_1}{B_1 \tan \alpha} \qquad . \quad (225)$$

The velocity of sliding may be found by applying the velocity vector method as in Art. 130 and as shown in Fig. 308, or it may be found from the specific sliding S_1 and S_2. From Fig. 308 the velocity of sliding on the pinion, $V_{c_2c_1}$, is represented in magnitude and direction by the vector c_1c_2, and the velocity of sliding on the gear, $V_{c_1c_2}$, by the vector c_2c_1. The velocity of sliding on the pinion and on the gear may also be found by multiplying the respective values of the specific sliding, by the tangential component of the velocity of c_1. Since B_1 and ρ_1 have the same angular velocity, the tangential component of the velocity of c_1 is to that of b_1 as ρ_1 is to B_1. Taking V_P as the pitch line velocity and V_{b_1} as the velocity of b_1, then

$$V_{b_1} = V_P \frac{B_1}{R_1} = V_P \cos \alpha$$

and the tangential component V of the velocity of c_1 becomes

$$V = V_{b_1} \frac{\rho_1}{B_1} = V_P \frac{\rho_1}{B_1} \cos \alpha \qquad . \quad . \quad . \quad . \quad (226)$$

Multiplying (222) by the value of V from (226) gives the velocity of sliding, V_1, on the pinion, or

$$V_1 = V_P \cos \alpha \left(\frac{\rho_1 N_2 - \rho_2 N_1}{B_1 N_2} \right) \qquad . \quad . \quad . \quad (227)$$

and multiplying (223) by the value of V from (226) gives the velocity of sliding V_2 on the gear, or

$$V_2 = V_P \cos \alpha \left(\frac{\rho_2 N_1 - \rho_1 N_2}{B_2 N_1} \right) \qquad . \quad . \quad . \quad (228)$$

Remembering that $B_1 N_2 = B_2 N_1$, it is to be noted that V_1 and $\dot{V}_2$ are equal in magnitude but opposite in sign. In a similar manner, V_1 and V_2 for a pinion and rack may be obtained by multiplying (224) and (225) by the value of V from (226).

145. Non-Interchangeable, or Unequal Addendum, Involute Gears. For interchangeability, as was shown in Art. 139, involute gears of the

same pitch must have the same angle of obliquity, the same addendum, and a tooth thickness equal to one-half the circular pitch. Under any given interchangeable system, this merely means that the teeth of all gears of a given pitch will fit and mesh properly at a distance between centers proportional to the number of teeth in the mating gears. If, in addition, the width of face and the diameter and length of the hubs of such gears are properly related to the pitch and number of teeth, the gears are reduced to the status of standardized commercial parts. While there is a considerable field for such gears, there are many situations where such gears cannot be used. The reason for this is that the service requirements are often such as to demand gears of a given pitch and number of teeth which differ widely in the bore, diameter, length, position, and form of the hubs, if not to differ in other respects. This can be verified by the inspection of the gears of almost any machine. There are, however, certain advantages in cutting the teeth of gears to an interchangeable system even though the gears are not otherwise interchangeable. The principal advantages are ease and convenience in calculations and economy in tools, especially if formed milling cutters are used to cut the gear teeth. To secure these advantages, there are certain disadvantages that must be accepted. A sufficiently large angle of obliquity must be chosen to permit the smallest gear of the set to mesh with the basic rack without interference and without undue undercutting. This means a greater angle of obliquity for the smaller gears than is necessary or desirable for the larger gears of the set. Thus an angle of obliquity of 20 degrees, while satisfactory for gears having a small number of teeth, is larger than necessary for gears having 40 or more teeth. The requirement of a fixed relation between the addendum and the pitch also mitigates against realizing to the greatest degree the advantages of the involute form of tooth profile. To realize to the fullest extent the benefits of this form of profile, we must be free to choose for each pair of mating gears the most suitable angle of obliquity and tooth proportions. For gears that are not otherwise interchangeable, there can be little objection to proportioning the teeth on a non-interchangeable basis. By so doing, mating pairs of gears have been designed to run satisfactorily at higher pitch line velocities than is possible for gears whose teeth are on an interchangeable basis. If proportioning teeth on a non-interchangeable basis meant a special cutter for each gear to be cut, it would, of course, be prohibitive; but, as will be shown, any cutter used to generate standard equal addendum teeth may be used to generate unequal addendum teeth. Thus a hob or a pinion-shaped or a rack-shaped cutter may be used.

In Art. 144 it was shown that the nearer the active portion of a tooth

profile is to the base circle the more sensitive it is as to form and the more difficult it is to produce with acceptable accuracy. It was also shown that to reduce wear the specific sliding should be as low as possible These advantages can be secured by varying the angle of obliquity and tooth proportions to suit the number of teeth of the mating gears. For doing this either one of two methods may be followed: the addendum and angle of obliquity may be varied without changing the pitch and distance between centers; or the pitch and distance between centers may be varied as well as the addendum and angle of obliquity. The first is called the *range cutter* method and the second the *variable center distance* method.

146. Range Cutter Method for Non-Interchangeable Gears. An interchangeable system requires only one generating cutter for each pitch. The first method mentioned above would require a series of generating cutters for each pitch with the addendum and angle of obliquity of each cutter selected with due regard to the reduction of the specific sliding and the sensitiveness of the tooth profiles. Because of the number of cutters required, this method of solving the problem, so far as the writer knows, has not been used commercially. What an orderly solution on this basis would mean is indicated by Table 6, the data for which were taken from "Spur Gears"* by Earle Buckingham. It is to be noted that five different angles of obliquity and three different proportions for the addendum are proposed for gears having ten or more teeth. This would mean five different generating cutters for each pitch; that is, five hobs or five pinion or five rack-shaped cutters for each pitch. While gears cut with one cutter would not be interchangeable with those cut with any other cutter of the set, the gears cut by each cutter would be interchangeable within the range of the sum of the numbers of teeth for mating gears given in the table. Under a non-interchangeable system organized in accordance with Table 6, all gears would have the same pitch and tooth thickness, and the distance between the centers of any two mating gears would correspond to the pitch and number of teeth in the gears as for interchangeable gears. Thus a pair of gears of 4 pitch and 16 and 24 teeth, respectively, would, in accordance with the table, be cut by a 4 pitch cutter corresponding to an angle of obliquity of 22½ degrees. The pitch diameters of the gears would be 4 and 6 inches, respectively, with 5 inches between centers as for 4 pitch interchangeable gears of 16 and 24 teeth. Two 10-tooth pinions in accordance with the table would have a contact ratio of 1.058, and would therefore drive positively and continuously. So far as concerns the reduction of the specific sliding and the sensitiveness of the tooth profiles, the

* Published by the McGraw-Hill Book Company.

range cutter method gives practically as good results as the second, or variable center distance, method, besides making the calculations much more simple.

TABLE 6

Angle of obliquity in degrees.......	α	25	$22\frac{1}{2}$	20	17	$14\frac{1}{2}$
Addendum in inches...............	a	$\dfrac{0.80}{p'_c}$	$\dfrac{0\ 90}{p'_c}$	$\dfrac{1.00}{p'_c}$	$\dfrac{1.00}{p'_c}$	$\dfrac{1.00}{p'_c}$
Working depth in inches...........	$h-c$	$\dfrac{1.60}{p'_c}$	$\dfrac{1.80}{p'_c}$	$\dfrac{2.00}{p'_c}$	$\dfrac{2.00}{p'_c}$	$\dfrac{2.00}{p'_c}$
Clearance in inches...............	c	$\dfrac{0.20}{p'_c}$	$\dfrac{0\ 20}{p'_c}$	$\dfrac{0\ 20}{p'_c}$	$\dfrac{0.20}{p'_c}$	$\dfrac{0\ 20}{p'_c}$
Whole depth in inches.............	h	$\dfrac{1.80}{p'_c}$	$\dfrac{2\ 00}{p'_c}$	$\dfrac{2.20}{p'_c}$	$\dfrac{2.20}{p'_c}$	$\dfrac{2\ 20}{p'_c}$
Smallest number of teeth in equal pinions........................	N	10	16	21	30	40
Range of the sum of the number of teeth in a pair of mating gears.....	N_1+N_2	20–31	32–41	42–59*	60–78†	80 and up

* Except that a pinion having 15, 16, 17, 18, or 19 teeth may mesh with a gear having 45 teeth or less.

† Except that a pinion having 34, 35, 36, 37, 38, or 39 teeth may mesh with a gear having 45 teeth or less.

147. Variable Center Distance Method for Non-Interchangeable Gears. (a) *General Considerations.* The second, or variable center distance, method would vary the pitch and distance between centers as well as the addendum and angle of obliquity. This is possible since, as shown in Art. 138, the position of the pitch point and the size of the pitch circles depend entirely on the size of the base circles and the distance between centers. Any generating cutter used to cut inter-changeable gears may be used to cut such gears. The problem is to determine the size of the blanks, the distance between centers, and the proper distance from the center of each blank to the cutter. Whatever the form of the generating cutter, whether an involute hob or a pinion-shaped or a rack-shaped cutter, the necessary calculations are most easily based on the basic rack tooth form to which the cutter corresponds. To illustrate the method, the generating cutter will be assumed, as in

"Spur Gears" by Earle Buckingham, to correspond to the full depth $14\frac{1}{2}$-degree basic rack shown in Fig. 309. Since equal addendum gears having an obliquity of $14\frac{1}{2}$ degrees are satisfactory for gears having 40 or more teeth, the variable center distance method may be confined to mating gears having pinions of less than 40 teeth. Assuming the smallest pinion to have 10 teeth, and that it is not to be undercut, makes it possible to establish a regular way of varying the root radius, or the distance from the center of the blank to the cutter.

(b) *Relation of Basic Rack to Pinion or Gear.* The relation of a 10-tooth pinion to the basic rack is shown in Fig. 310. To avoid undercutting, point b on the basic rack must be no nearer the center line O_1c than the point a on the base circle. Assuming this position gives a root radius H of the pinion of 4.48655 in. for a nominal diametral pitch of unity, and gives the thickness of the pinion teeth at the nominal pitch radius of 5.0 in. as the nominal circu-

FIG. 309

FIG. 310

lar pitch minus t, or 1.9259 in. It might be mentioned that a pair of 10-tooth pinions will not drive continuously since their contact ratio is only 0.961. For a pinion having 40 or more teeth, the nominal pitch line of the rack is taken, as intimated above, tangent to the pitch circle of the pinion. Hence, for a 40-tooth pinion having a diametral pitch of unity, the root radius becomes $(20 - 1.20) = 18.80$ in. If the rack were placed with b in line with a on the base circle of the 40-tooth pinion, the corresponding root radius $H = (A - f)$ would be 18.5463 in., which is 0.2537 in. less than 18.80 in. This difference of 0.2537 in.
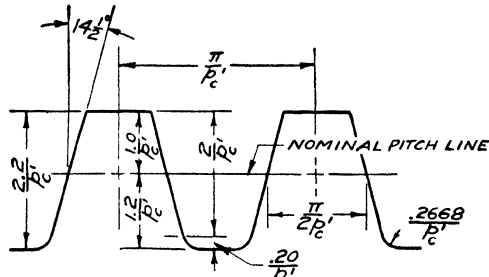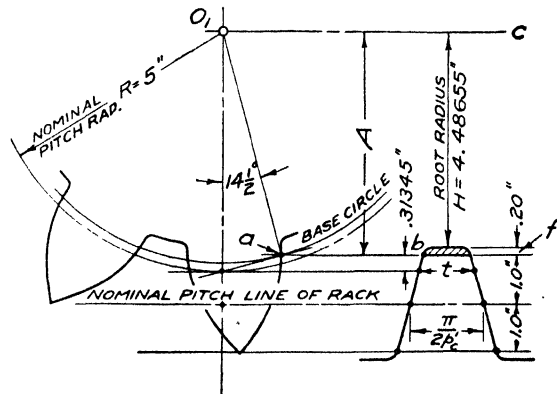
suggests a regular way of fixing the root radii of pinions having from 10 to 40 teeth. Thus for a pinion having N teeth, the root radius H, for a nominal diametral pitch of unity, may be taken,

$$H = (A - f) + 0.2537\left(\frac{N - 10}{40 - 10}\right)$$

$$= \left(\frac{N \cos^2 14\frac{1}{2}}{2} - 0.20\right) + 0.2537\left(\frac{N - 10}{30}\right) \qquad . \quad (229)$$

The distance G from the center of the gear to the nominal pitch line of the rack becomes

$$G = H + 1.20 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

The tooth thickness at the nominal pitch radius of the gear becomes

$$t' = (\pi - t) = \frac{\pi}{2} + 2(G - R) \tan 14\frac{1}{2} \quad . \quad . \quad . \quad (231)$$

(c) *Equation of the Involute.* Having the tooth thickness at the nominal pitch radii of two mating gears, it is necessary to have the equation of the involute to find the angle of obliquity, the distance between centers, the actual pitch radii, the tooth thicknesses at any other radii, etc. In Fig. 311 is shown an involute generated from a base circle of radius B. For a point P at any radius R:

$$\cos \alpha = \frac{B}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

$$\rho = B(\theta + \alpha) \quad \text{and} \quad \rho = B \tan \alpha$$

Equating and solving for θ gives, in radians,

$$\theta = \tan \alpha - \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

Also,

$$\tan \alpha = \frac{\rho}{B} = \frac{\sqrt{R^2 - B^2}}{B} = \sqrt{\left(\frac{R}{B}\right)^2 - 1} \quad . \quad . \quad (234)$$

Substituting in (233) the values of $\tan \alpha$ and α from equation (234) gives the polar equation of the involute as

$$\theta = \sqrt{\left(\frac{R}{B}\right)^2 - 1} - \tan^{-1}\sqrt{\left(\frac{R}{B}\right)^2 - 1} \quad . \quad (235)$$

(d) *Pair of Unequal Addendum Gears.* Fig. 312 shows two gears, *1* and *2*, in mesh for which R'_1 and R'_2 are the nominal pitch radii, and

B'_1 and B'_2 the respective base circle radii corresponding to an angle of obliquity $\alpha'_1 = \alpha'_2 = 14\frac{1}{2}$ degrees. Without backlash,

$$t''_1 + t''_2 = p_c = \frac{2\pi R''_1}{N_1} = \frac{2\pi R''_2}{N_2} \quad . \quad . \quad . \quad . \quad (236)$$
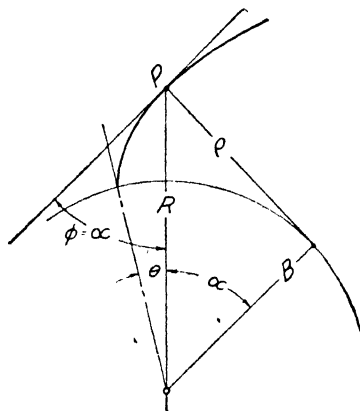
FIG. 311
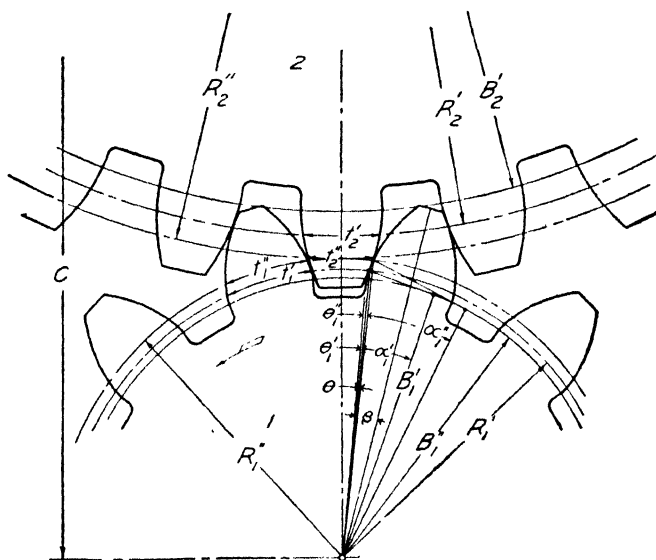
FIG. 312

From Fig. 312,

$$\beta - \frac{t'_1}{2R'_1} = \theta'_1 \quad \text{and} \quad \beta - \frac{t''_1}{2R''_1} = \theta''_1$$

or,

$$\theta = \theta''_1 - \theta'_1 = \left(\beta - \frac{t''_1}{2R''_1}\right) - \left(\beta - \frac{t'_1}{2R'_1}\right) = \frac{t'_1}{2R'_1} - \frac{t''_1}{2R''_1}$$

Whence,

$$t''_1 = 2R''_1\left[\frac{t'_1}{2R'_1} + (\theta'_1 - \theta''_1)\right] \quad \ldots \ldots \quad (237)$$

An expression for t''_2 may be obtained in a similar manner. Since $\alpha'_2 = \alpha'_1$ and $\alpha''_2 = \alpha''_1$, it should be remembered in determining t''_2 that, by equation (233), $\theta'_2 = \theta'_1$ and $\theta''_2 = \theta''_1$. Hence,

$$t''_2 = 2R''_2\left[\frac{t'_2}{2R'_2} + (\theta'_1 - \theta''_1)\right] \quad \ldots \ldots \quad (238)$$

But,

$$\frac{R''_2}{R''_1} = \frac{N_2}{N_1} \quad \text{and} \quad \frac{R'_2}{R'_1} = \frac{N_2}{N_1}$$

or,

$$R''_2 = R''_1\frac{N_2}{N_1} \quad \text{and} \quad R'_2 = R'_1\frac{N_2}{N_1}$$

Substituting these values of R''_2 and R'_2 in (238) gives

$$t''_2 = 2R''_1\frac{N_2}{N_1}\left[\frac{t'_2N_1}{2R'_1N_2} + (\theta'_1 - \theta''_1)\right] \quad \ldots \quad (239)$$

Adding (237) and (239) and substituting $\dfrac{2\pi R''_1}{N_1}$ for $(t''_1 + t''_2)$ from (236) and solving for θ''_1 gives

$$\theta''_1 = \frac{(t'_1 + t'_2)N_1 - 2\pi R'_1}{2R'_1(N_1 + N_2)} + \theta'_1 \quad \ldots \ldots \quad (240)$$

Substituting the values of θ''_1 and θ'_1 in radians from (233) gives

$$\tan \alpha''_1 - \alpha''_1 = \frac{(t'_1 + t'_2)N_1 - 2\pi R'_1}{2R'_1(N_1 + N_2)} + (\tan \alpha'_1 - \alpha'_1) \quad (241)$$

The expression $(\tan \alpha - \alpha)$ is called the involute function of α and is generally written inv. α. If the value of α''_1 is to be obtained from equation (241) otherwise than by trial, recourse must be had to a table of involute functions. Such a table in increments of a minute is to be found in "Spur Gears" by Buckingham.

To get the thickness t'' of an involute tooth at any radius R'' having

the thickness t' at a radius R' for an angle of obliquity α', we have from (237) and (233),

$$t'' = 2R'' \left[\frac{t'}{2R'} + (\tan \alpha' - \alpha') - (\tan \alpha'' - \alpha'') \right] \quad . \quad (242)$$

where, by (232), $\cos \alpha'' = \dfrac{B}{R''}$.

Equation (242) is also convenient in finding at what radius R'' the thickness t'' is zero, or the radius at which the tooth comes to a point.

(e) *Illustrative Problem.* Assuming as an illustration that gears *1* and *2* in Fig. 312 have, respectively, 12 and 30 teeth of unity diametral pitch, then,

$N_1 = 12$	$N_2 = 30$
$R'_1 = 6$ in.	$R'_2 = 15$ in.
$H_1 = 5.44080$ in.	$H_2 = 14.02885$ in. (Eq. 229)
$G_1 = 6.64080$ in.	$G_2 = 15.22885$ in. (Eq. 230)
$t'_1 = 1.90225$ in.	$t'_2 = 1.68917$ in. (Eq. 231)

Substituting in (241) gives the angle of obliquity at which the two gears will act:

$$\tan \alpha''_1 - \alpha''_1 = 0.016255 \quad \text{or} \quad \alpha''_1 = \alpha''_2 = 20° 34'$$

The distance C between centers is

$$C = R''_1 + R''_2 = \frac{B'_1}{\cos \alpha''_1} + \frac{B'_2}{\cos \alpha''_1}$$

$$= \frac{(R'_1 + R'_2)\cos 14\frac{1}{2}}{\cos \alpha''_1} = 21.71517 \text{ in.}$$

or the pitch radii are

$$R''_1 = \frac{R'_1 \cos 14\frac{1}{2}}{\cos \alpha''_1} = 6.2043 \text{ in.} \quad R''_2 = 15.5108 \text{ in.}$$

Having R''_1 and R''_2, the circular pitch, by equation (236), is

$$p_c = 3.2485 \text{ in.}$$

which is seen to be 0.1069 in. greater than the nominal circular pitch of 3.1416 in. Subtracting the sum of the root radii from the distance between centers gives the whole depth of the teeth plus clearance. For the basic rack shown in Fig. 309 the ratio of the whole depth to

the whole depth plus clearance is $\dfrac{2.2}{2.4} = \dfrac{11}{12}$. *Using this same proportion,* the whole depth of the gear teeth becomes

$$h = \tfrac{11}{12}(C - H_1 - H_2) = 2.0584 \text{ in.}$$

Hence,

$$\text{Clearance} = (C - H_1 - H_2 - h) = 0.1871 \text{ in.}$$

The external radii of the gears become

$$E_1 = H_1 + h = 7.4992 \text{ in.} \quad \text{and} \quad E_2 = H_2 + h = 16.0872 \text{ in.}$$

By equation (242) the respective tooth thicknesses of pinion and gear at the pitch radii R''_1 and R''_2 are

$$t''_1 = 1.8341 \text{ in.} \quad \text{and} \quad t''_2 = 1.4144 \text{ in.}$$

The proportions found for the above illustrative problem are summarized in Table 7. For any other nominal diametral pitch than unity, divide the linear dimensions by the desired pitch.

TABLE 7

Dimensions in Inches		Pinion	Gear
Number of teeth	N	12	30
Root radius	H	5.4408	14.02885
Pitch radius	R	6.2043	15.5108*
External radius	E	7.4992	16.0872
Whole depth of teeth	h	2.0584	2.0584
Clearance	c	0.1871	0.1871
Addendum	a	1.2949	0.5764
Circular pitch	p_c	3.2485	3.2485
Tooth thickness	t	1.8341	1.4144
Angle of obliquity	α	20° 34'	

(f) Comparison of Pairs of Equal and Unequal Addendum Gears. In Table 8 two pairs of equal pinions each having 12 teeth and a diametral pitch of unity are contrasted. It is to be noted that the specific sliding, particularly on the dedendum, is very much higher for the 20-degree standard stub tooth than for the 14½-degree variable center distance pinions. The latter pair of pinions would therefore wear much better and their active profiles could be more easily produced with acceptable accuracy. These advantages more than compensate for the fact that the duration of contact is greater and the angle of obliquity less for the stub tooth than for the variable center distance pinions.

TABLE 8

For Two Equal Pinions Having 12 Teeth and a Diametral Pitch of Unity. (Pinions having 12-20° Stub Teeth are Slightly Undercut)	20° Standard Stub Tooth System	14½° Basic Rack Variable Center Distance System
Number of teeth.	12	12
Maximum radius of curvature of active profile, inches.	3.8014	4.4122
Minimum radius of curvature of active profile, inches.	0.3028	1.2073
Active profile above pitch line, inches.	0.8000	0.7976
Active profile below pitch line, inches.	0.3538	0.5190
Total height of active profile, inches.	1.1538	1.3166
Specific sliding on the addendum.	+0.92	+0.72
Specific sliding on the dedendum.	−11.55	−2.65
Duration of contact in tooth intervals.	1.185	1.052
Angle of obliquity.	20°	25°-49'
Distance between centers in inches.	12	12.9039
Tooth depth in inches.	1.80	1.8538

In Table 9 two pair of mating gears, each pair having 12 and 30 teeth, respectively, and a diametral pitch of unity, are contrasted. It is to be noted that, with respect to specific sliding and duration of contact,
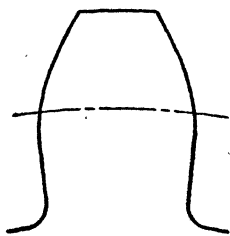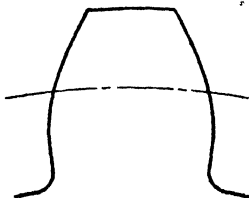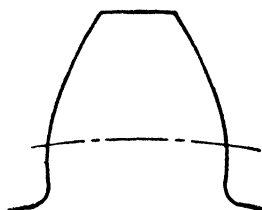
FIG. 313   FIG. 314   FIG. 315

the variable center distance pair is much superior to the pair having standard 20-degree stub teeth. It is also to be noted that the angle of obliquity is about the same for each pair.

In addition to the benefits just mentioned, varying the angle of obliquity and tooth proportions to suit the numbers of teeth in mating gears also results in teeth of stronger form. This is illustrated in Figs. 313, 314, and 315. In Fig. 313 is shown a standard 14½-degree composite tooth of a 14-tooth pinion of 2 diametral pitch; in Fig. 314 a standard 20-degree stub tooth for the same pitch and number of teeth; and in

TABLE 9

For a Pair of Mating Gears Having 12 and 30 Teeth and a Diametral Pitch of Unity. (Pinions having 12-20° Stub Teeth are Slightly Undercut)	20° Standard Stub Tooth System	14½° Basic Rack Variable Center Distance System
Number of teeth on pinion...	12	12
Maximum radius of curvature of active profile, inches................................. .	3 8014	4.7416
Minimum radius of curvature of active profile, inches.....`............................ ..	0.1682	0.7086
Active profile above pitch line, inches.........	0.8000	1.2943
Active profile below pitch line, inches.	0.3593	0.3522
Total height of active profile, inches....	1.1593	1.6465
Specific sliding on addendum of pinion... .. .	+0 64	+0.75
Specific sliding on dedendum of pinion .	−15 68	−2 90
Number of teeth on gear...	30	30
Maximum radius of curvature of active profile, inches	7.0142	6.9196
Minimum radius of curvature of active profile, inches	3.3810	2.8866
Active profile above pitch line, inches........	0.7441	0.5763
Active profile below pitch line, inches	0.5048	0.7039
Total height of active profile, inches	1.2489	1.2802
Specific sliding on addendum of gear........	+0.94	+0.74
Specific sliding on dedendum of gear........	−1.81	−3.10
Duration of contact in tooth intervals	1.230	1.325
Angle of obliquity...........................	20°	20°--34'
Distance between centers in inches...	21 00	21.7143
Tooth depth in inches	1.80	2.0576

Fig. 315 a 14½-degree variable center distance tooth for a pair of 14-tooth pinions of the same nominal pitch. It is to be noted that the form of the tooth in Fig. 314 is stronger than that of Fig. 313, and that the form of Fig. 315 is considerably stronger than either of the other two.

So far the discussion has been with reference to mating pairs of gears where the pinion had 10 or more teeth. For gear trains involving more than two gears, there are two possible solutions. If the distances between centers do not have to be proportional to the number of teeth in the gears, then all the gears of the train may be cut by the use of a 14½ degree cutter after the variable center distance method. If it is desired to have the center distances proportional to the number of

teeth in the gears, one of the range cutters may be selected which has an angle of obliquity suited to the smallest gear in the train. By this method all the gears of a train would have the same addendum and angle of obliquity; by the variable center distance method, the angle of obliquity of each pair would be different, and each gear of the train would have a different addendum.

Two 5-tooth equal addendum pinions, to drive continuously, would require an addendum of 0.722 divided by the diametral pitch and an angle of obliquity of $32\frac{1}{2}$ degrees, giving a contact ratio of 1.015 tooth intervals. On a variable center distance basis, using a $22\frac{1}{2}$-degree basic rack, a 19-tooth gear is the smallest gear that would give continuous driving with a 5-tooth pinion, the contact ratio being 1.035. For the continuous transmission of power, as before stated, the contact ratio should be at least 1.20.

While the variable center distance method using standard generating cutters had previously been applied in this and other countries, Maag was probably the first manufacturer to produce non-interchangeable gears on a commercial basis that were designed to realize fully the possibilities of the involute form of tooth profile. The principles discussed in this article are the same as applied by Maag.

148. Strength, Wear, and Noise of Gear Teeth. The purpose of gears is to transmit power safely and efficiently with as little noise and wear as possible. The strength of gear teeth and the amount of power that may safely be transmitted by them are questions that await further experimental investigation for satisfactory settlement.

It is evident from inspection of Figs. 313, 314, and 315 that the shape and strength of the teeth of gears having the same pitch and number of teeth, vary with the system. In addition, the shape and strength for any given system and pitch vary with the number of teeth in the gear. The factors affecting the strength of gear teeth may be listed as follows:

Physical characteristics of the material;

Pitch of the teeth;

Shape of the teeth;

Inaccuracies of tooth profiles and spacing causing variations of the angular velocity ratio resulting in more or less shock;

Point of application of the load;

Character of the load; that is, whether the load is steadily, or suddenly, or impulsively applied;

Influence of the velocity in increasing the load to be transmitted from tooth to tooth;

Influence of the inertia of the rotating masses in increasing the load to be transmitted from tooth to tooth;

Whether more than one tooth takes the load;

Factor of safety to cover unknown factors and contingencies.

As intimated above, the experimental work that has been done has not been sufficiently extensive to enable the influence of the above factors to be stated quantitatively in the form of an equation. The equation developed by Wilfred Lewis in 1892 has been extensively used and is still frequently used in estimating the strength of gear teeth. The equation assumes that the load on a gear is carried by a single tooth acting as a cantilever beam. Since a cantilever of constant width and of uniform strength is parabolic in outline, the weakest section of a tooth was assumed to be where a parabola would be tangent to the outlines of the tooth, the parabola being drawn through the point of intersection of the line of action of the load and the center line of the tooth. For any given system, the shape of the tooth and its weakest section was found to be a function of the circular pitch and number of teeth. Hence a tooth shape factor y, which for any given system varies with the number of teeth, appears in the equation. The Lewis equation is:

$$W = Sp_c by \qquad \ldots \ldots \ldots \ldots \ldots \ldots \quad (243)$$

Where W = the useful load in pounds assumed to be taken by one tooth;

p_c = the circular pitch in inches;

b = the width of face of the gear in inches;

y = the tooth shape factor in terms of the number of teeth N in the gear

$\quad = \left(0.124 - \dfrac{0.684}{N}\right)$ for the 14½-degree composite system, addendum equal to unity divided by the diametral pitch

$\quad = \left(0.170 - \dfrac{0.90}{N}\right)$ for the 20-degree stub tooth system, addendum equal to 0.80 divided by the diametral pitch

$\quad = \left(0.148 - \dfrac{0.90}{N}\right)$ for the 20-degree full depth tooth, addendum equal to unity divided by the diametral pitch

$\quad = \left(0.083 + \dfrac{0.152}{\sqrt{N}}\right)$ for 10 to 40 teeth and $\left(0.124 - \dfrac{0.684}{N}\right)$ for over 40 teeth for the 14½-degree variable center distance system;

S = the safe working stress = $\left(\dfrac{U_t}{k}\right)g$

Where U_t = ultimate tensile strength of the material in pounds per
square inch
= 24000 for cast-iron
= 36000 for semi-steel
= 36000 for bronze
= 45000 for malleable iron
= 60000 for steel castings (S.A.E. 1235)
= 70000 for steel forgings (S.A.E. 1030)
= 90000 for steel forgings (S.A.E. 1045)
= 120000 for steel forgings (S.A.E. 3245);

k = the factor of safety
= 3 for steady loads on single pairs of gears
= 4 for suddenly applied loads on single pairs of gears
= 5 for steady loads on gears of a train beyond the first mesh
= 6 for suddenly applied loads on gears of a train beyond
the first mesh;

g = the velocity influence factor
$= \dfrac{600}{600 + V}$ for ordinary commercial cut gears for pitch line
velocities V up to 2000 feet per minute
$= \dfrac{1200}{1200 + V}$ for carefully cut gears for pitch line velocities
V from 1000 to 4000 feet per minute
$= \dfrac{78}{78 + \sqrt{V}}$ for high class gears for pitch line velocities V
from 4000 to 10000 feet per minute.

It is to be noted that in the notation as defined practically all of the
factors affecting the strength of gear teeth have been considered except
the question of the distribution of the load over more than one tooth.
The assumption that the load is taken by one tooth is on the side of
safety. For convenience this load is taken as the useful load tangent
to the pitch circle, which is slightly greater than the component of the
load at the tip of the tooth acting normal to the axis of the tooth. By
using different values of the velocity factor g for different grades of
work, the influence of accuracy as well as velocity is taken into
account.

For non-metallic pinions of rawhide, fiber, Bakelite, Micarta, and
Fabroil, the low elasticity of the materials very largely nullifies the
effects of inaccuracies, distortions under load, and inertia of the rotating
masses. The teeth of non-metallic pinions soon become conjugate to
the mating metallic teeth, especially if the ratio of the gear to the pinion
is a whole number. For pinions of non-metallic materials for velocities

up to 3000 feet per minute, the value of the working stress in the Lewis equation may be taken as

$$S = 6000\left(\frac{150}{200 + V} + 0.25\right) \quad \text{. . . .} \quad (244)$$

The experimental work that has thus far been done has not been sufficiently extensive to establish for any given material the pressure per inch of face beyond which wear would be unacceptably rapid. It seems, however, quite well established that accurately cut gears when properly lubricated will not wear appreciably if the maximum stress at contact is kept within the elastic limit of the material in compression. For two convex cylinders in contact under a total load W the maximum compressive stress induced at contact is, according to Hertz,

$$S^2 = \frac{0.35W\left(\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}\right)}{b\left(\dfrac{1}{E_1} + \dfrac{1}{E_2}\right)} \quad \text{. . . . .} \quad (245)$$

Where W = the total load in pounds;
    b = the length in inches of the cylinders in contact;
    ρ_1 = the radius of curvature in inches of the first cylinder at contact;
    ρ_2 = the radius of curvature in inches of the second cylinder at contact;
    E_1 = the modulus of elasticity of the material of the first cylinder;
    E_2 = the modulus of elasticity of the material of the second cylinder.

In applying equation (245) to gears, it would appear, from the nature of the wear of gear teeth, that the use of the radii of curvature at the pitch points of the teeth in contact is justified. Where, as in internal involute gears, a convex surface is in contact with a concave, the plus sign in the numerator of (245) should be changed to minus. Besides holding the compressive stress at contact within the elastic limit of the material and making proper provision for lubrication, it is very important, for minimum wear, that the tooth profiles be smooth and accurately shaped and spaced.

Minimum vibration and noise demand good design and good workmanship. This applies both to the gears themselves and the case or carrier in which they are mounted. Rough tooth profiles cause a high-pitched squeal or scream, and inaccurate profiles and spacing cause a grumble of varying intensity which is one of the most disagreeable of gear noises.

To have and to maintain accuracy of alignment is also very important. With accurate gears having smooth profiles, the sound emitted will depend upon the number of tooth engagements per-second, the human ear being capable of hearing sounds of from 32 to 38000 vibrations per second. If the gear is so shaped that its natural frequency corresponds to the number of tooth engagements per second, the sound will be augmented. If, in turn, the gear case or carrier has the same natural frequency, it will act as a resonator and further augment the sound. Hence, neither the case nor the gears themselves should be resonators having a natural frequency corresponding to the number of tooth engagements per second. The resonance of gears may often be reduced by the use of ribs or by making them in parts, and the resonance of gear casings by avoiding simple shapes and extensive flat surfaces. Where two or more pairs of gears are housed in the same casing, consonance as well as resonance must be considered. In such cases the smaller the numbers expressing the ratios of the vibrations of the different pairs, the more harmonious will be the sound. For this reason countershaft gears of automobile transmissions have been made with the same number of teeth, the pitch being varied to secure the desired reduction ratios.

149. Comparison of Involute and Cycloidal Tooth Forms. One of the most important advantages of the involute over the cycloidal form of tooth is that involute cutters of all kinds are much easier and cheaper to produce than cycloidal cutters. For the same pitch and the same degree of accuracy, cycloidal teeth require about three times as many formed cutters as involute teeth. For all kinds of generating cutters the two tooth forms are on a par so far as the required number of cutters is concerned, but the cycloidal cutters, as previously stated, would be more difficult and expensive to make. Since the involute is not sensitive as to center distance, equal and unequal addendum gears may, as demonstrated in Art. 145, be cut with the same generating cutter. This gives the involute tooth form a decided advantage over the cycloidal.

As was shown in Art. 145 the most sensitive and difficult portions of the involute may, by proper design, be eliminated from the active portion of involute tooth profiles. Since the highly sensitive portions of cycloidal tooth profiles occur at and near the pitch circle, they cannot, by any expedient, be eliminated from the active profiles. However, being at and near the pitch circle, the inclusion of the sensitive portion in the active profile does not result in high specific sliding as with the involute.

The next most important advantage of the involute over the cycloidal form of tooth is that, for the transmission of motion at a constant angular velocity ratio, the distance between the centers of mating gears is not fixed for the former, while it is fixed for the latter. A still further

advantage of less importance is that the angle of obliquity during action is constant for the involute while variable for the cycloidal form. Hence, for constant power, the pressure at the bearings of a pair of involute gears will be constant in magnitude and direction, while for a pair of cycloidal gears both the magnitude and direction of the pressure at the bearings will vary somewhat during the action of the gears.

An advantage of the cycloidal over the involute tooth form for external gears is that with the former a convex surface is always in contact with a concave, while with the involute form a convex surface

20 TEETH

32 TEETH

SCALE 3"=12"
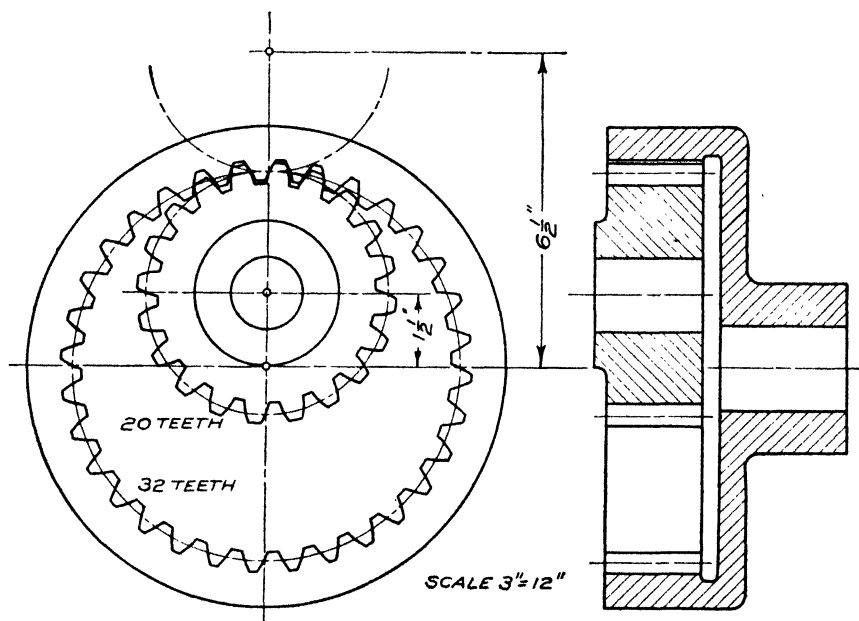
FIG. 316

is always in contact with a convex surface, or at best, with a flat surface. With the cycloidal form there is, therefore, a nearer approach to surface contact than with the involute, a condition more favorable to effective lubrication. Another advantage of this difference in contact is that for the same pressure the induced stress at contact would, as shown by equation (245), be less for cycloidal than for involute teeth.

150. Internal Involute Gears. For the same pitch and number of teeth, internal involute gears are superior in many respects to external involute gears. Only in recent years has this been fully appreciated and applied. Because of their efficiency and compactness such gears

are now employed for truck and tractor drives, and as reduction gears in air drills, hoists, aeroplane propellers, etc. Two points of superiority are shown in Fig. 316; greater compactness and greater safety. For the pinion and internal gear shown, the distance between the centers of the shafts is $1\frac{1}{2}$ in. and the overall covering dimension $9\frac{7}{8}$ in. For the same pitch and number of teeth, these dimensions for a pair of external gears would be $6\frac{1}{2}$ in. and 13.4 in., respectively. In addition, the internal gear, due to its form, acts as a protecting guard, while, for external gears, a separate cover must be furnished as a guard. Other advantages
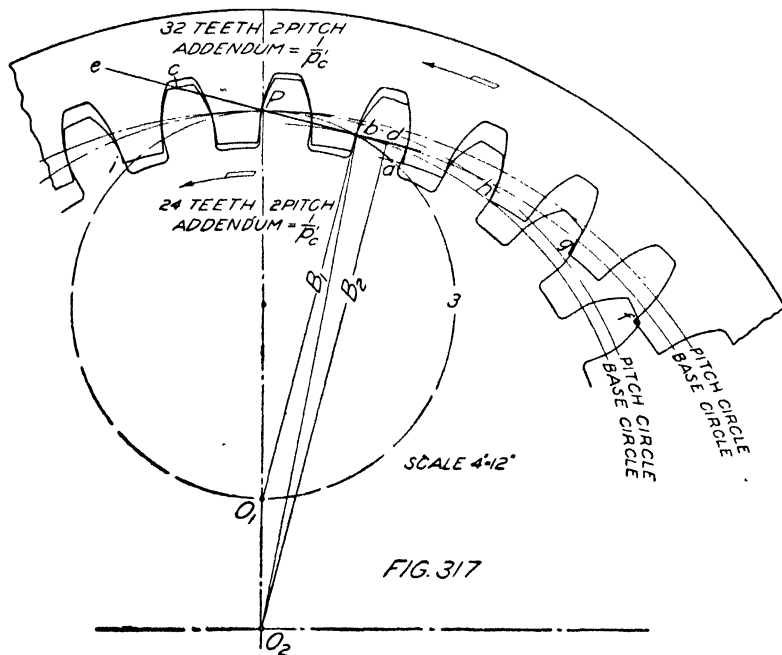
FIG. 317

of internal gears over external gears are, for the same tooth proportions, greater length of the path of contact, greater strength of the gear teeth, and lower specific sliding resulting in less wear and longer life of the teeth.

Fig. 317 shows some of the teeth of a 24-tooth pinion in mesh with a 32-tooth internal gear. The base circles of the pinion and gear correspond to an angle of obliquity of $14\frac{1}{2}$ degrees, and the teeth have an addendum of unity divided by the diametral pitch. The points of tangency of the line of obliquity de with the respective base circles are b and d. Since radius O_2b is greater than the base circle radius B_2 of

the gear, pure involute action cannot begin before b is reached. The possible action that may take place before this point is reached will depend upon the shape assigned to the pinion tooth profiles inside the base circle, or upon the shape assigned to the internal gear tooth profiles inside the circle of radius O_2b. In the figure, the flanks of the pinion teeth inside the base circle have been made radial. These portions of the flanks of the pinion teeth are therefore hypocycloids generated by circle 3 having a diameter equal to the radius of the pinion. Hence, for proper action between the tooth profiles of the pinion and gear, the faces of the gear teeth inside of the circle of radius O_2b must be hypocycloids generated by rolling 3 on the inside of the pitch circle of the gear. The teeth are thus corrected so far as concerns the interference of the active profiles. Contact between each pair of teeth would therefore begin at a and cease at c where the outside radius of the pinion intersects the line of obliquity de, the resulting path of contact being abc. However, the pinion and gear are so nearly of the same size that the *inactive* profiles of the teeth will interfere, or foul, as they go in and come out as shown by the shaded areas at f, g, h, and i. It is to be noted, for the data chosen, that fouling would not be eliminated even if the addendum of the teeth of the gear were reduced by making the addendum circle equal to O_2b. The trouble is that there is not sufficient difference between the numbers of teeth on the gear and pinion. For $14\frac{1}{2}$-degree obliquity and a pinion addendum of unity divided by the diametral pitch, the gear should, in general, have at least 12 more teeth than the pinion if fouling is to be avoided. For 20-degree obliquity and a·pinion addendum equal to 0.75 divided by the diametral pitch, the gear should, in general, have at least 7 more teeth than the pinion.

While for tooth space equal to tooth thickness, a 4-tooth pinion cannot be made to drive a rack continuously, a 4-tooth pinion with proper obliquity and pointed teeth can be made to drive continuously an internal gear having 6 or more teeth up to about 10,000. By using pointed pinion teeth and making the tooth thickness of the pinion greater than the tooth space, a 3-tooth pinion can be made to drive a 5-tooth internal gear.

It is obvious from an inspection of Fig. 319 that, for the involute pinion shown, an external gear having the same addendum and pitch radius as the internal gear would have a shorter path of contact than that shown for the pinion and internal gear. It is also obvious that the outlines of the teeth of the external gear would correspond to the outlines of the spaces of the internal gear. The teeth of an internal gear are therefore stronger in form than the teeth of an external gear of the same pitch diameter and number of teeth.

Generally speaking, the portions of the tooth profiles of a pair of external gears that are in contact at the beginning and end of action are opposite in sensitiveness, while for an internal gear pair they are of the same sensitiveness. This means that the contact radii of curvature of the teeth of an internal gear pair differ less in length than the contact radii of the teeth of a pair of external gears of the same pitch and number of teeth. According to Art. 144, the specific sliding and consequent wear would therefore be less for an internal gear pair than for a pair of external gears.

The tooth contact for a pair of internal involute gears is between convex and concave surfaces. For the same pitch and tooth load, this, according to Art. 148, means that the maximum compressive
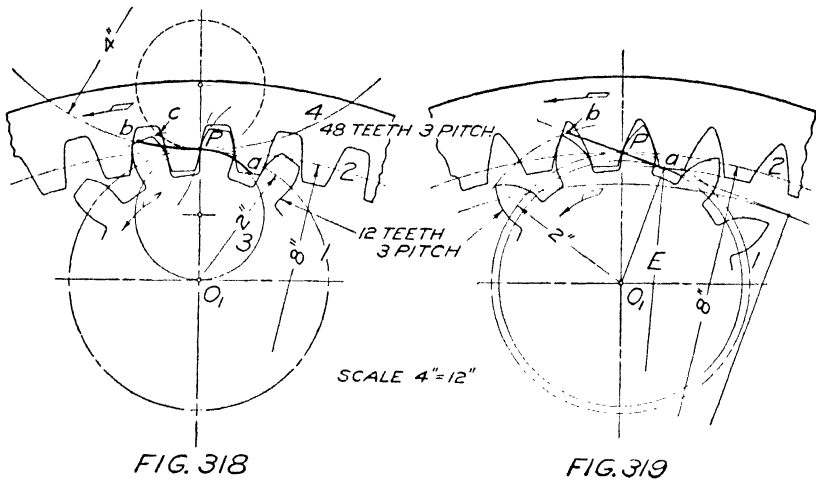
FIG. 318   FIG. 319

stress induced at contact would be appreciably lower for an internal gear pair than for a pair of external gears of the same number of teeth.

The cycloidal pinion and internal gear shown in Fig. 318 have 12 and 48 teeth, respectively, of 3 diametral pitch and an addendum of unity divided by the diametral pitch. The respective flanks and faces of the pinion and gear teeth have been generated by circle 3. According to Art. 134 the radius of the generating circle of the respective flanks and faces of the gear and pinion could be made as great as 5 in. without causing the inactive profiles to foul. In the figure a radius of 4 in. was used, thus giving a path of contact aPb 1.884 in. long. A generating circle the same size as 3 would have given a shorter path of contact aPc 1.611 in. long. For the teeth as generated, the angle of obliquity at the beginning of action is 25° 35′ 10″ and 7° 2′ at the end of action. The 20-degree involute pinion and internal gear shown in Fig. 319 have

12 and 48 teeth, respectively, of 3 diametral pitch. To contrast the length of the path of contact with that of Fig. 318, the addendum radius of the gear was taken as small as involute action permitted, namely E, and the working depth of the tooth the same as Fig. 318, giving a pinion addendum of 0.4587 in. It is to be noted that for this addendum the pinion teeth come almost to a point. For the proportions shown, the path of contact aPb is 1.492 in., or 0.392 in. shorter than the path of the cycloidal pair shown in Fig. 318.
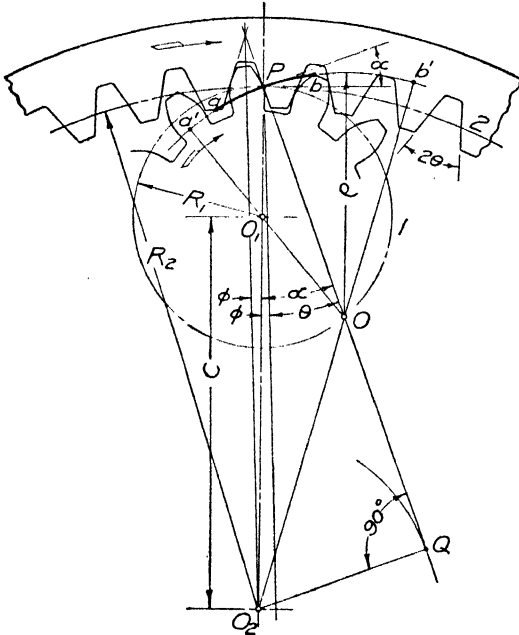
151. Williams' Tooth Form for Internal Gears. Williams proposed that the teeth of internal gears be made with straight profiles as shown in Fig. 320. Having assumed straight profiles corresponding to some desired angle of obliquity α for contact at the pitch point P, the path of the point of contact and the conjugate profile of the pinion teeth may be found as outlined in Art. 126. As shown in Fig. 320, the path of the point of contact closely approximates the arc of a circle of radius ρ. With reference to the figure:

FIG. 320

N_1 = the number of teeth on the pinion;

N_2 = the number of teeth on the gear;

α = the angle of obliquity for contact at the pitch point;

ϕ = the angle corresponding to half of a tooth space

$$= \frac{1}{4}\left(\frac{360}{N_2}\right) = \frac{90}{N_2};$$

θ = half the included angle of a tooth space = $(\alpha - \phi)$;

$O_2Q = R_2 \sin \alpha$;

$PQ = R_2 \cos \alpha$;

ρ = the radius approximating the path of contact

$$= \frac{1}{2}(PQ) = \frac{R_2 \cos \alpha}{2}$$

Evidently a circle with O_1 as a center would be tangent to the path of contact at a'. Since a given point on the face of a pinion tooth can make contact properly with only one point on the profile of a gear tooth, a' marks the earliest beginning of contact possible and fixes the limiting value of the addendum of the gear teeth. In a similar manner b' marks the extreme end of contact and fixes the limiting value of the addendum of the pinion teeth, provided they do not come to a point within this limit.

For the purpose of comparison, Fig. 320 was drawn for the same pitch, the same numbers of teeth, and the same working depth, using equal addenda, as was used in Figs. 318 and 319. The angle of obliquity at the pitch point was taken 20 degrees, making the gear tooth space angle $2\theta = 36° 15'$. For these data the resulting length of the path of contact aPb is 1.645 in., or 0.153 in. greater than for the 20-degree involute pair of Fig. 319. The angle of obliquity at the beginning of action at a is 26° 21' 30'', and 13° 50' at the end of the action at b.

The simple form of the tooth spaces of the internal gear and the resulting form of the path of contact constitute the advantages of the Williams tooth form. Cutters for the internal gear are easily made with

Fig. 321

precision, and may be used in existing gear-cutting machines. For satisfactory results the teeth of the pinion should be generated. This would require a cutter of the same shape as a section of the gear. However, such a cutter, while easily made, could not be used on any existing gear-cutting machine. With the Williams tooth form the path of contact is longer for the same working depth than for the involute tooth form,
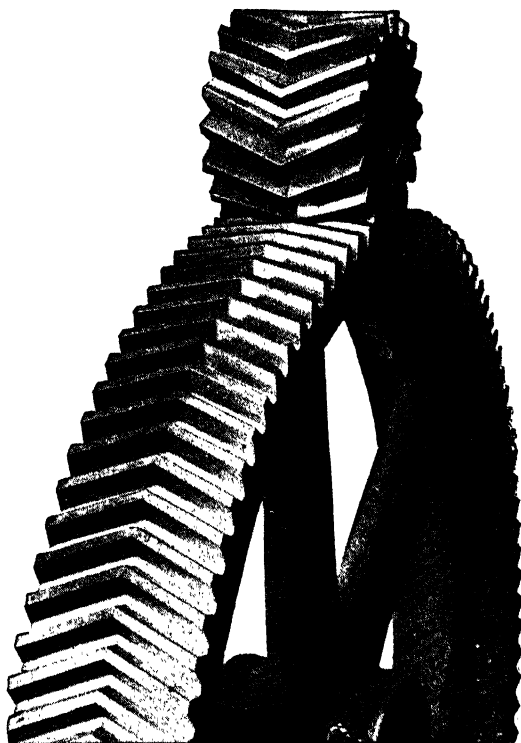
and smaller pinions are possible for equal tooth space and thickness and equal addenda for the pinion and gear.

152. Stepped and Helical Spur Gears. Thus far only straight spur gears, the pitch elements of whose teeth are straight lines parallel to the axes of rotation, have been considered. Spur gears, the pitch elements of whose teeth are not straight and parallel to the axes of rotation, are called *curved* spur gears. While theoretically the pitch elements of the teeth of such gears may be of any form, they are, for ease and convenience in cutting or casting, almost invariably regular helices or combinations of regular helices. Where the pitch elements of the teeth are regular helices, the gear is called a *helical* spur gear. A pair of such gears is shown at *3* and *4* in Fig. 270. It is evident that a pair of such gears, when transmitting power, would develop an axial thrust. Bearings to take this thrust must therefore be provided. A pair of *herringbone* spur gears is shown in Fig. 321. It is to be noted that a herringbone gear is equivalent to two equal helical spur gears of opposite twist. Hence, the axial thrust on one half of the gear is counterbalanced by an equal and opposite axial thrust on the other half, thus relieving the supporting bearings of axial thrust.

FIG. 322

A straight line in a plane which rolls on a base cylinder will generate the involute tooth surface of a straight spur gear if the line is parallel to the axis of the cylinder. If the straight line is oblique to the axis of the base cylinder, it will generate the involute tooth surface of a helical spur gear when the plane is rolled on the base cylinder. From the method of generation it follows that the contact between the teeth of a pair of helical spur gears will be a straight diagonal line. It also follows that the tooth sections are involute in outline only in the plane of rotation.
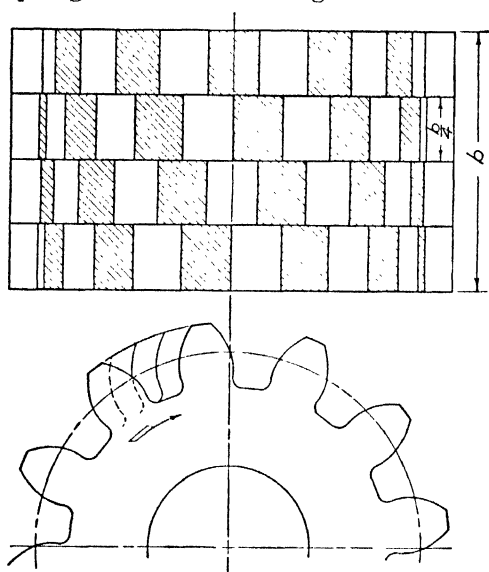
Fig. 322 shows four equal straight spur gears of equal width having a relative angular displacement between them of one-quarter the circular pitch. By fastening the laminations together, a *stepped* spur gear is

formed. To make the figure more easily read, the plan was projected from the pitch circle and the pitch sections of the teeth cross-hatched. It is to be noted that the stepped gear would rotate through a pitch arc equal to one-quarter of the circular pitch from the time a tooth on one lamination leaves contact until a tooth on the next lamination comes into contact in the axial plane. Hence, contact in the axial plane, or pure rolling action between mating teeth, is four times as frequent for the stepped gear as it would be for a straight spur gear of width b. If, therefore, the laminations could be accurately spaced, stepped gears would be quieter and smoother in action than straight spur gears. However, since accurately spaced stepped gears would be more difficult and expensive to make than helical spur gears, they are never used to secure quiet running. Cast gears having a small number of teeth are sometimes stepped to secure continuous driving. Cast gears that would otherwise have a contact ratio of less than unity are made in two steps.

A helical spur gear may be looked upon as a stepped spur gear having an infinite number of laminations which have been given a uniform relative twist. Obviously, if the total twist for a pair of mating gears is made equal to or greater than the circular pitch, there will be continuous contact in the axial plane as the gears rotate. Hence, helical gears are much quieter and smoother running than straight spur gears and can be run successfully at much higher velocities, pitch line velocities as high as 12,000 feet per minute being not uncommon. Double helical and herringbone gears are therefore used for pitch line velocities that would be too high for straight spur gears; and, because they run more quietly and smoothly, they are being used in preference to straight spur gears for lower velocities.

Another advantage of helical over straight spur gears is that continuous positive driving can be secured by the use of smaller pinions. For continuous positive driving with straight spur gears, the arc of action must be greater than the circular pitch. With helical spur gears, the arc of action plus the twist of the teeth must be equal to or greater than the circular pitch. Hence, for a pair of helical spur gears, continuous driving is, within limits, independent of the number of teeth in the gears. It is possible, though not practical, to have a helical pinion of one tooth work properly with a helical gear.

If the faces of the teeth of a pair of helical gears were relieved from the pitch elements outward, the contact between the teeth would, theoretically, be reduced to a point, which, during action, would traverse the pitch elements of the mating teeth. Hence, tooth contact throughout rotation would remain in the plane of the axes of the gears, and there would be only pure rolling between the teeth. While such a pair of

gears, because of having point contact, may not be considered of any practical importance, they may be considered as probably constituting the only example of pure rolling and constant angular velocity ratio combined with positive driving.

In Fig. 323 is shown the pitch cylinder of a helical gear with the corresponding pitch elements of adjacent teeth crossing the elements of the pitch cylinder at an angle ϕ, called the helix angle. The circular pitch measured in the plane of rotation is p_c. The normal circular pitch p_n, or, briefly, the normal pitch, is the distance between the corresponding pitch .elements of adjacent teeth measured along the normal helix cd. Obviously,

$$p_n = p_c \cos \phi \qquad \ldots \ldots \ldots \quad (246)$$
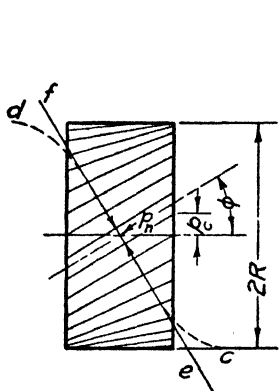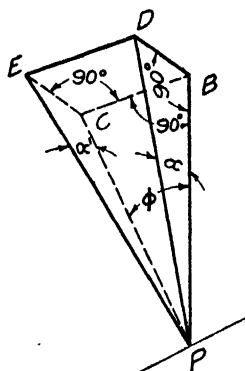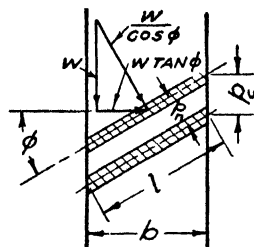
FIG. 323   FIG. 324   FIG. 325

Since the product of the corresponding circular and diametral pitches is equal to π, the relation of the diametral pitches p'_n and p'_c becomes

$$p'_n = \frac{p'_c}{\cos \phi} \qquad \ldots \ldots \ldots \quad (247)$$

As pointed out above, the tooth profiles of helical involute gears are involute in outline in the plane of rotation but not in any other plane. While not involutes, the outlines of the normal sections of the teeth will approximate involutes for a pitch radius corresponding to the radius of curvature of the normal helix.

The angle of obliquity α' in the normal plane is related to the helix angle ϕ and the angle of obliquity α in the plane of rotation. In Fig. 324, PBD represents the plane of rotation of a helical gear and is at right angles to plane PBC which is tangent to the pitch cylinder of the

gear; *PCE* is the normal plane, and plane *PDE* is at an angle α with the tangent plane *PBC* measured in the plane of rotation *PBD*. Obviously,

$$BD = CE = PB \tan \alpha$$

and

$$PC = \frac{PB}{\cos \phi}$$

Hence,

$$\tan \alpha' = \frac{CE}{PC} = \cos \phi \tan \alpha \quad . \quad . \quad . \quad (248)$$

The angle of obliquity α' is therefore always less than the angle of obliquity α in the plane of rotation.

The above equations apply also to helical gears, sometimes erroneously called *spiral* gears. Helical spur gears should not, however, be confused with helical gears. Even though they appear alike and are cut in the same way, they differ in design and in the action between mating teeth. Helical spur gears have parallel axes, their angular velocities are inversely as their pitch diameters, their teeth make line contact, and the action between mating teeth consists of a combination of rolling and sliding in the plane of rotation as in straight spur gears. There is no component of sliding along the teeth. Helical gears have non-parallel, non-intersecting axes, their angular velocities are not inversely as their pitch diameters, their teeth make point contact, and, in action, there is a component of sliding along the teeth as well as a combination of rolling and sliding similar to that for helical spur gears.

If *PE* in Fig. 324 represents the load on a tooth in the normal plane, then *PD* represents the component of the load in the plane of rotation, and *ED* the component parallel to the axis of the gear. If we deal only with the tooth load in the tangent plane *PBC*, then *PB* is the component in the plane of rotation and *CB* the component parallel to the axis. In Fig. 325, the shaded bands represent sections of teeth at the pitch surface of a helical spur gear. If in this figure *W* corresponds to the load *PB* of Fig. 324, then the axial thrust becomes $W \tan \phi$, and the normal load becomes $W \div \cos \phi$. Noting that the width of face b equals $l \cos \phi$, where l is the length of the tooth, the load per inch of length of tooth becomes,

$$w = \frac{W}{\cos \phi} \div l = \frac{W}{b} \quad . \quad . \quad . \quad . \quad (249)$$

Hence the load per inch of length of tooth for a helical spur is the same as for a straight spur gear of the same width of face b. However, the

thickness of a helical gear tooth is proportional to p_n, while the thickness of the corresponding straight spur gear tooth would be proportional to p_c. The helical gear tooth would therefore appear to be the weaker tooth. For usual values of ϕ and b it is, however, stronger, owing to the fact that the line of contact runs diagonally across the tooth and can never be straight across the tip of a tooth as in straight spur gears.

It is to be noted that the larger the helix angle ϕ, the greater the axial component $W \tan \phi$, and the greater the thrust on the bearings of helical gears, or the greater the wedging action of the teeth of herringbone gears. The helix angle should therefore be made no greater than necessary. Since, as before pointed out, the twist should be equal to or greater than the circular pitch, b and ϕ are related thus:

$$\frac{b}{p_c} > \frac{1}{\tan \phi} \text{ or } > \cot \phi . \quad . \quad . \quad . \quad . \quad . \quad (250)$$

Hence the smaller b the greater must be the helix angle ϕ. For herringbone gears the active width of face and therefore the ratio of b to p_c must be twice that given above, or $2 \cot \phi$. The American Gear Manufacturers' Association recommend a minimum factor of 2.3 instead of 2.0, or a minimum width of face

$$b = \frac{2.3 p_c}{\tan \phi} = 2.3 p_c \cot \phi \quad . \quad . \quad . \quad . \quad (251)$$

This Association also recommends 45 and 20 degrees as maximum and minimum values for the helix angle. As a rule, large helix angles are confined to high speeds and light loads.

TABLE 10

		Minimum	Maximum
Addendum....................	a	$0.7/p'_c$	$1.0/p'_c$
Clearance....................	c	$0.157/p'_c$	$0.30/p'_c$
Backlash, industrial gears.......	.....	$0.030/p'_c$ but not $< 0.002''$	
high-speed gears......	.....	$0.040/p'_c$ but not $< 0.003''$	
Angle of obliquity.............	α	$15°\text{-}23'$	$25°$
Helix angle, or angle of cut.....	ϕ	$20°$	$45°$
Active face of herringbone gears.	b	$2.3 \, p_c \cot \phi$	

When helical and herringbone gears are cut with the tools for straight spur gears, their tooth proportions are necessarily referred to the normal section. The general designer should always leave the details of tooth

proportions, helix angle, etc., to the gear manufacturer. In the design of helical and herringbone gears, the American Gear Manufacturers' Association recommend that the diametral pitch and tooth proportions be referred to and calculated on the basis of the diametral pitch in the plane of rotation. The further recommendations of the Association are given in Table 10.

Helical and herringbone gears that are to be cut on a Fellows gear shaper are usually designed for a helix angle ϕ of approximately 15 degrees or for a helix angle of approximately 23 degrees.

153. Non-Circular Spur Gears. Gears for the transmission of motion between parallel shafts are called spur gears. Thus far only circular spur gears have been treated. The pitch surfaces of such gears are right cylinders, their pitch curves are circles, and their purpose is to transmit motion between parallel shafts at a constant angular velocity ratio. The purpose of non-circular spur gears is to transmit motion between parallel shafts at a varying angular velocity ratio. Rolling sectors and regular and irregular uni- and multi-lobed wheels having plane motion about parallel axes were fully treated in Chapter VI. As was there pointed out, such rolling surfaces and their corresponding rolling curves may be used as the pitch surfaces or curves of non-circular spur gears. Many non-circular pitch curves are shown in Figs. 231 to 252. A pair of elliptical gears, Fig. 235, is perhaps one of the most familiar examples of a pair of non-circular spur gears. ,

Since the pitch curves of non-circular gears are not of uniform curvature, it follows from Arts. 127, 129, and 136 that the tooth spaces of any such gear must necessarily differ in shape. To cut such a gear accurately by the use of formed cutters would therefore require as many cutters as there are tooth spaces. On account of the limited demand for such gears, machines to cut them by a process of generation have not been developed. Such a machine is not available even for elliptical gears, the most common form of non-circular gears. The tooth profiles of non-circular gears, unless cut by a generating process, must, therefore, be only approximately correct in outline.

A method of cutting approximately correct teeth on non-circular gears by the use of formed cutters may be derived from the graphical methods of describing the outlines of gear teeth as given in Arts. 129 and 136. If, as outlined in Art. 129, circles are used as generators and are rolled on the pitch curves, cycloidal tooth profiles will be described. Since it is required that the generating curve be capable of rolling on the pitch curves, the circle, being the simplest, is, theoretically, the best generator for non-circular spur gear teeth. The radius of the generating circle must, of course, be less than the minimum radius of curvature of

the pitch curve on which it is to roll. If a diameter of generating circle corresponding to available cycloidal formed cutters can be used, a cutter may be selected for each tooth space to correspond closely to the radius of curvature of the pitch curve at that tooth space.

If involute teeth are desired, the generator becomes a straight line, and the procedure of Art. 137 would be followed. As for pitch circles, the base curves of non-circular pitch curves are curves drawn tangent to obliquity lines through various points on the pitch curve, or the pitch curves and corresponding base curves may be looked upon as made up of a series of circular arcs. It was thus that the base curves and the corresponding involute outlines of the teeth were found for the elliptical gears shown in Fig. 235. Centers on the minor and major axes for arcs that would closely approximate the pitch ellipses were first found, and then the corresponding base arcs were drawn from the same centers for the desired angle of obliquity. This same approximate method may be applied to other non-circular pitch curves, and an involute formed cutter selected to correspond more or less closely to the radius of curvature of the pitch curve at each tooth space.

CHAPTER VIII

CUTTING OF STRAIGHT AND HELICAL SPUR GEARS

154. Cast and Rolled Gears. The first toothed wheels were made of wood, and the first metallic gears were cast. Although the production of cut gears has been greatly increased and the cost greatly reduced, cast gears, both small and large, are still found, in certain situations, to be cheaper than cut gears and to give satisfactory service. Cast gears are still used in machines and appliances of low grade, and in places where they are exposed to the elements and operated only occasionally. Such gears are, for example, still used in agricultural machinery.

Cast gears may be pattern molded or machine molded. For pattern molding a complete pattern is employed. Except for the core prints on the hub, such patterns are, in appearance, exactly like the gears desired. To allow for the possible warping and twisting of the pattern, for imperfections in the form and spacing of the teeth, and for rapping the pattern preparatory to drawing it from the sand, backlash must be provided. The backlash, or difference between the width of the tooth spaces and the thickness of the teeth, is generally made about 0.045 times the circular pitch.

Since patterns are costly and liable to warp and twist, special machines for molding gears were devised to reduce the amount and cost of the pattern work and, at the same time, to produce more accurate gears. The principle of such machines may be illustrated by the use of Fig. 326. About a stake, or arbor, a circular pit having a depth equal to the width of face of the gear, and a diameter greater than the outside diameter of the gear, is first swept up in the sand. On the central stake is then mounted an indexing head carrying a radial arm to the outer end of which is attached the vertical support d which carries at its lower end a pattern of at least two teeth. The vertical support d and pattern block can be moved up and down or in and out radially. As shown in the plan view, it is moved radially outward to the proper radius, and molding sand is filled in between the pattern block and the outer edge bc of the pit. Owing to the shape of the teeth and the fact

that the pattern block can be moved radially inward, no draft is required, thus leaving the teeth more accurate in form than the teeth of a complete gear pattern. Having filled the space between the pattern block and pit with sand, the pattern block is withdrawn radially and accurately rotated, or indexed, about the central arbor through an angle corresponding to the circular pitch. The pattern block is then run out to the proper radius as before, and sand is filled in between it and the outer edge of the pit. After completing the molding of the tooth spaces in this manner, the molding machine is lifted out and the mold for the gear completed by placing the cores to form the inner part of the rim and the arms and hub.

Small gears and other products are frequently die cast. In die
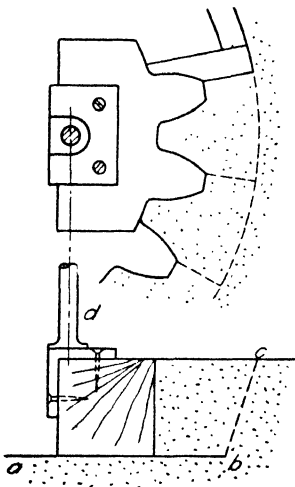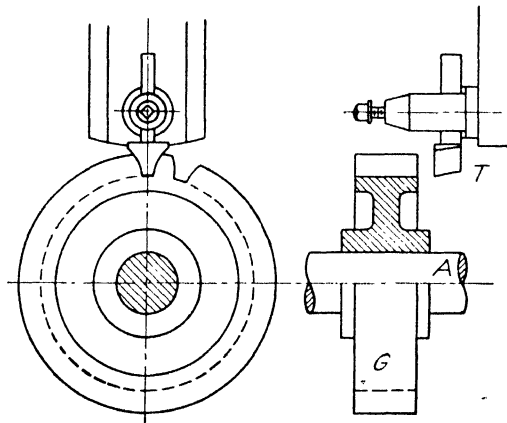
FIG. 326   *FIG. 327*

casting the molten metal is forced into steel molds under pressure. In general the process is limited to melting temperatures of about 1200° F., or to the use of aluminum, zinc, tin, lead, and antimony, and their alloys. Such castings may be made so accurate as to size and form as to be used directly without machining. Certain non-metallic materials, such as Bakelite, which, under moderate heat, are sufficiently plastic, are pressed to form in dies. Small gears and other products are thus made.

The teeth of metallic gears, especially of steel, have been rolled to form by a molding process of generation. The metal blank was heated until plastic and then rolled under pressure in contact with a rolling die, which generated the teeth. Straight and curved spur and bevel gears

were produced by this process which was used for a time and then abandoned.

155. Gear-Cutting Machines. If the teeth of a gear are to be cut, a gear blank is first turned to the finished dimensions of the gear. The teeth are then formed in the heavy rim of the blank by cutting away the metal to form tooth spaces of the required shape. The development of gear-cutting machines to do this work at a reasonable cost has required a vast amount of experimentation and the designing and redesigning of many different types of machines and cutters. No class of machine tools has called for more ingenuity in design than gear-cutting machines. In the many machines that have been designed, the cutting is either a planing or a milling operation. In the planing machines the cutter has a reciprocating motion and makes many cutting and return strokes for each tooth space formed. Machines that mill the tooth spaces use rotating cutters, and the cutting is continuous for at least each tooth space that is cut. The principal types of gear-cutting machines may be classified as non-generating machines, template machines, and generating machines. The non-generating machines use a cutter formed to the shape of the tooth space to be cut. The cutter may be in the form of a planing tool as shown in Fig. 327, or in the form of a milling cutter as shown in Figs. 328 and 329. In the template machines, a template of proper form is used to guide the cutting tool in shaping the tooth spaces. In the generating machines, the gear teeth are generated by properly relating the motion of the gear blank and the motion of the cutter. For want of space only the principles of the machines and not the machines themselves will be discussed.

156. Formed Cutter Method of Cutting Gear Teeth. Quite naturally, the use of a planing tool, corresponding in shape to the tooth space to be cut, was one of the first methods to suggest itself for cutting gears. The manner in which such a cutter is used to cut a straight spur gear is shown in Fig. 327. The gear blank G is mounted on the arbor, or mandrel, A; the tool T is mounted in the tool post and slide with its midplane in the radial plane of the blank. As the tool reciprocates parallel to the axis of the blank, it is fed, for each cutting stroke, radially inward toward the axis of the blank. After each tooth space has been finished, the arbor and blank are turned, or indexed, through an angle corresponding to the circular pitch of the gear. This indexing must be accurately done if the teeth are to be equally spaced.

To cut a helical spur gear by this method, the blank must be oscillated about its axis in unison with the reciprocation of the tool. If the helix angle is 23 degrees, the blank must rotate through a pitch arc equal to the axial travel of the tool times the tangent of 23 degrees.

Because of the many idle return strokes of the tool for each tooth space that is cut, the above method of cutting gear teeth is slow. A further objection to the method is that all the cutting is done by a single cutting face or edge. For this reason, except for small teeth, the tooth spaces of gears to be cut by this method are first rough cut to approximately correct shape.
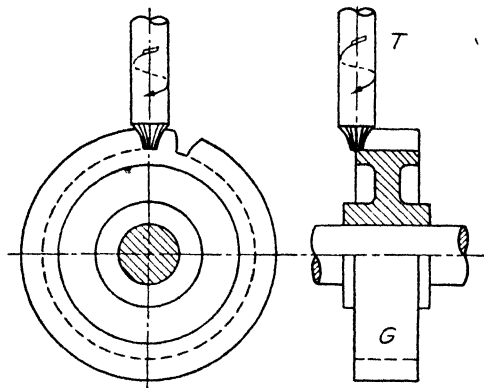
FIG. 328

As pointed out in Art. 123, the first milling cutter was made about 150 years ago and was an end milling tool of the type shown at T in Fig. 328. It is mounted with its axis of rotation in the radial plane of the blank and at a distance from the axis of the blank equal to the root radius of the gear. As the cutter rotates it is fed parallel to the axis of the blank; or, as the cutter rotates, the blank is fed parallel to its own axis. After each tooth space has been finished, the blank and arbor are indexed through an angle corresponding to the circular pitch. To cut a helical gear by this method, the blank must be rotated about its axis as the cutter is fed parallel to the axis of the blank. It is to be noted that this method of cutting is superior to the planing method mentioned above. Instead of having only one, the tool has many cutting edges; and instead of there being many, there is only one idle stroke
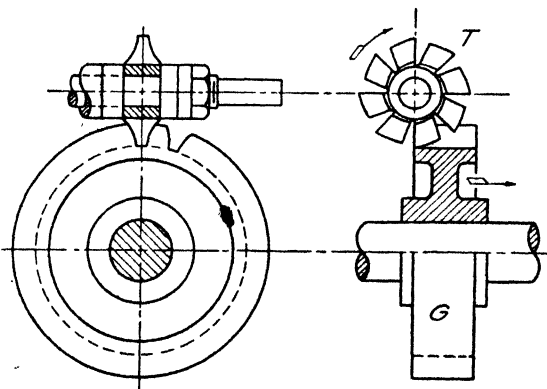
FIG. 329

for each space that is cut. This type of cutter, while still used in Europe, is not used in this country.

The type of formed cutter shown in Fig. 329 was invented by Joseph R. Brown in 1864; it can be sharpened by grinding without changing

the shape of its cutting face. The cutter is mounted with its midplane in the radial plane of the blank at a distance from the axis of the blank equal to the root radius of the gear. As the cutter rotates, the blank is fed axially into the cutter; or the blank may be held stationary and the rotating cutter fed parallel to the axis of the blank. After each tooth space has been finished, the blank and arbor are indexed through an angle corresponding to the circular pitch of the gear. It is to be noted that this method of cutting has the same advantages over the planing method as those mentioned for the method of Fig. 328.

Except for large spur gears of heavy pitch, comparatively few of the gears now produced are cut by formed cutters. The use of milling and other machines employing the Brown type of cutter is largely confined to small repair and jobbing shops. For the standard $14\frac{1}{2}$-degree composite system of tooth profiles, sets of formed cutters of the Brown type are available for each of the diametral pitches given in Table 11.

TABLE 11

DIAMETRAL PITCHES

Stock Cutters					Made to Order				
2	6	11	20	30	1	$2\frac{1}{4}$	$4\frac{1}{2}$	34	60
$2\frac{1}{2}$	7	12	22	32	$1\frac{1}{4}$	$2\frac{3}{4}$	$5\frac{1}{2}$	38	64
3	8	14	24	36	$1\frac{1}{2}$	$3\frac{1}{4}$	13	44	70
4	9	16	26	40	$1\frac{3}{4}$	$3\frac{1}{2}$	15	50	80
5	10	18	28	48	....	$3\frac{3}{4}$	....	56	120

For each diametral pitch, a set of 15 cutters is available to cut from 12 teeth to a rack. For the same pitch and same degree of accuracy, as stated in Art. 149, the cycloidal system requires about three times as many cutters as the involute system.

A given number of cutter can, of course, be correct only for a given number of teeth. In Table 12 each cutter is of correct shape for the minimum number of teeth of its range. Where a gear has a number of teeth falling within one of the ranges given in the table, it is customary to use the cutter corresponding to that range. It can be shown, however, that the action of a pair of mating gears can be improved by using for one gear a cutter corresponding to a range higher than would include the number of teeth in the gear, and for the other gear the customary cutter. The depths to which the spaces are cut should then be varied accordingly. Buckingham in his book " Spur Gears " gives a table

TABLE 12

CUTTERS FOR STANDARD $14\frac{1}{2}°$ COMPOSITE SYSTEM

Number Cutter	Will Cut Gears	Number Cutter	Will Cut Gears
1	from 135 teeth to rack	5	having 21 or 22 teeth
$1\frac{1}{2}$	" 80 " " 134	$5\frac{1}{2}$	" 19 or 20 "
2	" 55 " " 79	6	" 17 or 18 "
$2\frac{1}{2}$	" 42 " " 54	$6\frac{1}{2}$	" 15 or 16 "
3	" 35 " " 41	7	" 14 teeth
$3\frac{1}{2}$	" 30 " " 34	$7\frac{1}{2}$	" 13 "
4	" 26 " " 29	8	" 12 "
$4\frac{1}{2}$	" 23 " " 25		

showing the cutters and the depths to which the tooth spaces should be cut for different pairs of gears.

If a formed cutter of the Brown type is to be used to cut a helical

FIG. 330   FIG. 331

spur gear, the cutter is rotated in a plane, making the helix angle ϕ with the radial plane of the blank G, as shown in Fig. 330. With the cutter at a distance from the axis of the blank equal to the root radius of the gear, the blank is fed axially toward the rotating cutter. While

being fed axially the blank is simultaneously rotated, counter-clockwise in the present instance, the ratio of the pitch arc to the axial feed being equal to the tangent of ϕ. On the completion of each tooth space the blank is indexed through an angle corresponding to the circular pitch of the gear.

As stated in Art. 152, the normal sections of the teeth of a helical involute gear are not involute in outline. The outlines correspond approximately to involutes for a pitch radius equal to the radius of curvature of the normal helix. This radius of curvature is equal to the radius of curvature at the end of the minor axis of an ellipse formed by the intersection of a plane ef with the pitch cylinder when drawn tangent to the normal helix cd as shown in Fig. 323. The minor and major axes of such an ellipse are, respectively, $2R$ and $\dfrac{2R}{\cos \phi}$, for which the radius of curvature ρ at the end of the minor axis is

$$\rho = \frac{R}{\cos^2 \phi} \quad \text{. . . . .} \quad (252)$$

Or, if the gear has N teeth, the shape of the normal section of a tooth and space will correspond approximately to a formative number of teeth N_f whose value is

$$N_f = \frac{2\pi\rho}{p_n} = \frac{2\pi R}{p_n \cos^2 \phi} = \frac{2\pi R}{p_c \cos^3 \phi} = \frac{N}{\cos^3 \phi} \quad \text{. .} \quad (253)$$

On account of the warped nature of the helical surface of the tooth space, a cutter selected to correspond to N_f and the diametral pitch p'_n of the gear would undercut the flanks and relieve the faces of the tooth profiles.* To most closely approximate the tooth space, the cutter should be selected for the diametral pitch p'_n and a number of teeth N'_f instead of N_f. The value of N'_f is

$$N'_f = \frac{N}{\cos^3 \phi} + p'_n d \tan^2 \phi \quad \text{. . . . .} \quad (254)$$

where N = the number of teeth in the gear;

p'_n = the diametral pitch of the cutter or the normal diametral pitch of the gear;

d = the pitch diameter of the cutter;

ϕ = the helix angle.

---

* See "Selection of Milling Cutters for Helical Involute Teeth," by Ernest Wildhaber, *American Machinist*, Dec. 20, 1923.

157. Template Method of Cutting Gear Teeth. The principle of the template method of cutting spur gear teeth may be illustrated by Fig. 331, which shows the gear blank G and the planing tool T. This tool is mounted in a tool holder which can be reciprocated parallel to the axes of the blank and also fed normal to the axis of the blank. As the tool head is fed perpendicular to the axis of the blank, the follower roller R will traverse the cam, or template, C, and the cutting point of the tool, moving upward as it advances, will trace a curve in exact agreement with the line described by the center of the roller. In using template machines the tooth spaces of the gear to be cut are first either cast or rough machined approximately to shape. Employing a template of proper form, one side of each tooth is then finished, the gear being indexed as required. Having finished one side of the teeth, the tool
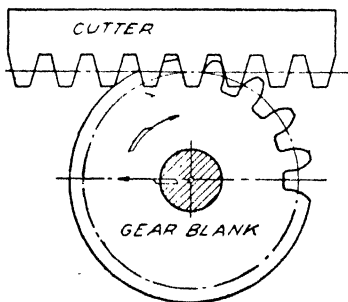
FIG. 332
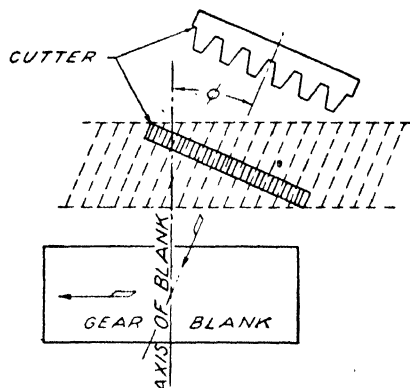
FIG. 333

and template are changed and the other side of the teeth is finished. In some machines the wheel is reversed to finish the second side of the teeth. There are machines which employ two tools and two templates operating simultaneously. The use of template machines is practically confined to gears having large teeth, say, to teeth having a circular pitch of 3 inches or more. In planing helical and herringbone gears by this method, the gear must, of course, be oscillated as the planing tool reciprocates.

158. Shaping Method of Generating Gear Teeth. As before stated, generating machines employ planing or rotating cutters, the gear teeth being generated by properly relating the motion of the gear blank and the motion of the cutter. The planing tools used are of two types: a cutter in the form of a rack, and a cutter in the form of a pinion. The hob, which is the rotating cutter employed, is very well adapted to

repetition work. It is not only widely used as a generating cutter but is also much used as a roughing-out tool.

The generation of gear teeth by the use of a cutter of basic rack form may be illustrated by Fig. 332. For straight spur gears, the blank is mounted on an arbor with its axis parallel to the pitch plane of the rack, or cutter, and the cutter is reciprocated parallel to the axis of the blank. The blank is first fed into the cutter until the pitch circle of the blank is tangent to the pitch line of the cutter; then, as the cutter reciprocates, the blank is slowly fed parallel to the pitch line of the cutter and at the same time rotated about its axis with a motion equivalent to the pure rolling of the pitch circle of the blank on the pitch line of the cutter. If the cutter contained a few more teeth than the gear to be cut, all of the teeth of the blank would be generated by an uninterrupted feeding movement to the left equal to the pitch circumference of the blank. Since such a long cutter would not be practical, a shorter one is used; and the blank, after moving to the left a distance equal to some multiple of the circular pitch, is returned to its original position to start again the generating movement to the left. While being returned to its original position, the blank is, of course, not rotated. How often this interrupted generation must be repeated depends upon the number of teeth to be cut.

The cutting of a helical spur gear may be illustrated by Fig. 333. The dotted construction shown above the gear blank represents a rack having the same helix angle ϕ and the same normal pitch p_n as the gear to be cut, and, hence, represents a rack on which the finished gear could roll. The cutter to be used, shown by full lines in the figure, may be regarded as cut from the imaginary rack by planes passed normal to the teeth of the rack. The teeth of the gear may be generated by reciprocating such a cutter in a plane parallel to but along a path at an angle ϕ with the axis of the blank. As in cutting straight spur gears, the blank is fed to the left parallel to the pitch plane of the rack and at the same time rotated about its axis with a motion equivalent to pure rolling with the imaginary rack. After rolling to the left some multiple of the circular pitch, the blank is returned, without rolling, to its original position and the generating process continued as before and the same repeated until the gear is finished.

The related motions of cutter and blank as just described for the cutting of straight and helical spur gears are in accordance with the operation of the Maag gear shaper which uses a rack-shaped cutter properly relieved to facilitate cutting and to provide for sharpening without change of shape. Such cutters are easily made with precision. In some other machines using this type of cutter, the blank is merely

rotated about its axis, while the reciprocating cutter is translated as a rack. In such machines it is the cutter and not the blank that is set back after each interval of generation.

The Fellows gear shaper, the Sykes gear-cutting machine, and the universal gear shaper made by the National Tool Company use pinion-shaped cutters. Such cutters are properly relieved to facilitate cutting and to provide for sharpening without change of form. A cutter for generating straight spur gears is shown in Fig. 334. The work, or gear blank, is mounted on an arbor whose axis is parallel to the axis of the spindle of the ram on which the pinion-shaped cutter is mounted. The
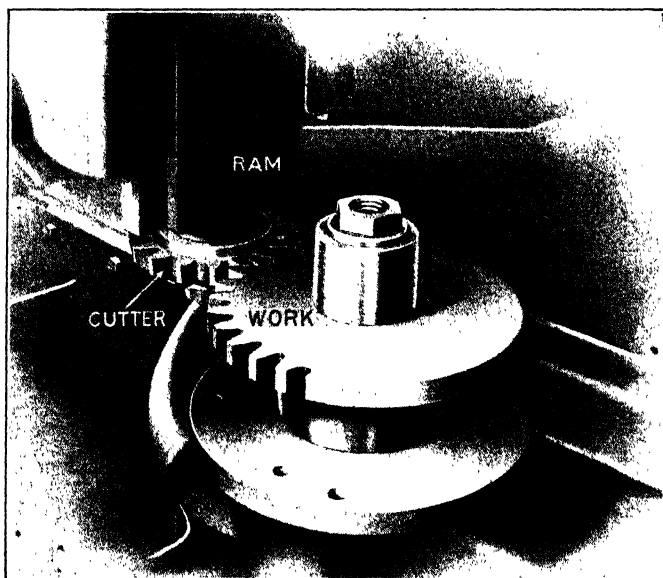
FIG. 334

gear blank is first fed into the reciprocating cutter until the nominal pitch circles of blank and cutter are tangent, after which the cutter and blank are given a feed rotation equivalent to the pure rolling of their pitch circles. All the teeth of the gear are thus generated in one complete revolution of the blank.

The cutter used for generating helical or herringbone spur gears is in the form of a helical spur pinion as shown in Fig. 335. The process of generation is the same as for straight spur gears except that the cutter, while making the cutting stroke, rotates so that its teeth may follow the helix angle. Hence, the cutter rotates through a pitch arc equal to the travel of the ram times the tangent of ϕ.

With the Fellows gear shaper, straight and helical internal as well as external gears may be cut. The machine is extensively used in the
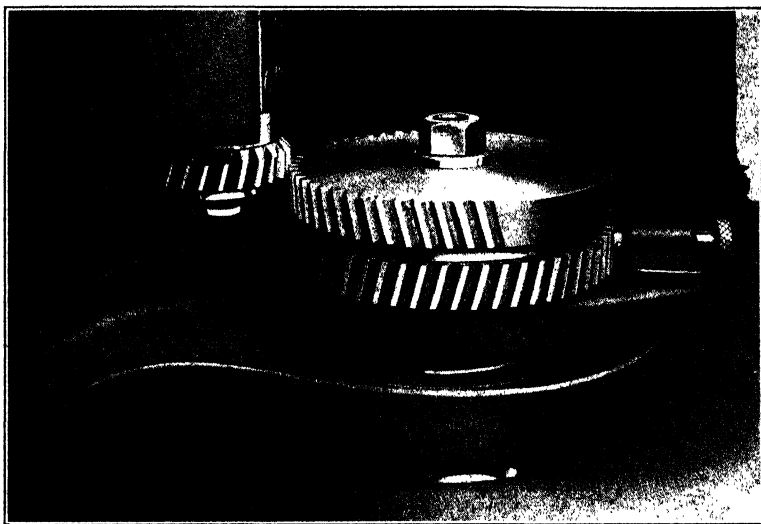
FIG. 335

automotive industry for finishing gears that have been roughed out on gear milling or hobbing machines. The pitch diameter of the pinion cutters is small, 3 in. and 4 in. for external gears, and nominally $3\frac{1}{4}$ in. and $5\frac{1}{2}$ in. for internal gears. For practical tooth proportions and obliquity of action, these small pitch diameters limit the size of the teeth that can be cut, the largest tooth being 3 diametral pitch or approximately one inch circular pitch. Cutters of unity diametral pitch have been used in the Sykes gear shaper.
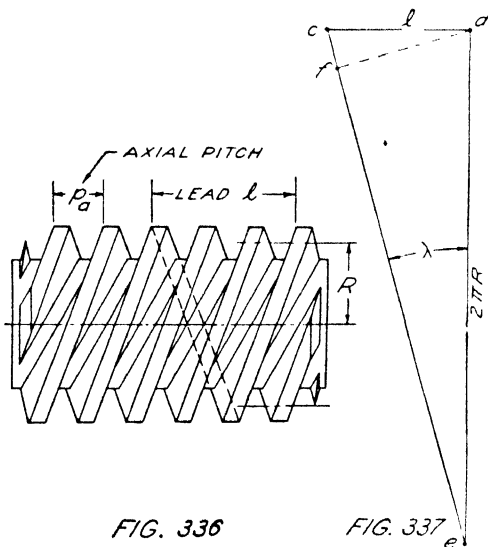
FIG. 336

FIG. 337

159. Hobbing Method of Generating Gear Teeth. In Fig. 336 is shown a triple-thread screw, the lead l of which is 3 times the axial pitch p_a. By lead is meant the

axial advance of the thread per turn. It is the distance from a point on one thread to the corresponding point on the same thread measured in the axial plane of the screw. The pitch is the axial distance from a point on one thread to the corresponding point on the next thread. For a single-thread screw the lead and pitch are equal. Assuming the threads of proper form, the screw of Fig. 336 could be used to drive a gear, in which case, it would be called a worm and R would be the pitch radius of the worm. If the right triangle cde in Fig. 337, the respective legs of which represent the lead and pitch circumference of the worm, were wrapped clockwise on the pitch cylinder of the worm, the hypotenuse ce would follow the pitch helix of the worm. The angle λ is called the lead angle of the worm. Obviously,

$$\tan \lambda = \frac{l}{2\pi R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (255)$$

For a multiple-thread worm of N threads the axial pitch is

$$p_a = \frac{l}{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (256)$$

and the normal pitch p_n is

$$p_n = \frac{(df)}{N} = \frac{l \cos \lambda}{N} = p_a \cos \lambda \quad . \quad . \quad . \quad (257)$$

For the worm to drive a straight spur gear, the normal pitch p_n of the worm must be equal to the circular pitch of the gear. Hence, the axial pitch of the worm must be, from (257),
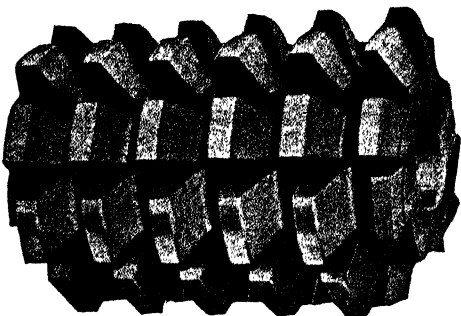
$$p_a = \frac{p_n}{\cos \lambda} = \frac{p_c}{\cos \lambda} \quad (258)$$

Fig. 338 shows a single-thread tool-steel screw that has been made into a hobbing cutter, or hob, by cutting straight radial gashes in the screw and properly relieving

FIG. 338

the remaining portions of the thread to facilitate their acting as cutters. A hob having more than one thread is called a multi-thread hob. Since such hobs cut more rapidly than single-thread hobs, they are much used as roughing-out tools in cutting gears. Where a hob is employed as a finishing tool, a single-thread hob is generally used since greater accuracy can be secured with less difficulty than with multi-thread hobs.

Assuming that the thread surfaces of the screw from which the hob was made are of proper form to generate involute teeth, the manner of applying such a hob to the generation of straight spur gear teeth may be illustrated by Fig. 339. In the figure are shown the pitch surface, the pitch helix, and the lead angle λ of the hob *1* and the pitch cylinder *2* and corresponding pitch elements of adjacent teeth of a straight spur gear. The pitch helix of the hob will be tangent to a pitch element of the gear when the pitch surfaces of hob and gear are tangent and the axis of the hob is at an angle λ with the plane of rotation of the gear. Assuming the hob to be single threaded and to have a normal pitch equal to the circular pitch of the gear, the hob, if acting as a worm, would have to make as many revolutions as there are teeth in the gear to cause the gear to make one revolution. A double-thread hob would have to make only one-half as many revolutions. Hence, if the gear is to have N_2 teeth and the hob has N_1 threads, the angular velocity ratio of the hob to the gear must be
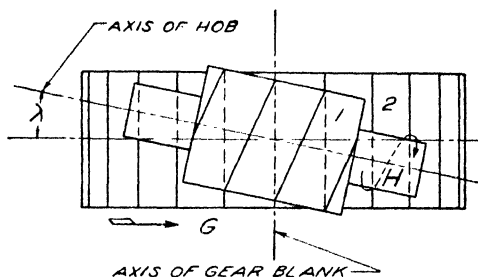
FIG. 339

$$\frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} \qquad \cdots \cdots \cdots \quad (259)$$

If, in Fig. 339, N_2 teeth are to be cut by a hob having N_1 threads, the sequence of operations would be as follows: First the hob would be mounted with its axis at an angle λ with the plane of rotation of the blank and at a distance from the axis of the blank equal to the sum of the pitch radii of hob and blank. Then, as the hob and gear are positively driven at an angular velocity ratio of $\dfrac{N_2}{N_1}$, the hob would be fed parallel to the axis of the blank. In this manner it is possible to generate all the teeth of the blank in one complete axial pass of the hob across the face of the blank. The hob, having left-hand threads, would rotate in the direction indicated by arrow H, the gear in the direction indicated by arrow G, and the feed of the hob would be upwards parallel to the axis of the blank.

In Fig. 340 are shown the pitch cylinder of a hob *1* having a right-hand thread and the pitch cylinder of a helical gear *2* having right-hand helical teeth, λ being the lead angle of the hob and ϕ the helix angle

of the gear teeth.   For the pitch helix of the hob to be tangent to the pitch element of a gear tooth, the pitch cylinders must be tangent and the axis of the hob must be at an angle $\theta = (\phi - \lambda)$ with the plane of rotation of the gear as shown. If the hob and gear were both left-hand instead of right-hand, the angle between the axis of the hob and the plane of rotation of the gear would be the same, $\theta = (\phi - \lambda)$, but rotated $(180 - 2\theta)$ from the position shown. If either one were left-hand and the other right-hand, the angle θ would be $(\phi + \lambda)$.
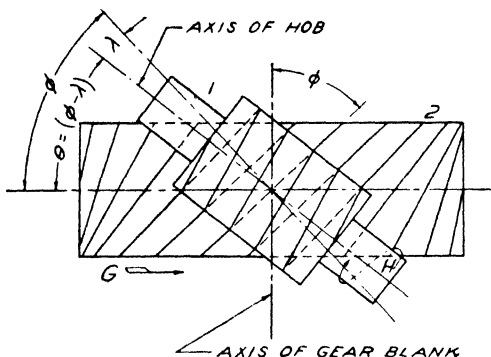
FIG. 340

It has just been shown that in cutting a straight spur gear having N_2 teeth using a hob having N_1 threads, the hob must make $\dfrac{N_2}{N_1}$ revolutions to one of the blank as the hob is fed across the face parallel to the axis of the blank.   In cutting the teeth of a helical gear by the use of a hob, the angular velocity ratio of hob and blank is not independent of the feed of the hob, and will be greater or less than $\dfrac{N_2}{N_1}$, depending upon the direction of the feed and the hand of the hob and gear.   In Fig. 341 let $ABJL$ represent the development of a helical gear having N_2 right-hand helical teeth, ϕ being the helix angle and l_2 the lead of the teeth. Assuming that the hob is of the same hand and that it has N_1 threads and a lead angle λ, the axis of the hob would be set at an angle $\theta = (\phi - \lambda)$ with the plane of rotation of the blank as shown in Fig. 340.   The hob and gear would be positively driven in the directions shown by the respective arrows H and G in Fig. 340, and the hob fed downward parallel to the axis of the blank.   The direction of the downward feed of the hob is represented by the arrow F in Fig. 341.   If G
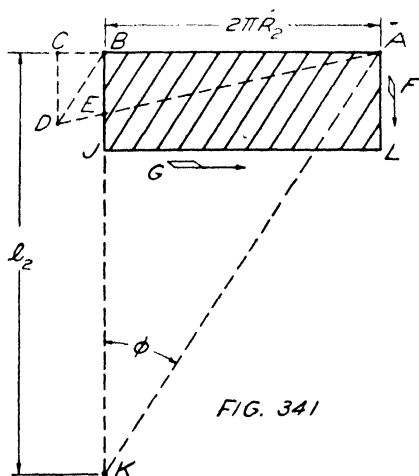
FIG. 341

is the direction of motion of the blank, then, relatively, the hob moves from right to left, at the same time that it moves downward, AE being its line of movement relative to the blank. If in exactly one revolution of the blank, represented by AB in Fig. 341, the hob is fed downward a distance BE, it is to be noted that the cutter would not be working on the same tooth it was at A. The cutter will not be working on the same tooth until it has been fed downward a distance CD and the blank has been rotated through the pitch arc AC. Hence to cut N_2 teeth for a feed equal to CD for $\dfrac{N_2}{N_1}$ rotations of the hob, the gear blank must be rotated through $\dfrac{AC}{AB}$ turns. The angular velocity ratio of the hob to the blank must, therefore, be

$$\frac{\omega_1}{\omega_2} = \left(\frac{N_2}{N_1}\right) \div \left(\frac{AC}{AB}\right) = \left(\frac{N_2}{N_1}\right) \div \left(\frac{AB + BC}{AB}\right) = \frac{N_2}{N_1\left(1 + \dfrac{BC}{AB}\right)} \quad (260)$$

From similar triangles,

$$\frac{BC}{AB} = \frac{CD}{l_2}$$

But,

$$l_2 = \frac{AB}{\tan \phi} = \frac{2\pi R_2}{\tan \phi}$$

Hence,

$$\frac{BC}{AB} = \frac{CD}{l_2} = \frac{CD \tan \phi}{2\pi R_2}$$

where CD represents the feed of the hob parallel to the axis of the blank for $\dfrac{N_2}{N_1}$ revolutions of the hob. Calling this feed f and substituting the above value of $\dfrac{BC}{AB}$ in (260) gives

$$\frac{\omega_1}{\omega_2} = \frac{N_2}{N_1\left(1 + \dfrac{f \tan \phi}{2\pi R_2}\right)} \quad . \quad . \quad . \quad . \quad (261)$$

Equation (261) holds where the hob and gear are of the same hand. Where the hob and gear are of opposite hand, the plus sign in the denominator must be changed to a minus sign.

Since helical gears are of the same form as helical spur gears, the

method outlined above for the hobbing of helical spur gears applies also to the hobbing of helical gears.

In the discussion above it was assumed that the threads of the hob were of such form as to generate involute teeth. To generate teeth of helical spur and helical gears that will be conjugate, the hob teeth must be of such form as to generate the straight-sided teeth of the basic rack. It has been common to assume that the teeth of the hob should be straight sided in the axial plane of the hob. Properly, the sides of the teeth should be straight not in the axial plane, but in a plane tangent to a base cylinder of radius r.* The value of this radius is

$$r = \frac{R \sin \lambda}{\sqrt{\tan^2 \alpha + \sin^2 \lambda}} \quad \ldots \ldots \quad (262)$$
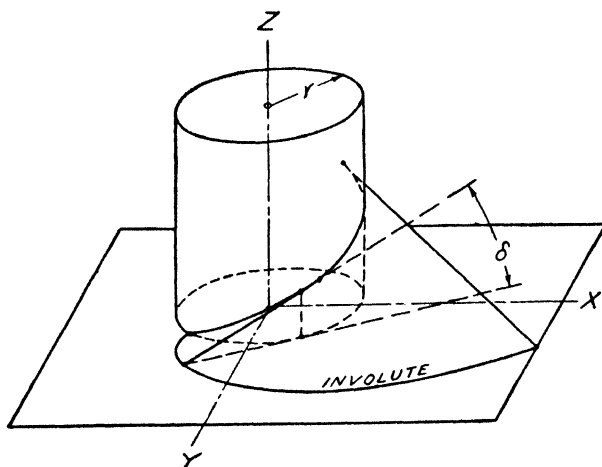
FIG. 342

where R = the pitch radius of the hob;
    λ = the lead angle of the hob;
    α = the angle of obliquity or angle of the sides of the teeth of the basic rack.

The involute helicoidal surface of the threads of a hob may be considered as generated by a line rolling on a base cylinder of radius r while remaining tangent to a helix having a lead l equal to that of the hob. Hence the generating line would make a constant angle δ with a plane

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* For a discussion of the theory, use, and manufacture of involute hobs, see "Spur Gears," by Buckingham, McGraw-Hill Book Company; and "Problem of the Theoretically Correct Involute Hob," by Nikola Trobjevich, in *Machinery* for January, 1919.

perpendicular to the axis of the base cylinder and would describe in this plane an involute curve as shown in Fig. 342. As defined,

$$\tan \delta = \frac{l}{2\pi r} \qquad \cdots \cdots \cdots \quad (263)$$

In finishing the threads of a screw for a hob the tool may be set as shown at A or B in the Fig. 343.

In the above description of the use of a hob in generating gear teeth, the pitch helix of the hob was assumed tangent to the pitch element of the tooth to be cut. In this position the hob will cut a width of space equal to the normal pitch thickness of the hob tooth. If the angle of
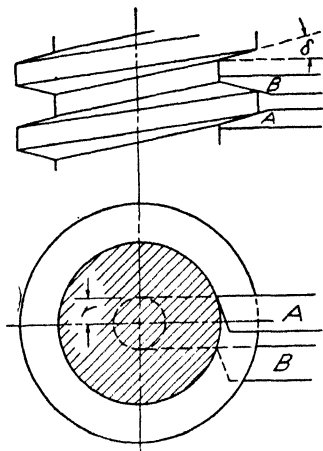
FIG. 343
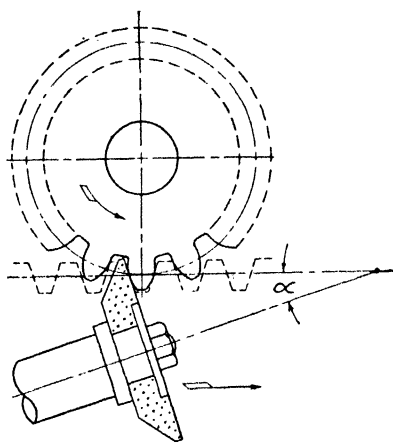
FIG. 344

set of the hob is altered, the thickness of the space and tooth produced will be changed and the height of fillet at the root of the tooth, but there will be no other change.* The sections of the gear teeth in the plane of rotation will still be involute in outline. The hob, may, therefore, be set to a depth and angle by trial to secure the tooth thickness desired. This freedom of choice of the set of the hob is of great practical value in gear cutting.

160. Grinding Gear Teeth. When grinding is employed as a final operation on metal parts it is used either to improve the quality of the finish of the surfaces or to improve both the quality of the finish and the accuracy of form of the surfaces. Grinding is about the only method

* For a discussion of the theory, use, and manufacture of involute hobs, see "Spur Gears," by Buckingham, McGraw-Hill Book Company.

of restoring accuracy of form to parts that have been distorted in heat treating. This is the chief reason for grinding hardened or heat-treated gears. Unhardened gears are often ground not so much to improve the accuracy of form of the teeth as to secure more highly finished tooth profiles.

In grinding gear teeth, generating methods are generally used, although formed grinding wheels are used to some extent. One of the generating methods employed in grinding the teeth of straight involute spur gears is shown in Fig. 344. The gear to be ground is mounted on an arbor, and the grinding wheel is so mounted that the plane of its active face, which is normal to its axis, will coincide with the side of a tooth of the imaginary basic rack, shown dotted in the figure. The axis of the grinding wheel spindle is therefore at the angle of obliquity α with the pitch line of the rack. As the grinding wheel is moved forward parallel to the pitch line of the rack, the gear is rotated about its axis as though actuated by the imaginary rack which is moving in unison with the grinding wheel. With a grinding wheel of small diameter, this movement of the gear and grinding wheel is an interrupted feeding movement. Between successive feeds, the grinding wheel is given a grinding and a return stroke parallel to the axis of the gear. In some grinding machines movement parallel to the axis of the gear is eliminated by using a grinding wheel large enough in diameter to cover the whole width of face of the teeth.

CHAPTER IX

STRAIGHT AND SPIRAL BEVEL GEARS

CUTTING OF BEVEL GEARS

161. Cycloidal Bevel Gear Teeth. While, as stated in Art. 115, rolling cones are not necessarily right circular cones, the use of other than such cones is so rare that only rolling right circular cones will be treated. By a pair of such cones rolling together without slipping, motion may be transmitted between intersecting shafts at a constant angular velocity ratio. The method of determining a pair of cones for any given angular velocity ratio and angle between intersecting shafts was given in Art. 117. As there pointed out, such rolling cones may be used as the pitch surfaces of bevel gears. By supplying the conical pitch surfaces with teeth of proper form, motion may be positively transmitted between intersecting shafts which will be equivalent to the rolling of the pitch cones.

In the case of spur gears, the motion is about parallel axes, and therefore falls, according to Art. 27, under the head of plane motion. Hence, in dealing with the relative motion of such gears, the gears may be represented by plane sections. This simple treatment cannot, however, be applied to bevel gears. While each gear taken separately has plane motion of rotation about its axis, the relative motion of corresponding sections of a pair of gears is in a spherical surface. The tooth profiles are, therefore, not generated on a plane but on a spherical surface, making two projections necessary in determining the profiles of the teeth.

Just as an element of a describing cylinder in rolling on a pitch cylinder generates the cycloidal tooth surface of a spur gear, so an element of a describing cone in rolling on a pitch cone generates the cycloidal tooth surface of a straight bevel gear. In Fig. 345 are shown a pitch cone _1_ and two describing cones _3_ and _4_ having the same slant height, or cone distance, and a common apex _O_. The bases of the cones are, therefore, small circles of a sphere having a radius equal to the common cone distance of the cones. If cone _3_ is rolled to the right on the outside of _1_ from contact along _aO_ to contact along _PO_, the

element gO will generate the ruled surface aOg, and point g will describe, in the surface of the sphere, the curve ag. This curve, being analogous to the epicycloid treated in Art. 128, may be called a spherical epicycloid. In a similar manner, if 4 is rolled to the right on the inside of 1 from contact along $a'O$ to contact along $P'O$, the element $g'O$ will generate the ruled surface $a'Og'$, and the point g' will describe, in the surface of the sphere, the curve $a'g'$, which may be called a spherical hypocycloid. The generated surfaces aOg and $a'Og'$ are of proper form for the respective faces and flanks of gear teeth for the pitch cone 1. In a similar manner the describing cones 3 and 4 could be used to generate the teeth of another pitch cone having the same cone distance as
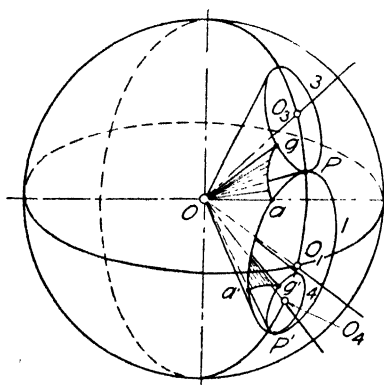
FIG. 345
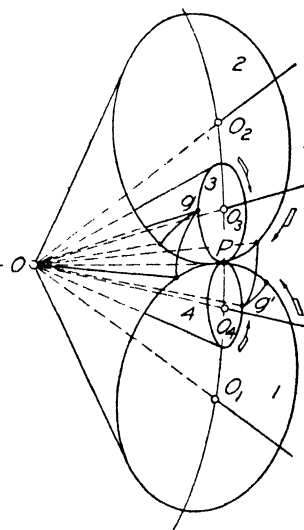
FIG. 346

1. For the second toothed cone to engage properly with the first, the faces of the teeth of one and the flanks of the teeth of the other pitch cone must be generated by the same describing cone. The two pitch cones with teeth thus generated would then work together properly and would positively transmit motion from one to the other equivalent to the rolling of the pitch cones.

In Fig. 346 are shown two pitch cones *1* and *2* and two describing cones *3* and *4*. These cones have the same cone distance, their axes are in the same meridian plane of the enveloping sphere, and they make contact along the common element OP. If these four cones are rolled together simultaneously about their fixed axes OO_1, OO_2, OO_3, and OO_4, element gO of *3* will generate the tooth face of *1* and the flank of

2 as shown, and element *g'O* of *4* will generate the tooth flank of *1* and the face of *2*. Since *gO* will always be the common element of the tooth face of *1* and the flank of *2*, a plane through the three points *P*, *g*, and *O* will be the common normal plane of the generated surfaces and will always pass through the common element *PO* of the pitch cones. Likewise, since *g'O* will always be the common element of the tooth flank of *1* and the face of *2*, a plane through *P*, *g'*, and *O* will be the common normal plane of the generated surfaces and will always pass through the common element *PO* of the pitch cones. Since the common normal plane always passes through the contact element of the pitch cones, it therefore follows that teeth whose surfaces are thus generated will transmit motion positively about axes OO_1 and OO_2 at a constant angular velocity ratio. It is to be noted that the sections of bevel gear teeth are spherical, and that the path of contact of mating sections follows the boundaries of the corresponding spherical sections of the describing cones. Thus, in Fig. 346, the path of contact of the outer sections of the teeth is the line *g'Pg* in the surface of the sphere of radius *OP*. While the same fundamental law applies both to cycloidal bevel and cycloidal spur gear teeth, it is quite evident that the former are much more difficult to draw or to describe mechanically than the latter.

162. Involute Bevel Gear Teeth. When a plane is rolled on a base cylinder, a straight line in the plane parallel to the axis of the cylinder will generate the involute tooth surface of a straight spur gear, and a curved line in the plane would generate the involute tooth surface of a curved spur gear. In a similar manner, a curved line in a circular plane will generate the spherical involute tooth surface of a curved bevel gear when the plane is rolled on the base cone, and a radial line in the circular plane would generate the spherical involute tooth surface of a straight bevel gear. In Fig. 347 the cone distance element *AO* of the pitch cone *AOB* is shown in a horizontal plane. In Fig. 348 a plane *EF* has been passed through the pitch element *AO* at an angle of obliquity *α* with a horizontal plane through the same pitch element. The base cone for angle *α* is, therefore, the cone tangent to plane *EF*, or cone *COD*. The projection of this base cone, but not of the pitch cone, is shown in Fig. 348, the plane *EF* being tangent to it along the element *GO*. If the circular plane *EF*, whose radius is equal to the cone distance of the pitch and base cones, is rolled on the base cone, the element *GO* will generate a spherical involute surface, and the point *G* will describe, in the surface of the sphere, the spherical involute *GHKL*. The spherical involute surface thus generated is of proper form and a suitable portion of it could be used for the surface of the teeth of a straight

bevel gear having the pitch and base cones shown. For another pitch cone of the same cone distance as AOB, a corresponding base cone could be similarly determined for the same angle of obliquity, and a spherical involute surface could be generated as before by rolling the circular plane on the base cone. If the two base cones were supplied with teeth having surfaces corresponding to the surfaces thus generated, they would work together properly as straight involute bevel gears, and would transmit motion at a constant angular velocity ratio equivalent to the rolling of the pitch or base cones. It is to be noted that the sections of the teeth are spherical and that the path of contact of mating sections follows the boundary of the obliquity plane in the corresponding spherical section.
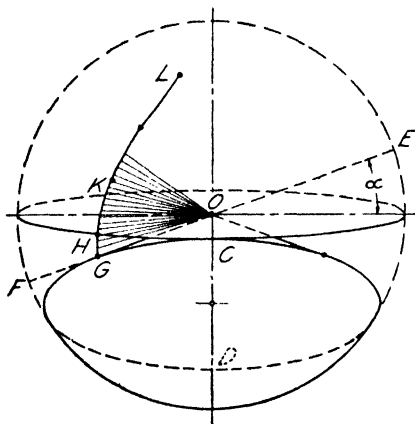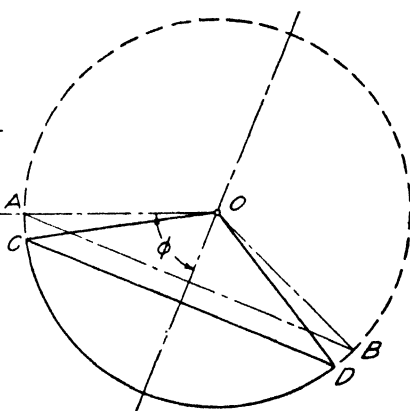
FIG. 348   FIG. 347

In Fig. 347 the pitch angle ϕ of the pitch cone was purposely taken less than 90 degrees. For a pitch angle of 90 degrees, the elements of the pitch cone would lie in a plane of radius AO, and the resulting bevel gear would be what is called a *crown gear*. It is to be noted that while the elements of the pitch cone would lie in a plane, the elements of the corresponding base cone would not. Therefore, while the crown gear is to bevel gears what the rack is to spur gears, the sides of the teeth of an involute crown gear are not flat as are the teeth of a rack.

163. Octoid Bevel Gear Teeth. Cycloidal and involute bevel gear teeth are so complex in form that no successful attempts have been made to design and build gear-cutting machines to cut them. Practically all gear-cutting machines capable of cutting bevel gear teeth with accuracy are machines which generate teeth that are conjugate to crown gear teeth having flat or plane sides. This form of tooth together

with a machine to generate conjugate bevel gear teeth was invented by Hugo Bilgram in 1884.

In Fig. 349, AB represents the pitch plane of a crown gear. The plane sides of the teeth lie in meridian planes of the sphere at the desired angle with the pitch plane as shown by plane KL for the side HPG of a tooth. Meridian planes perpendicular to KL will pass through an axis QR perpendicular to KL. As KL is revolved about axis MN, the line of contact of plane KL with a conjugate surface will lie in a plane through QR perpendicular to KL; and the path of the point of contact in the surface of the sphere will follow the figure-of-eight curve $PCDPFEP$. Because of the shape of this curve the corresponding teeth are referred to as *octoid* teeth. For teeth of the height shown, the path of contact is either TS or $T'S'$, depending on the direction of rotation. Since the path of contact is symmetrical with respect to the great circle

FIG. 349

MN, a pair of bevel gears having teeth conjugate to the crown gear teeth will themselves be conjugate.
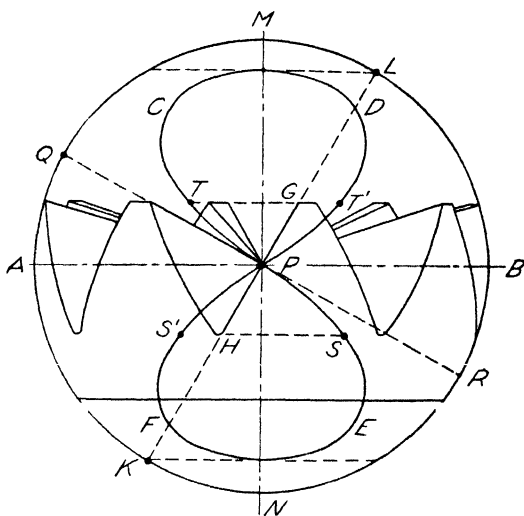
164. Tredgold's Approximate Method of Drawing Bevel Gear Teeth. As before mentioned, it is difficult to construct spherical involutes and epicycloids and to represent the teeth of bevel gears on paper. In practice a method known as Tredgold's approximation is always used.

Fig. 350 shows the projection of two pitch cones *1* and *2* on a plane parallel to the plane of the axes OO_1 and OO_2 of the cones. PM and PN represent the bases of the pitch cones, and the common cone distance PO is the radius of a circumscribing sphere. O_1O_2 is drawn perpendicular to the pitch element PO, and O_1M and O_2N perpendicular, respectively, to the pitch elements OM and ON; hence, PO_1M and PO_2N may be assumed to represent the projections of two cones constructed about the axes OO_1 and OO_2. These cones are called back cones, and any element of one of these cones is perpendicular to an

element of the pitch cone having the same base. As shown in the figure, the surfaces of these back cones closely approximate the surface of the circumscribing sphere for a short space either side of the base pitch circles. These conical surfaces have a practical advantage over the spherical surface in that they can be developed upon a plane for the construction of the tooth profiles. Tredgold's approximate method consists in describing tooth outlines on the developed surfaces of these back cones and then of wrapping these surfaces into their original positions. In the development of the surfaces of the back cones, PO_1M and PO_2N, the development of their base boundaries, PM' and PN',

FIG. 350
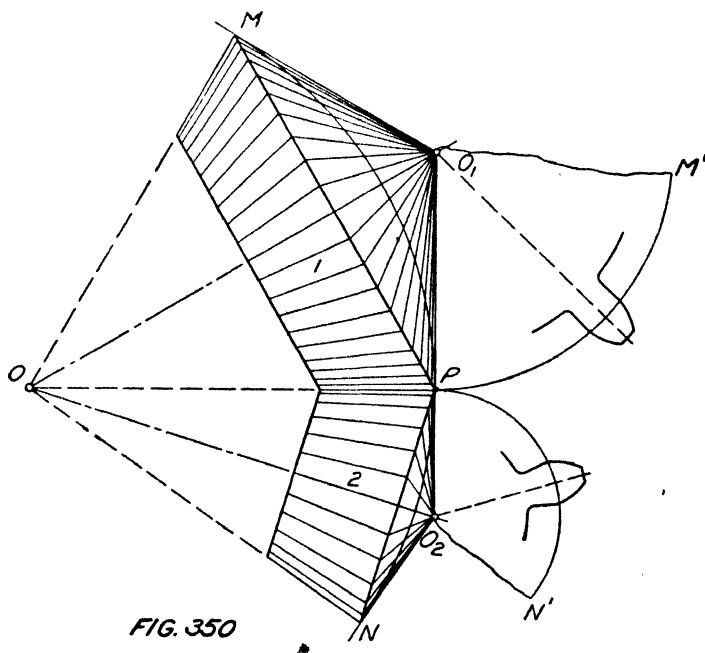
may be treated as the pitch arcs of spur gears and tooth outlines drawn upon them by any of the methods previously applied to such gears. When the developed surfaces are rolled back into the back cones, the tooth outlines described will represent the large ends of the teeth of the bevel gears; and a straight line through O following such profiles would sweep up the tooth surfaces of the bevel gears. As can be seen, such teeth would agree very closely with teeth of truly correct form. With the tooth form determined, it is merely a matter of projection to draw a picture of a bevel gear. Two elevations of a bevel gear are shown in Fig. 351, a brief study of which will make the method of projection clear.

In Fig. 351, OMM is the projection on an axial plane of the pitch cone of the bevel gear. The angle ϕ is called the *pitch angle* of the gear and R the *pitch radius*. As before shown, A represents very closely the correct shape of the outer ends of the bevel gear teeth. It is to be noted that this shape does not correspond to the shape of a spur gear tooth for a pitch radius R but corresponds in shape to a pitch radius R_f equal to the cone distance of the back cone. Since R_f determines the form of the tooth, it is called the *formative radius*. Obviously,

$$R_f = \frac{R}{\cos \phi} \qquad \ldots \ldots \ldots \ldots \quad (264)$$
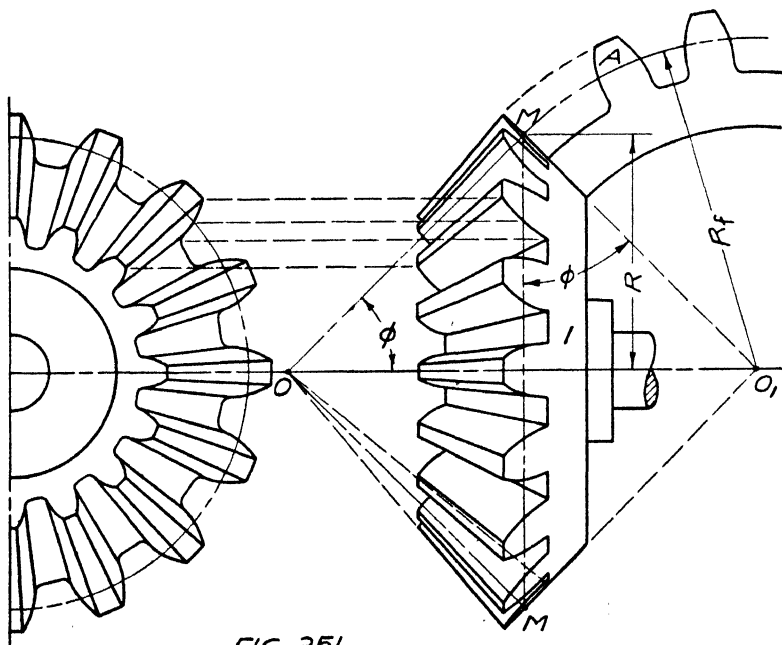
FIG. 351

and the formative number of teeth N_f is,

$$N_f = 2p'_c R_f = \frac{2p'_c R}{\cos \phi} = \frac{N}{\cos \phi} \qquad \ldots \quad (265)$$

where N is the actual number of teeth in the bevel gear.

165. Relation of the Elements and Angles of Bevel Gears. In Fig. 352 a bevel pinion *1* is represented in mesh with a bevel gear *2* for an angle θ between shafts. In Art. 117 it was shown how to locate

the contact element of a pair of pitch cones for any given angular velocity ratio and angle between shafts. If ω_1 is the angular velocity of the pinion and ω_2 the desired angular velocity of the gear, then the contact element of the pitch cones is located as follows: From the point of intersection O of the axes of the shafts lay off distances x and y along the axes such that,

$$\frac{x}{y} = \frac{\omega_1}{\omega_2}$$

The intersection P of lines parallel to the shafts through the points
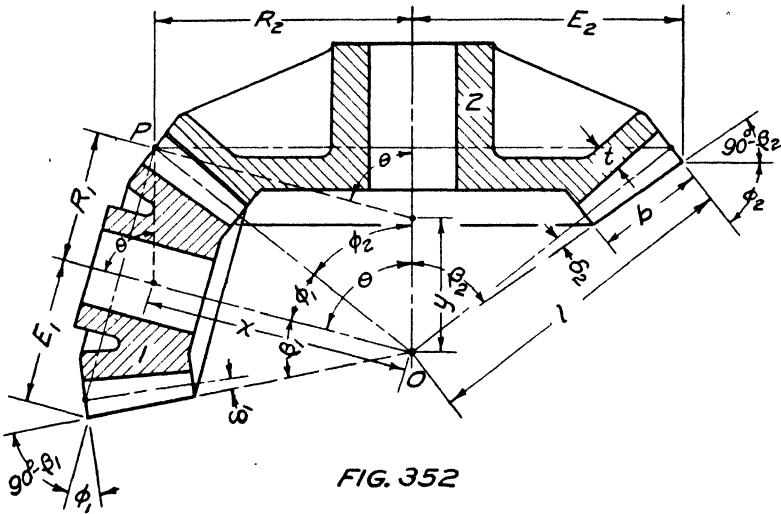
FIG. 352

thus located determines the contact element OP of the pitch cones. With further reference to Fig. 352, let

R_1 and R_2 = the respective pitch radii of the pinion and gear;
E_1 and E_2 = the respective external radii of the pinion and gear;
N_1 and N_2 = the respective numbers of teeth of the pinion and gear;
p_c = the circular pitch in inches;
p'_c = the diametral pitch;
a_1 and a_2 = the respective addenda of the pinion and gear;
θ = the angle between the center lines of the shafts of the pinion and gear;
ϕ_1 and ϕ_2 = the respective pitch angles of the pinion and gear;
β_1 and β_2 = the respective face angles of the pinion and gear;
δ_1 and δ_2 = the respective addendum angles of the pinion and gear.

Obviously,

$$x = \frac{R_2}{\sin \theta} \qquad y = \frac{R_1}{\sin \theta} \quad \text{and} \quad \frac{R_1}{R_2} = \frac{N_1}{N_2}$$

Hence,

$$\tan \phi_2 = \frac{R_2}{y + x \cos \theta} = \frac{R_2}{\dfrac{R_1}{\sin \theta} + \dfrac{R_2 \cos \theta}{\sin \theta}} = \frac{R_2 \sin \theta}{R_1 + R_2 \cos \theta}$$

$$= \frac{\sin \theta}{\dfrac{R_1}{R_2} + \cos \theta} = \frac{\sin \theta}{\dfrac{N_1}{N_2} + \cos \theta} \qquad \cdots \cdots \quad (266)$$

Similarly,

$$\tan \phi_1 = \frac{\sin \theta}{\dfrac{N_2}{N_1} + \cos \theta} \qquad \cdots \cdots \cdots \cdots \quad (267)$$

also,

$$\tan \delta_1 = \frac{a_1}{l} = \frac{a_1 \sin \phi_1}{R_1} \quad \text{and} \quad \tan \delta_2 = \frac{a_2}{l} = \frac{a_2 \sin \phi_2}{R_2} \quad (268)$$

Assuming the addenda of pinion and gear to be equal, that is, assuming $a_1 = a_2 = a = \dfrac{k}{p'_c}$, then,

$$\tan \delta_1 = \tan \delta_2 = \frac{k \sin \phi_1}{p'_c R_1} = \frac{2k \sin \phi_1}{N_1} = \frac{2k \sin \phi_2}{N_2} \quad (269)$$

For equal addendum full depth teeth, $k = 1.0$, and for stub teeth $k = 0.80$.

The face angles are:

$$\beta_1 = (\phi_1 + \delta_1) \qquad \text{and} \quad \beta_2 = (\phi_2 + \delta_2) \quad \cdot \quad \cdot \quad (270)$$

The external radii are:

$$E_1 = R_1 + a_1 \cos \phi_1 \quad \text{and} \quad E_2 = R_2 + a_2 \cos \phi_2 \quad (271)$$

By means of the above relations the dimensions and angles necessary in making bevel gear patterns and in machining the blanks may be found. The width of face b is rarely made greater than one-third the cone distance PO because of the difference in the size of the tooth sections at the large and small ends. The thickness t of the rim is generally made about equal to the thickness of the root of a tooth at the large end.

In the great majority of cases where bevel gears are used, the shafts are at right angles. For such cases, stock gears can frequently be

obtained from gear manufacturers if the proportions and angular velocity ratio are not unusual. Where the shafts are at right angles and the gears of the mating pair are of the same size, the gears are called *miter gears*.

166. Action of Bevel Gears. Due mainly to the greater number of teeth in contact, spur gears having a large number of teeth will run together more quietly and smoothly than those having a smaller number of teeth. The fact that a pair of bevel gears will run more quietly and smoothly than a pair of spur gears of the same number of teeth may, therefore, be explained. It was shown in Art. 164, equation (265), that a bevel gear having N teeth is equivalent to a spur gear having $N_f = \dfrac{N}{\cos \phi}$ teeth. Since bevel gears are equivalent to spur gears having a greater number of teeth, it therefore follows that a pair of bevel gears will run more quietly than a pair of spur gears having the same number of teeth as the bevel gears.

The action between bevel gear teeth consists of a combination of rolling and sliding between spherical sections of the teeth about the apex of the pitch cones as a center. As in spur gears, there is no sliding along the elements of the teeth. For continuous positive driving, the pitch arc of action must be greater than the circular pitch. This ratio can be very closely approximated for a pair of cycloidal or involute bevel gears by reference to a pair of equivalent spur gears as determined by Tredgold's approximation.

167. Interchangeability of Bevel Gears. For a set of bevel gears of the same pitch to be interchangeable, they must belong to the same odontic system and have a path of the point of contact that is symmetrical with respect to the plane of the axes, and they must have the same cone distance, the same addendum, and a tooth thickness of one half the circular pitch. When, from a set of interchangeable bevel gears, a substitution is made for one of a pair, the angular velocity ratio and the angle between shafts are changed. If the angle between shafts is fixed, then, for different angular velocity ratios, the gears must be made in pairs. Generally, stock, commercial gears are made for common angular velocity ratios and for shafts at 90 degrees.

168. Spiral and Skew Bevel Gears. Thus far only straight bevel gears have been discussed. Bevel gears the pitch elements of whose teeth are curved lines and not straight lines through the common apex of the pitch cones may be called *curved* bevel gears. While theoretically the pitch elements of the teeth of curved bevels may be of any form, the actual curve is governed by what is possible to cut. Curved bevels having curved oblique teeth are called *spiral* bevels, while those having

straight oblique teeth are called *skew* bevels. A pair of Gleason* *spiral*
bevel gears is shown in Fig. 353. These bevel gears have teeth that are
curved on the arc of a circle. They were developed primarily for auto-
mobile drives. The action between the teeth of spiral and skew bevels
is the same as between the teeth of straight bevels. There is no sliding
along the teeth.

Curved bevel gears possess the same advantages over straight bevel
gears that curved spur gears possess over straight spur gears. Obviously,
if the teeth, Fig. 354, are given a lead equal to or greater than the
circular pitch, there would be continuous contact in the plane of the
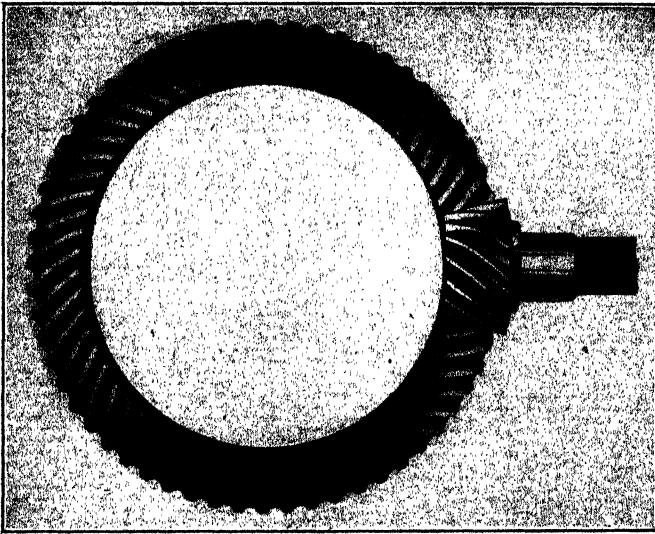
Fig. 353

axes of the pair of gears as they rotate; and the different pairs of teeth
would come into contact gradually instead of coming at once into full
line contact across the entire width of face as in straight bevel gears.
For these reasons curved bevels run more quietly and smoothly than
straight bevels. For best results the Gleason Works recommend a lead
of from 1.15 to 1.40 times the circular pitch and recommend also a
spiral angle of from 30 to 40 degrees for small and from 15 to 20 degrees
for large bevel gears. For high velocity ratios and pinions having a
small number of teeth, the addenda of the pinion and gear are made
unequal to avoid undue undercutting, the addendum of the pinion
teeth being made about 0.70 and the addendum of the gear teeth about

* Gleason Works, Rochester, N. Y.

0.30 the working depth of the teeth.   Another advantage of curved over straight bevels is that continuous positive driving can be secured with smaller pinions. For continuous driving with straight bevels, the arc of action must be greater than the circular pitch, while with curved bevels all that is necessary is that the lead of the teeth be greater than the circular pitch. Hence, for a pair of curved bevels, continuous positive driving is, within limits, independent of the number of teeth in the gears.   However, ten is about the minimum number of teeth for a pinion if undue undercutting is to be avoided and sufficient durability secured.

For spiral bevel gears there is an axial thrust due to the angle of the spiral.   In addition, there is an axial thrust, as for straight bevels, tending to separate the gears.   This thrust is independent of the direc-
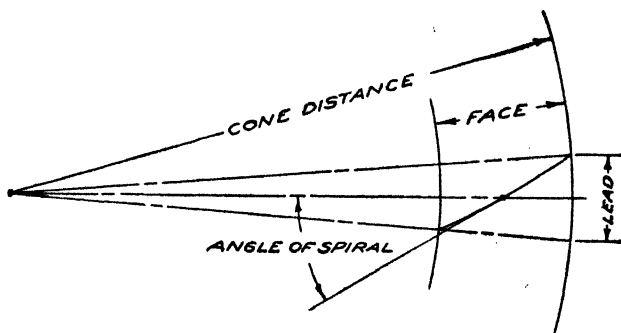
FIG. 354

tion of rotation and is due to the cone angle and obliquity of action. Because of the obliquity of action there is a force component in the axial plane of the gears. Since the gears are conical, this force component has components parallel to the axes of the gears tending to separate them. Since the axial thrust due to the angle of the spiral may be in either direction, the resultant thrust is either the sum or the difference of the thrusts due to the angle of the spiral and the obliquity of action. The hand of the pinion, being more evident than the hand of the gear, is usually used to designate the hand of a pair of spiral bevel gears. The pinion and gear in Fig. 353 would be called right-hand spiral bevels. For the pinion as the driver and for clockwise rotation when viewed from the large end, the thrust on the pinion due to the spiral angle would be toward the center of the gear.   Reversing the rotation of the pinion would reverse this thrust.   For left-hand spiral bevels, the thrusts for clockwise and counter-clockwise rotation of the pinion

would be just the reverse of those given for the right-hand spiral bevels. As to hand and thrust, skew bevel gears are similar to spiral bevel gears.

169. Formed Cutter Method of Cutting Bevel Gear Teeth. Before generating machines were developed, straight bevel gears were cut on milling machines by the use of formed cutters. Occasionally pairs of bevel gears are still cut by this method. As will be shown, the method is inherently incorrect and should never be used where the service requirements for the gears are at all exacting.

While there are two methods of using formed cutters of the Brown type, only the preferred method will be described. Standard spur gear milling cutters are used for the parallel depth method, while special

FIG. 356

FIG. 355

bevel gear milling cutters are required for the offset method of cutting bevel gear tooth spaces that vary characteristically in depth as well as width. It is the use of this latter type of cutter that will be described.
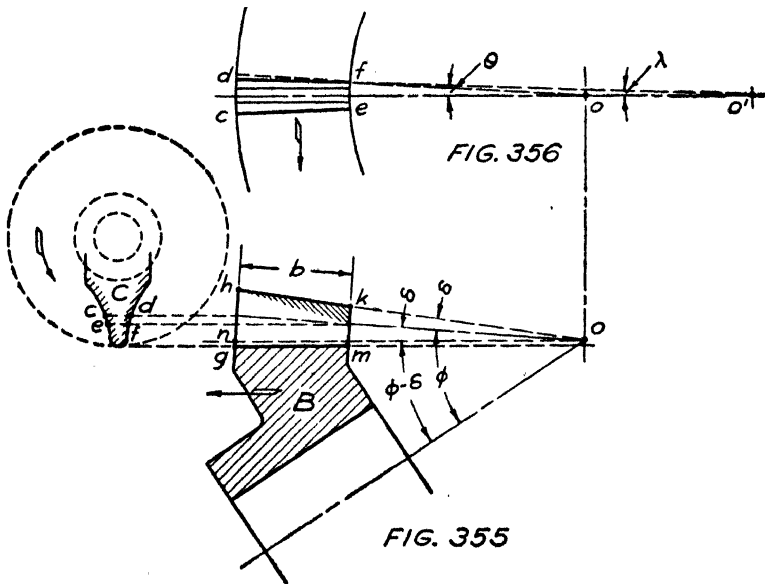
It was shown in Art. 164 that the teeth of a bevel gear correspond closely to those of a spur gear of a pitch radius and number of teeth equal to the formative radius R_f and formative number of teeth N_f given by equations (264) and (265). The cutting face of such a bevel gear milling cutter is shown at C in Fig. 355. It has the same shape as the sides of the large end of the tooth spaces to be cut, but is made sufficiently thin to pass through the small end of the tooth spaces. The bevel gear blank B is mounted on an arbor of the milling machine in-

dexing head with its axis in a plane perpendicular to the axis of rotation of the cutter. The blank is then adjusted until its axis is at an angle $(\phi - \delta)$ with the horizontal travel of the milling machine table, ϕ being the pitch and δ the addendum angle of the blank. With the axis of the blank in the mid plane of the cutter, and the cutter adjusted to the proper depth, counter-clockwise rotation of the cutter as the blank is fed to the left would cut a slot in the blank, the bottom of which would be parallel to nO. In Fig. 356 is shown a projection of the slot at and below the surface of the pitch cone, the width of the slot at the pitch surface of the cone being cd at the large end and ef at the small end. It is to be noted that df extended does not intersect the center line at O but at O'. If, in finishing the side $ghkm$ of the tooth space, the cutter is to follow the pitch element as required, then the relative positions of blank and cutter must be changed. To cause the extension of df to pass through O, the blank must be turned forward through an angle $(\theta - \lambda)$, as indicated by the arrow in Fig. 356. The blank is then moved backward parallel to the axis of the cutter a distance called the offset. The value of this offset s is

$$s = \frac{cd}{2} - \left(\frac{cd - ef}{2}\right)\frac{l}{b} \qquad . \quad . \quad . \quad . \quad . \quad (272)$$

where l is the cone distance of the pitch cone and b the width of face of the gear. Having completed these adjustments, the rotating cutter, as the blank is fed to the left into it, will follow the pitch element of the side of the tooth space. After finishing the side of a tooth space, the blank is indexed and the same side of the next tooth space is finished. In a similar manner, the same side of all the other tooth spaces are finished and the blank and cutter then positioned to finish the opposite side of all the tooth spaces.

While teeth cut as just described will be of proper thickness in the pitch cone, they cannot be of correct shape throughout their length. The curve of the cutter being correct for the large end of the tooth spaces, its change of curvature is not rapid enough for correct shape at the small end of the spaces. Hence the points of the teeth grow more and more defective in form as the small end is approached, the points being too wide. The teeth must, therefore, be relieved by filing or grinding, as suggested by the shaded triangular area in Fig. 355.

In applying the above method to steel gears, the tooth spaces are roughed out before positioning the blank for the finishing cuts. The tooth spaces of cast-iron gears are not roughed out unless the teeth are larger than about 5 diametral pitch.

170. Template Method of Cutting Bevel Gear Teeth. In general this method is applied only to large bevel gears which are beyond the capacity of generating machines. The tooth spaces are first roughed out and then finished by a reciprocating planing tool whose line of travel on each cutting stroke is determined by a template, or former.

The principle of the template method of cutting bevel gear teeth may be illustrated by Fig. 357. In this figure O is the apex of the bevel gear blank B, and EO is the center line of roller R and the path of travel of the cutting point of the tool C. This tool is held in the tool slide S which is caused to reciprocate on the guide GG. The roller R on arm GG is maintained in contact with the template T whose guiding surface is parallel to an enlarged profile of the tooth to be cut. The guiding surface of the template must be of such form that, as the roller is moved downward in contact with it, the center line EO of the roller
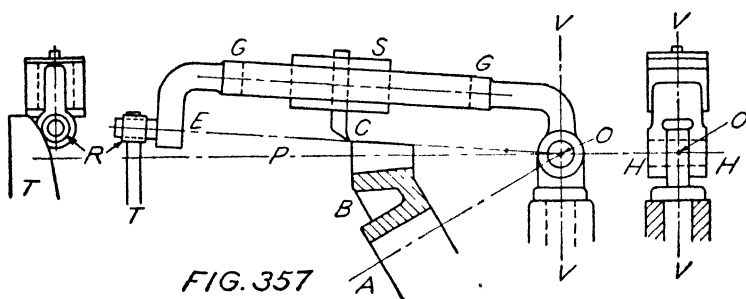
FIG. 357

will generate the tooth surface of the bevel gear about O as a center. Thus one side of a tooth is finished by feeding the guiding arm GG of the cutting tool downward after each cutting stroke until the bottom of the tooth space is reached. The gear is then indexed and the same side of the next tooth is finished, and, in a similar manner, the same side of the remaining teeth. The cutting tool and guide are then positioned for finishing the other side of the teeth.

171. Shaping Method of Generating Bevel Gear Teeth. The shaping method of generating bevel gear teeth is generally employed as a finishing process, the tooth spaces having first been roughed out. Practically all of the machines using this method are built to generate teeth that are conjugate to a crown gear having straight or flat-sided teeth. Hence the generating tools will have straight sides and can be easily made with precision.

The Gleason* method of generating the teeth of straight bevel

* Gleason Works, Rochester, N. Y.

gears is illustrated in Fig. 358. The horizontal line in the figure represents the pitch plane of an imaginary crown gear. The bevel gear blank, with its tooth spaces roughed out, is placed in the machine with its apex at the center of and its pitch cone tangent to the pitch plane of the crown gear. The space between the cutting edges of the tools C and C corresponds to the thickness of the tooth to be cut. These tools reciprocate, their cutting edges following planes which correspond to the sides of a tooth space of the basic crown gear. The beginning of the generation of a tooth is shown at (a), the mid-position at (b), and the end of the process at (c). Starting with the related positions shown at (a), the blank and tools are given a generating roll as the cutting tools reciprocate. This roll of the blank and the reciprocating tools is
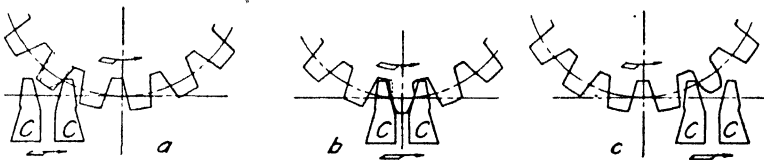
FIG. 358
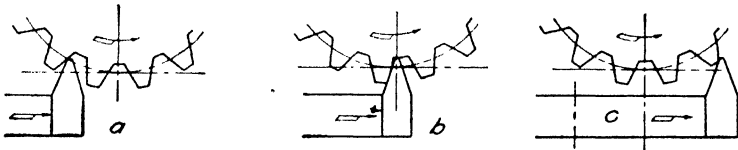
FIG. 359

sufficient for the generation of a tooth, and corresponds to the rolling together of the pitch cone of the blank and the pitch surface of the imaginary crown gear. Having generated a tooth, the tools are withdrawn, and the blank is indexed for the next tooth while the blank and tools are rolled back to their original positions, the second and succeeding teeth being generated in the same manner as the first tooth.

The Gleason* method of generating the teeth of spiral bevel gears is illustrated in Fig. 359. As before mentioned, Gleason spiral bevel gears have their teeth curved on the arc of a circle. The cutter is a circular steel disk having a number of blades fastened to its periphery whose cutting edges are straight. Owing to a correction for the spiral and dedendum angles of the gear, the angle which the cutting edges of the blades make with the plane of rotation of the cutter is not quite

* Gleason Works, Rochester, N. Y.

the same as the angle of the sides of the teeth of the imaginary straight crown gear, and the plane of rotation of the cutter is not quite parallel to the pitch plane of the crown gear. The horizontal line in Fig. 359 represents the pitch plane of the crown gear. As for straight bevel gears, the gear blank is mounted in the machine with its apex coincident with the center of and its pitch cone tangent to the pitch plane of the imaginary crown gear. The beginning of the generation of the side of a tooth is shown at (a), the mid position at (b), and the end of the process at (c). As the cutter rotates, the pitch cone of the blank and the pitch plane of the crown gear are rolled together, as indicated by the arrows, from the position at (a) to that at (c). During this rolling action, one side of a tooth is generated. The blank and cutter are then rolled back to the original position, the gear blank indexed, and a side of the second tooth is generated. In a similar manner, the same side of all the remaining teeth and then the opposite side of all the teeth are generated. The process as just described applies to the cutting of spiral bevel pinions. In generating the gear it is usual to employ a double row of cutters and to generate both sides of a tooth space simultaneously.

Large spiral bevels over 18 inches and up to 60 inches in diameter are generated on machines using a single straight-sided planing tool. In these machines the blank rotates continuously and the same cut is taken on all the teeth in the gear before the next cut is taken.

Generated spiral bevels may be run at much higher speeds than straight bevels cut by the template method without objectionable noise and vibration. The limiting peripheral velocity for straight bevels cut by the template method is about 1000 feet per minute, while generated spiral bevels have been run satisfactorily at peripheral velocities as high as 5000 feet per minute.

CHAPTER X

HELICAL AND HYPERBOLOIDAL GEARS

CUTTING OF HELICAL GEARS

172. Helical Gears. By means of helical gears motion may be transmitted at a constant angular velocity ratio between non-intersecting shafts at any angle. As mentioned in Art. 152, the pitch surfaces of such gears are right cylinders, the teeth are helicoidal, and, in the general case, there is point contact between the teeth. While the non-intersecting shafts may be at any angle, they are, as a rule, at right angles, as shown for the pair at *3* and *9* in Fig. 270 or as shown in Fig. 360.
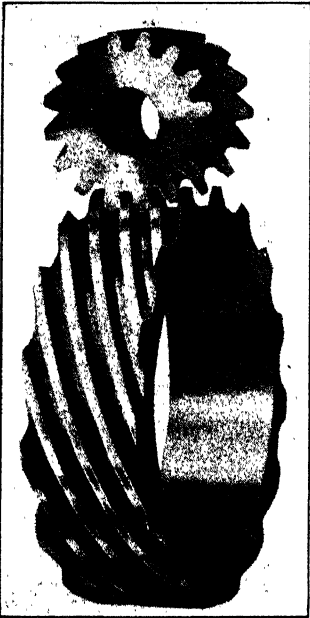
Where two cylinders are tangent to each other, they will have line contact when the axes are parallel, and point contact when the axes are not parallel. In the latter case, the two elements, one on each cylinder, through the point of contact determine a plane tangent to both cylinders. The contact radii of the cylinders are perpendicular to this plane at the coincident points of contact of the cylinders. When the cylinders are rotated about their axes, the direction of motion of the coincident points of contact at any instant is in the tangent plane at right angles to the axes of rotation. This is shown in Fig. 361 which

FIG. 360

represents the plan view of two cylinders *1* and *2* in contact at *P* whose non-intersecting axes O_1O_1 and O_2O_2 are at an angle θ. If V_1 and V_2 are the respective linear velocities of the coincident points of contact in *1* and *2*, then *V*, drawn perpendicular to *km*, represents their common component. The respective components of V_1 and V_2 perpendicular

to V are PT_1 and PT_2. The algebraic difference of these components represents the instantaneous velocity of sliding of the coincident points of contact in the direction T_1T_2, while V represents the velocity of rolling perpendicular to the sliding. The rolling and sliding action between the tangent cylinders at any instant is thus determined by their peripheral velocities V_1 and V_2. If V_1 and V_2 are constant, then the relative sliding and rolling will be constant. The direction of sliding will be tangent to a helix crossing the elements of 1 at an angle ϕ_1 and tangent to a helix crossing the elements of 2 at an angle ϕ_2. The rolling which is at right angles to the sliding will be between helices crossing the elements of the respective pitch cylinders 1 and 2 at angles of $(90 - \phi_1)$ and $(90 - \phi_2)$. Since these helices are at right angles
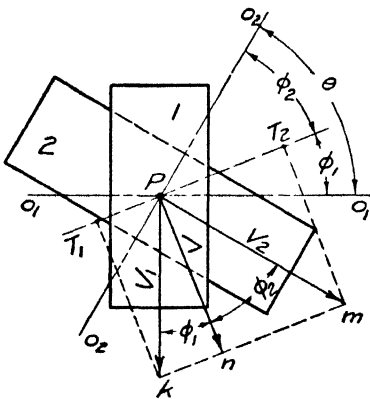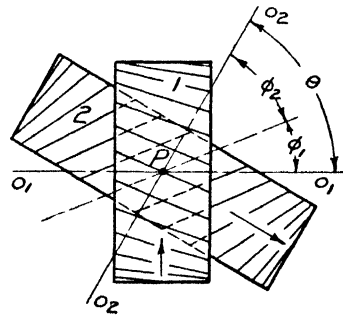
FIG.361   FIG.362

to the helices corresponding to the helix angles ϕ_1 and ϕ_2, they are called normal helices.

If cylinders 1 and 2 are used as the pitch surfaces of gears to have a constant angular velocity ratio corresponding to V_1 and V_2, the teeth of these gears must be of such form as to permit the sliding action and to transmit motion at an angular velocity ratio corresponding to the rolling action. Evidently the sliding may take place if the gears are supplied with teeth of uniform section whose pitch elements are helices crossing the elements of the respective pitch cylinders at angles ϕ_1 and ϕ_2, ϕ_1 being the helix angle of 1 and ϕ_2 the helix angle of 2. If helicoidal involute teeth are desired, the form of the teeth, as for helical involute spur gears, must be such as would be generated by a straight oblique line in a plane which rolls on the base cylinder of the gear.

If gears 1 and 2 were supplied with teeth of the form just described,

they would transmit motion positively from one shaft to the other at a constant angular velocity ratio. Each gear would be a helical gear and would appear and be cut in the same way as a helical spur gear. As before pointed out, helical gears should not, however, be confused with helical spur gears. The action between mating teeth is widely different. Helical spur gears are mounted on parallel shafts, their angular velocities are inversely as their pitch diameters, their teeth make line contact, and the action between mating teeth consists of a combination of rolling and sliding in the plane of rotation as for straight spur gears. There is no sliding along the teeth. With helical gears, the non-intersecting shafts may be at any angle, the angular velocities are not inversely as the pitch diameters, the teeth make point contact, and, in action, there is a component of sliding along the teeth as well as a combination of rolling and sliding similar to that in helical spur gears.

The hand of helical gears is designated the same as for screws. Thus in Fig. 362, in which the same proportions have been used as in Fig. 361, *1* and *2* are right-hand gears. If *1*, as viewed from the right, is rotated clockwise, then *2* will be driven counter-clockwise, as indicated by the arrows; or, if *1* is rotated counter-clockwise, *2* will be driven clockwise.

As pointed out in Art. 152, mating helical spur gears must be of opposite hand and have the same helix angle; mating helical gears may be of the same or of opposite hand, and the helix angles may or may not be the same. Since for mating helical spur gears the normal pitch and helix angle must be the same, it follows that they have the same circular pitch. For mating helical gears, while the normal pitch must be the same, the circular pitches are different unless the helix angles are equal.

Where the hands of mating helical gears are the same, the angle between the shafts, whether 90 degrees or less, is equal to the sum of the helix angles; that is, $\theta = (\phi_1 + \phi_2)$, as in Fig. 361. Where the hands of the gears are not the same, as may be the case for shafts at an acute angle, the angle between shafts is equal to the difference of the helix angles. Thus, in Fig. 361, if V_1 were so much less than V_2 as to make *Pkm* an obtuse angle, then V would fall to the left of V_1, making *1* and *2* of opposite hand and so increasing ϕ_2 that θ would equal $(\phi_2 - \phi_1)$. It is to be noted that where the shafts are at right angles, as they generally are, V must fall on or between V_1 and V_2, making both gears of the same hand, either right or left, and making θ always equal to $(\phi_1 + \phi_2)$.

173. Pitch and Form of Teeth of Helical Gears. In Fig. 363 is shown the pitch cylinder of a helical gear with the corresponding pitch elements of adjacent teeth crossing the elements of the pitch cylinder

at a helix angle ϕ. The circular pitch measured in the plane of rotation is p_c. The normal circular pitch p_n, or, briefly, the normal pitch, is the distance between the corresponding pitch elements of adjacent teeth measured along the normal helix cd. Obviously,

$$p_n = p_c \cos \phi$$

Since the product of corresponding circular and diametral pitches is equal to π, the relation of the diametral pitches p'_n and p'_c becomes

$$p'_n = \frac{p'_c}{\cos \phi}$$

In Fig. 364 is shown the development, to a reduced scale, of the helical gear shown in Fig. 363. Obviously, if the right triangle adb
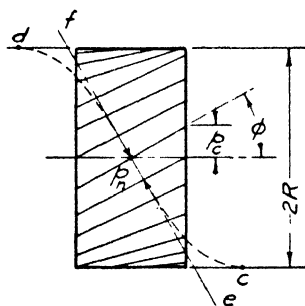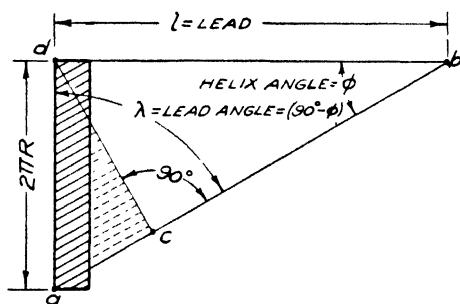
FIG. 363   FIG. 364

were wrapped around a pitch cylinder of radius R, the hypotenuse ab would form a helix crossing the elements of the cylinder at the helix angle ϕ, and line dc would form the normal helix. Evidently dc represents the length of the normal helix from one tooth around the cylinder to the same tooth extended. Hence the number of teeth N in the gear is

$$N = \frac{dc}{p_n} = \frac{(ad) \cos \phi}{p_n} = \frac{2\pi R \cos \phi}{p_n} = 2R p'_n \cos \phi \quad (273)$$

The lead l of the teeth of a helical gear is

$$l = 2\pi R \tan \lambda = 2\pi R \tan (90 - \phi) = \frac{2\pi R}{\tan \phi} \quad . \quad (274)$$

The radius B of the base cylinder for an angle of obliquity α is,

$$B = R \cos \alpha$$

For the same lead as the pitch cylinder, the tangent of the helix angle ϕ' of the base cylinder is

$$\tan \phi' = \frac{2\pi B}{l} = \frac{2\pi R \cos \alpha}{l} \qquad \ldots \ldots \ldots \quad (275)$$

Substituting the value of l from (274) gives

$$\tan \phi' = \cos \alpha \tan \phi \qquad \ldots \ldots \ldots \ldots \quad (276)$$

A straight line at an angle of ϕ' with the gear axis in a plane rolling on the base cylinder would generate the helicoidal involute tooth surface of the gear.

174. Angular Velocity Ratio of Helical Gears and Distance between Centers. As for all other classes of gears, the angular velocities of helical gears are inversely as the number of teeth. This relation, while self-evident, may be obtained if desired by combining equations (278) and (280) given below.

$$\frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} \qquad \ldots \ldots \ldots \ldots \ldots \quad (277)$$

Also, if R_1 and R_2 are the respective pitch radii of 1 and 2 in Fig. 361, and ω_1 and ω_2 the respective angular velocities, then the angular velocity ratio of 1 and 2 is

$$\frac{\omega_1}{\omega_2} = \left(\frac{V_1}{R_1}\right) \div \left(\frac{V_2}{R_2}\right) = \frac{V_1 R_2}{V_2 R_1}$$

But,

$$V = V_1 \cos \phi_1 = V_2 \cos \phi_2 \quad \text{or} \quad \frac{V_1}{V_2} = \frac{\cos \phi_2}{\cos \phi_1}$$

Hence,

$$\frac{\omega_1}{\omega_2} = \frac{R_2 \cos \phi_2}{R_1 \cos \phi_1} \qquad \ldots \ldots \ldots \ldots \quad (278)$$

The angular velocities of helical gears are not, therefore, inversely as their pitch radii, as for spur gears, unless the helix angles ϕ_1 and ϕ_2 are equal. From equation (273),

$$N_1 = 2R_1 p'_n \cos \phi_1 \quad \text{and} \quad N_2 = 2R_2 p'_n \cos \phi_2 \quad . \quad (279)$$

or, since the normal pitch of mating gears must be equal,

$$\frac{N_2}{N_1} = \frac{R_2 \cos \phi_2}{R_1 \cos \phi_1} \qquad \ldots \ldots \ldots \ldots \quad (280)$$

The distance C between the centers of a pair of helical gears is equal to the sum of the pitch radii of the pinion and gear. Hence, from (279),

$$C = R_1 + R_2 = \frac{1}{2p'_n}\left(\frac{N_1}{\cos \phi_1} + \frac{N_2}{\cos \phi_2}\right)$$

$$= \frac{N_1}{2p'_n}\left(\frac{1}{\cos \phi_1} + \frac{n}{\cos \phi_2}\right) \quad \ldots \ldots \quad (281)$$

where $n = \dfrac{\omega_1}{\omega_2} = \dfrac{N_2}{N_1}$. If the pinion and gear have the same helix angle, equation (281) reduces to

$$C = \frac{N_1}{2p'_n}\left(\frac{1 + n}{\cos \phi_1}\right) \quad \ldots \ldots \ldots \ldots \quad (282)$$

or, for $\phi_2 = \phi_1$,

$$N_1 = \frac{2p'_n C \cos \phi_1}{n + 1} \quad \ldots \ldots \ldots \ldots \quad (283)$$

175. Worm and Worm Wheel. A worm is a regular helical gear, while a worm wheel is not. The worm, like all helical gears, is formed by cutting helicoidal tooth spaces in a right cylinder, while the worm wheel is formed by hobbing tooth spaces in the face of the blank as shown in Fig. 367 and as described in Art. 178. In the case of a worm and worm wheel, it is the manner of hobbing the wheel that differentiates the pair from a pair of regular helical gears. Ordinarily a worm has one, two, three, or four threads, and is referred to as a single, double, triple, or quadruple thread worm. The worm is usually the driver and the wheel the follower. The combination is generally, though not necessarily, used between shafts at right angles where a great reduction in speed is desired.

For a worm to drive a straight spur gear, Fig. 339, the normal pitch of the worm must be equal to the circular pitch of the gear, and the angle between the worm and gear shafts must be equal to 90 degrees minus the lead angle λ of the worm. The teeth of the worm and gear will make point contact. For a worm to drive a helical gear, Fig. 340, the normal pitch of the worm must be equal to the normal pitch of the gear, and the angle between the worm and gear shafts must be $[90° - (\phi - \lambda)]$ if the worm and gear are of the same hand, and $[90° - (\phi + \lambda)]$ if they are of opposite hand. The teeth of the worm and gear will make point contact. If the worm and gear were of the same hand and $\phi = \lambda$, then the angle between the shafts would be 90 degrees; if of opposite hand and $\phi = \lambda$, the angle between shafts

would be (90° − 2λ). Where worms are used to drive involute spur or helical gears, the worms, to be conjugate, must be true involute helical gears. They must therefore have helical involute teeth of the form described in Art. 173; that is, the sections of the teeth in the planes of rotation must be involute in outline.
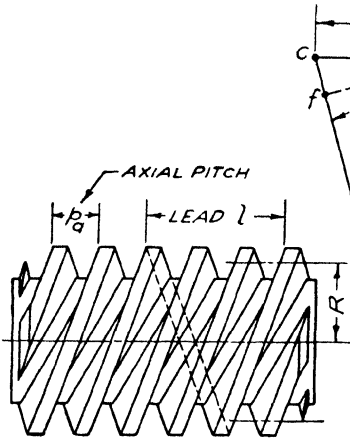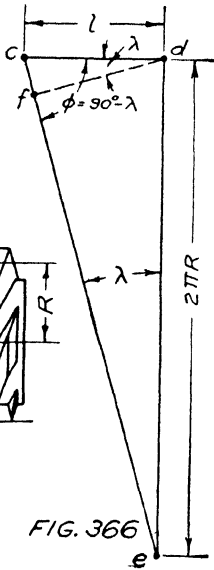
However, because of the point contact between the teeth, worms are very seldom mated with straight spur or with helical gears. To secure line contact, worms are almost invariably mated with hobbed worm wheels of the type shown in Fig. 367. For a worm and worm wheel to be conjugate, all that is required is that the hob used to cut the wheel will have the same diameter and form of teeth as the worm to be used. Hence the worm for a worm wheel is not necessarily a true involute helical gear the sections of whose teeth in the plane of rotation are involute in outline.

In Fig. 365 is shown a triple thread worm, the lead l of which is three times the axial pitch p_a. The respective legs of the right triangle cde in Fig. 366 represent the lead and pitch circumference of the worm. If this triangle were wrapped on the pitch cylinder of the worm, the hypotenuse ce would follow the pitch helix of the worm. The lead angle, helix

FIG. 365   FIG. 366

FIG. 367

angle, and pitch radius of the worm are, respectively, represented by λ, ϕ, and R. Obviously

$$\tan \lambda = \frac{l}{2\pi R} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (284)$$

For a multi-thread worm of N threads, the axial pitch is

$$p_a = \frac{l}{N} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (285)$$

and the normal pitch p_n is

$$p_n = \frac{(df)}{N} = \frac{l \cos \lambda}{N} = p_a \cos \lambda \quad \cdot \quad \cdot \quad (286)$$

Since the equations deduced in Arts. 173 and 174 do not involve the form of the teeth, they apply to the use of worms. Their application will be restricted to the usual case of shafts at right angles for which the helix angle ϕ of the wheel is equal to the lead angle λ of the worm. As for all kinds of gears,

$$\frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (287)$$

From (278)

$$\frac{\omega_1}{\omega_2} = \frac{R_2 \cos \phi_2}{R_1 \cos \phi_1} = \frac{R_2 \cos \lambda}{R_1 \cos (90 - \lambda)} = \frac{R_2}{R_1 \tan \lambda}$$

Substituting the value of $\tan \lambda$ from (284) gives

$$\frac{\omega_1}{\omega_2} = \frac{2\pi R_2}{l_1} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (288)$$

Equation (288) shows that the angular velocity ratio of a worm and wheel is independent of the pitch radius of the worm. For any given axial pitch and number of threads, the pitch radius of the worm should, however, be so chosen as to yield a lead angle as efficient as the velocity ratio and conditions of service will permit. Special attention should therefore be given to worms and wheels whose primary function is to transmit power. By a proper choice of the lead angle, efficiencies as high as 97 per cent have been attained in automotive worm and wheel drives. Experimental work is in agreement with the theory of screws in showing that worms are most efficient for a lead angle of about 45 degrees, and that there is little change in the efficiency for a variation of about 15 degrees above or below this angle. With respect to the efficiency of worms and wheels, it may be said that wear will be least

for lead angles from 60 to 30 degrees, will not be serious from 30 to 20 degrees, and will not be destructive from 20 to 12 degrees. The corresponding ranges for the helix angle are, therefore, 30 to 60, 60 to 70, and 70 to 78 degrees. Assuming the same variation to hold for the design of helical gears in general, the helix angle of the driver should, for best results, be from 30 to 60 degrees.

The American Gear Manufacturers' Association recommend for worm gearing for general commercial use, " that the thread form to be regarded as standard will be the form produced by a straight-sided milling cutter having a diameter not less than the outside diameter of the worm nor greater than 1.25 times the outside diameter of the worm, the sides of the cutter having an angle of obliquity of $14\frac{1}{2}$ degrees in the case of single and double thread worms and an angle of obliquity of 20 degrees in the case of triple and quadruple thread worms."
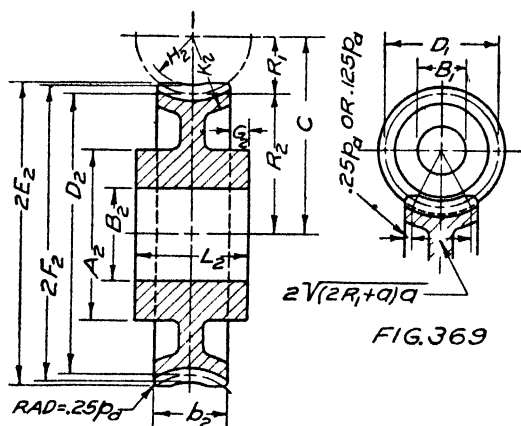
The recommendations of the A. G. M. A. for the standardization of commercial worm gearing for general purposes for shafts at 90 degrees are set forth in Table 13. Referring to Figs. 368 and 369, let

FIG. 369

FIG. 368

p_a = the axial pitch of the worm in inches = the circular pitch p_c of the worm wheel;

a = the addendum of the worm and worm wheel teeth;

h = the whole depth of the worm and worm wheel teeth;

l_1 = the lead of the worm in inches = $p_a N_1$;

λ = the lead angle of the worm;

α' = the angle of the side of the milling cutter. A very close approximation to the pressure angle α_n in the normal plane for contact at the pitch point is, in degrees,

$$\alpha_n = \left(\alpha' - \frac{90 \sin^3 \lambda}{N_1}\right);$$ and the tangent of the pressure angle α_a in the axial plane is

$\tan \alpha_a = \sec \lambda \tan \alpha_n;$

N_1 and N_2 = the respective numbers of teeth of worm and wheel;

R_1 and R_2 = the respective pitch radii of worm and wheel;

$\quad C$ = the distance between centers of worm and wheel

$\quad = R_1 + R_2$;

D_1 and D_2 = the respective pitch diameters of worm and wheel;

$\quad F_2$ = the radius at the throat of the worm wheel;

E_1 and E_2 = the respective outside radii of worm and wheel;

A_1 and A_2 = the respective diameters of the worm and wheel hubs;

B_1 and B_2 = the respective diameters of the bores of the worm and wheel hubs;

b_1 and b_2 = respectively, the threaded length of the worm and the width of the face of the worm wheel;

G_1 and G_2 = the respective extensions of the worm and wheel hubs;

L_1 and L_2 = the respective lengths of the worm and wheel hubs;

$\quad H_2$ = the radius of the wheel face;

$\quad K_2$ = the radius of the wheel rim.

176. Solution of Helical Gear Problems. As before stated, helical gears are used to transmit motion at a constant angular velocity ratio between non-intersecting shafts which are at an angle of 90 degrees or less, 90 degrees being the usual angle. In helical gear problems the angular velocity ratio and angle between shafts are always fixed, while the distance between shafts may or may not be fixed, the most difficult type of problem being for a fixed distance between shafts. As an example of the simpler type of problem let it be assumed that the angular velocity ratio of driver to follower is 2 for an angle between shafts of 60 degrees, and that the distance between shafts is to be approximately 5.25 in.; that is:

$$n = \frac{\omega_1}{\omega_2} = 2, \quad \theta = 60° \quad \text{and} \quad C = R_1 + R_2 = 5.25 \text{ in., approximately.}$$

A diametral pitch of 4 will be assumed suitable for the power to be transmitted. Assuming the same helix angle for driver and follower gives $\phi_2 = \phi_1 = \dfrac{\theta}{2} = 30°$, which, according to Art. 175, is an efficient angle. Hence, by equation (283) the number of teeth N_1 in the driver is

$$N_1 = \frac{2p'_n C \cos \phi_1}{n+1} = \frac{2(4)5.25(0.86603)}{2+1} = 12.124$$

Since C is not absolutely fixed, N_1 may be called 12, in which case, by equations (277) and (279), the remaining values become as follows:

TABLE 13

PROPORTIONS OF STANDARD COMMERCIAL WORMS AND WHEELS RECOMMENDED
BY THE AMERICAN GEAR MANUFACTURERS' ASSOCIATION
(GEAR RATIOS 10 TO 1 TO 100 TO 1)

Worm Dimension	Single and Double Thread Worms	Triple and Quadruple Thread Worms
$p_a =$	$\frac{1}{4}, \frac{5}{16}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, 1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4},$ or 2	$\frac{1}{4}, \frac{5}{16}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, 1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4},$ or 2
$a =$	$0.318 p_a$	$0.286 p_a$
$h =$	$0.686 p_a$	$0.623 p_a$
$\alpha' =$	$14.5°$	$20°$
$\lambda =$	$\tan \lambda = \dfrac{l_1}{2\pi R_1}$	$\tan \lambda = \dfrac{l_1}{2\pi R_1}$
$N_1 =$	1 or 2	3 or 4
$R_1 =$	$1.20 p_a + 0.55$	$1.20 p_a + 0.55$
$D_1 =$	$2.40 p_a + 1.10$	$2.40 p_a + 1.10$
$2E_1 =$	$3.036 p_a + 1.10$	$2.972 p_a + 1.10$
$A_1 =$	$1.664 p_a + 1.0$	$1.726 p_a + 1.0$
Max. $B_1 =$	$p_a + 0.625$	$p_a + 0.625$
$b_1 =$	$p_a\left(4.5 + \dfrac{N_2}{50}\right)$	$p_a\left(4.5 + \dfrac{N_2}{50}\right)$
$G_1 =$	p_a	p_a
$L_1 =$	$p_a\left(6\ 5 + \dfrac{N_2}{50}\right)$	$p_a\left(6.5 + \dfrac{N_2}{50}\right)$

Wheel Dimension	Single and Double Thread Worms	Triple and Quadruple Thread Worms
$p_c =$	p_a as for worms	p_a as for worms
$a, h,$ and $\alpha' =$	as above for worms	as above for worms
$N_2 =$	N_1 (Gear ratio)	N_1 (Gear ratio)
$R_2 =$	$\dfrac{p_a N_2}{2\pi}$	$\dfrac{p_a N_2}{2\pi}$
$D_2 =$	$\dfrac{p_a N_2}{\pi}$	$\dfrac{p_a N_2}{\pi}$
$2F_2 =$	$D_2 + 0.636 p_a$	$D_2 + 0.572 p_a$
$2E_2 =$	$D_2 + 1.1135 p_a$	$D_2 + 0.8903 p_a$
$A_2 =$	$1.875 B_2$	$1.875 B_2$
$b_2 =$	$2\sqrt{(D_1 + a)a} + 0.5 p_a$	$2\sqrt{(D_1 + a)a} + 0.25 p_a$
$G_2 =$	$0.25 B_2$	$0.25 B_2$
$L_2 =$	$b_2 + 2G_2$	$b_2 + 2G_2$
$H_2 =$	$0.882 p_a + 0.55$	$0.914 p_a + 0.55$
$K_2 =$	$2.20 p_a + 0.55$	$2.10 p_a + 0.55$

$$N_2 = nN_1 = 24$$

$$R_1 = \frac{N_1}{2p'_n \cos \phi_1} = 1.732 \text{ in.}$$

$$R_2 = \frac{N_2}{2p'_n \cos \phi_2} = 3.464 \text{ in.}$$

$$C = R_1 + R_2 = 5.196 \text{ in.}$$

If the distance of 5.25 in. between centers is to be adhered to, then a solution must be sought by varying the helix angles. Since 5.196 in. differs very little from the distance desired, the numbers of teeth, 12 and 24, may be retained. Remembering that $\phi_2 = \theta - \phi_1$, it is found, by substituting in equation (281), that

$$\left[\frac{1}{\cos \phi_1} + \frac{2}{\cos (\theta - \phi_1)} \right] \text{ must equal 3.5}$$

By trial, in which a table of reciprocals is convenient, it is found that $\phi_1 = 27° 25'$, and $\phi_2 = 32° 35'$. Hence,

$$R_1 = \frac{N_1}{2p'_n \cos \phi_1} = 1.6898 \text{ in.}$$

$$R_2 = \frac{N_2}{2p'_n \cos \phi_2} = 3.5603 \text{ in.}$$

$$C = R_1 + R_2 = 5.2501 \text{ in.}$$

The above values are satisfactory provided the gear-cutting machine can be geared for the required lead.

177. Graphical Solution of Helical Gear Problems. The relations between the dimen-

FIG. 370

sions of any pair of helical gears may be represented graphically. In Fig. 370 let ox and oy represent the axes of a pair of helical gears and θ the angle between them. On ox and oy, respectively, lay off oa and ob such that $\dfrac{oa}{ob} = \dfrac{\omega_1}{\omega_2} = n$, the angular velocity ratio of driver and follower. From o draw a line through the intersection d of lines bd and ad drawn parallel respectively to ox and oy. Then draw two parallel lines a distance apart equal to $2C$ at an angle ϕ_1 with ox, $2C$ being twice the distance between the axes of the gears and ϕ_1 the
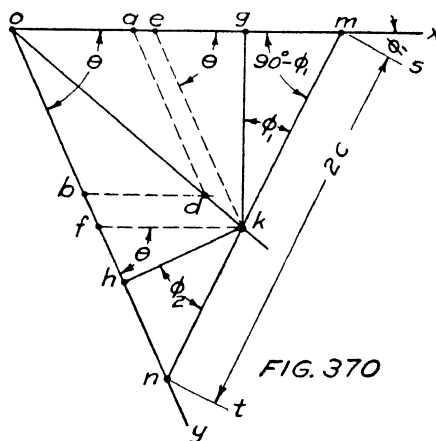
helix angle of the driver. These parallel lines will intersect ox and oy at m and n, the line mn intersecting od at k. Draw kg and kh perpendicular, respectively, to ox and oy. Then angle hkn will represent the helix angle ϕ_2 of the follower and km and kn the respective diameters D_1 and D_2 of the driver and follower.

Obviously, if angle $xms = \phi_1$, then angle tny must be equal to $(\theta - \phi_1)$ or to ϕ_2, the helix angle of the follower. By construction,

$$\frac{\omega_1}{\omega_2} = \frac{oa}{ob} = \frac{oe}{of} = \frac{kf}{ke} = \frac{kh}{kg} = \frac{(kn)\cos\phi_2}{(km)\cos\phi_1} \qquad . \quad . \quad (289)$$

But from equation (278),

$$\frac{\omega_1}{\omega_2} = \frac{D_2\cos\phi_2}{D_1\cos\phi_1} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (290)$$

Combining (289) and (290) gives

$$\frac{kn}{km} = \frac{D_2}{D_1} \quad \text{or} \quad \frac{kn + km}{km} = \frac{2C}{km} = \frac{D_2 + D_1}{D_1}$$

Hence,

$$km = D_1 \quad \text{and, therefore, } kn = D_2$$

From equations (279)

$$N_1 = p'_n D_1 \cos\phi_1 = p'_n(km)\cos\phi_1 = p'_n(kg)$$

$$N_2 = p'_n D_2 \cos\phi_2 = p'_n(kn)\cos\phi_2 = p'_n(kh)$$

FIG. 371
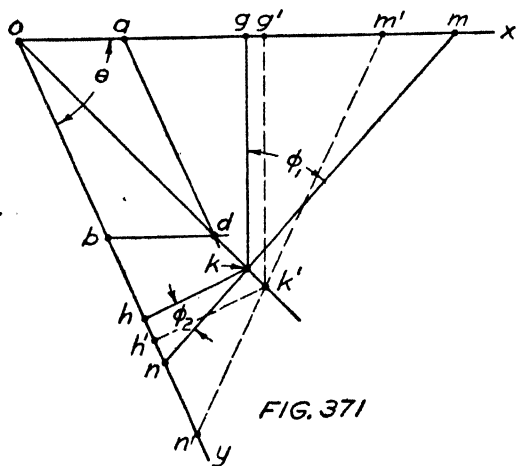
Lengths (kh) and (kg) may therefore be regarded as the pitch diameters of straight spur gears having the same pitch p'_n and the same numbers of teeth N_1 and N_2 as the pair of helical gears.

The method of applying the above construction to the solution of helical gear problems is shown in Fig. 371. Assuming θ, $\dfrac{\omega_1}{\omega_2}$, p'_n, and C known, the values N_1 and N_2, D_1 and D_2, and ϕ_1 and ϕ_2 may be determined.

Lines ox and oy are first drawn from o at the given angle θ, and the line od determined by drawing ad and bd parallel, respectively, to ox and oy, having laid off oa and ob to the ratio $\dfrac{oa}{ob} = \dfrac{\omega_1}{\omega_2} = n$. A line $m'n'$ equal in length to $2\,C$ is then drawn in such a position that it will intersect od extended at a maximum distance ok' from o. Since k' is a maximum distance from o, the perpendiculars $k'g'$ and $k'h'$ will be of maximum length and will give maximum approximate values of N_1 and N_2 when multiplied by p'_n. The nearest whole numbers lower than these maximum computed values should be taken for N_1 and N_2 that will satisfy the relation $\dfrac{\omega_1}{\omega_2} = \dfrac{N_2}{N_1}$. Having fixed on N_1 and N_2, the values $kg = \dfrac{N_1}{p'_n}$ and $kh = \dfrac{N_2}{p'_n}$ may be computed and drawn parallel to $k'g'$ and $k'h'$, thus locating k. Drawing through k the line $mn = m'n' = 2\,C$, determines the helix angles $mkg = \phi_1$ and $nkh = \phi_2$. If the first values of N_1 and N_2 that are tried do not give practical values of ϕ_1 and ϕ_2, the next lower values of N_1 and N_2 that would satisfy the given angular velocity ratio should be tried. If by such trials a satisfactory solution cannot be found, either the pitch or the distance between centers should be changed.

While the helix angles, determined as just outlined, may be very close approximations to the true results, it cannot be hoped that they will be exact. The exact values should be obtained by trial by the use of equation (281) as illustrated at the close of the previous article.

178. Cutting of Helical Gears. Since any one of a pair of helical gears is, geometrically, a helical spur gear, the cutting of a helical gear presents the same problem as the cutting of a helical spur gear. Helical gears are, therefore, cut in the same way as helical spur gears. The formed cutter method of cutting was described in Art. 156, the template method in Art. 157, the shaping method of generation in Art. 158, and the hobbing method of generation in Art. 159. It remains to explain the cutting of a worm wheel.

Worm wheels that are to have line contact with their worms are always hobbed. The hobbing of such a wheel is shown in Fig. 372. Single and double thread hobs are usually fluted parallel to their axes, as shown in Fig. 338; hobs having more than two threads should be fluted normal to the thread angle reckoned from the outside diameter of the hob. For shafts at 90 degrees, the hob is positioned with its axis at right angles to the axis of the arbor on which the worm wheel

blank is mounted.   As the hob and blank are driven positively, the hob
making $\dfrac{N_2}{N_1}$ revolutions to one of the blank, the hob is fed toward the
axis of the blank, or vice versa, until the axis of the hob is at the
proper distance from the axis of the blank.   One revolution of the
blank, after the hob has been fed to its proper depth, serves to generate
all the teeth in the worm wheel blank.

For large lead angles, hobs that taper off at one end like a tap are
sometimes used.   The axis of such a hob is set at once to the proper
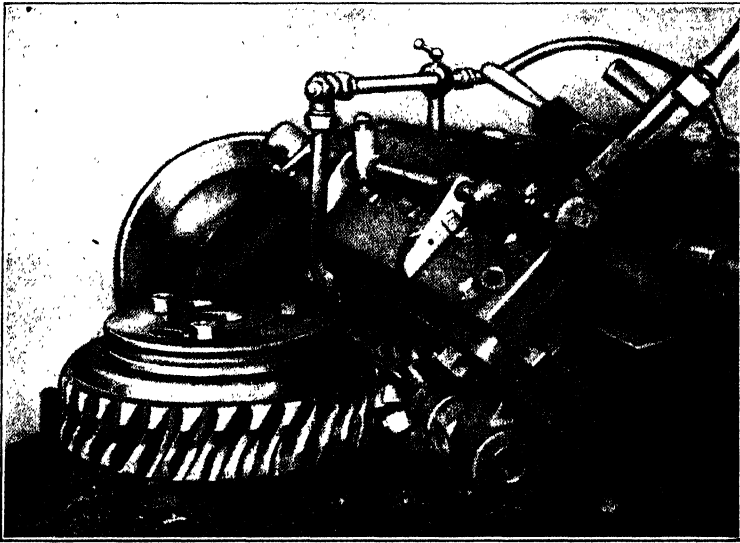
Fig. 372

distance from the axis of the blank, and, as the hob and blank rotate,
the hob is fed into the blank in the direction of its own axis.   To com-
pensate for this movement differential gearing superposes on the uniform
rotation of the hob an additional rotary movement proportional to the
axial advance of the hob.

179. Hyperboloidal Gears.   As defined in Arts. 115 and 122, hyper-
boloidal gears are toothed gears whose pitch surfaces are rolling hyper-
boloids.   In Fig. 373 are shown the pitch cylinders *1′* and *2′* of a pair
of helical gears and the hyperboloidal pitch surfaces *1* and *2* of a pair
of hyperboloidal gears mounted on the same shafts *AA* and *BB* which
are at an angle θ in parallel planes.   Motion may be transmitted from
one shaft to the other at the same angular velocity ratio either by the

pair of helical gears represented by *1'* and *2'* or by the pair of hyper-boloidal gears represented by *1* and *2*. While the two pairs of gears are mounted on the same shafts, it is to be noted that they are quite different in appearance. The hyperboloidal gears look like bevel gears and are often referred to as skew bevel gears. While the shafts may be at any angle, they are, in most engineering applications, at right angles.

Rolling hyperboloids were quite fully discussed in Art. 118, and a
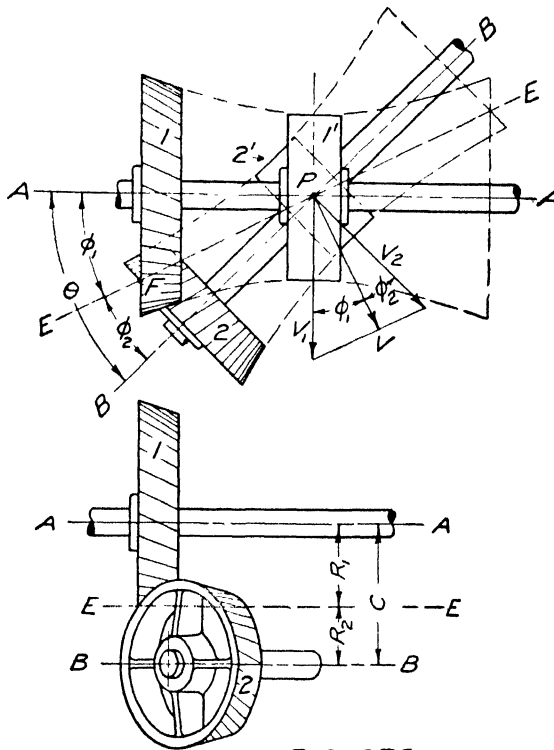
FIG. 373

graphical method was there developed for locating the contact element and for determining the respective radii of the gorge circles of a pair of rolling hyperboloids for a given angular velocity ratio $\frac{\omega_1}{\omega_2}$ and a given angle θ and a given distance C between shafts. Equations (198), (199), and (200), as there derived, apply to Fig. 373. For shafts at right angles, $\theta = 90°$, these equations become:

$$\frac{R_1}{R_2} = \tan^2 \phi_1 = \cot^2 \phi_2 \quad \ldots \ldots \quad (291)$$

$$\frac{\omega_1}{\omega_2} = \frac{R_2}{R_1} \tan \phi_1 = \frac{R_2}{R_1} \cot \phi_2 \quad \ldots \ldots \quad (292)$$

$$\frac{\omega_1}{\omega_2} = \cot \phi_1 = \tan \phi_2 \quad \ldots \ldots \ldots \quad (293)$$

By adding unity to each side of equation (292) and substituting n for $\frac{\omega_1}{\omega_2}$ and C for the distance $(R_1 + R_2)$ between shafts, the following expressions are obtained:

$$R_1 = \frac{C \tan \phi_1}{n + \tan \phi_1} = \frac{C}{n \tan \phi_2 + 1} \quad \ldots \quad (294)$$

$$R_2 = \frac{Cn}{\tan \phi_1 + n} = \frac{C}{\cot \phi_2 + n} \quad \ldots \ldots \quad (295)$$

As for all kinds of gears,

$$\frac{\omega_1}{\omega_2} = \frac{N_2}{N_1} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (296)$$

Two pair of Gleason* hyperboloidal gears are shown in Fig. 374. These gears are known to the trade as Hypoid gears. The method of accurately generating the teeth of such gears was developed at the Gleason Works.† Like spiral bevel gears, Hypoid gears were developed primarily for automobile drives. They are now applied to other purposes, and Hypoid gears 55 inches in diameter have been made.

For hyperboloidal as for helical gears there is a component of sliding along the teeth as well as a combination of rolling and sliding similar to that in spur gears. However, hyperboloidal gear teeth have line contact, while the teeth of helical gears have point contact. Hyperboloidal gears are, therefore, more durable and suitable for the transmission of power than helical gears. The sliding along the teeth sufficiently assists lubrication so that Hypoid gears, in spite of this additional sliding, wear as well as spiral bevel gears. Their most important ad-

* Gleason Works, Rochester, N. Y.

† See U. S. patents to Ernest Wildhaber: 1,622,555; 1,676,419; 1,673,540; 1,705,886; 1,676,371; etc.

See also "Design, Production, and Application of the Hypoid Rear-Axle Gear," by Arthur L. Stewart and Ernest Wildhaber, in the *Journal of the Society of Automotive Engineers* for June, 1926.

vantage over spiral bevel gears is due to the fact that their shafts do not intersect. This makes it possible, as shown in Fig. 374, to continue the shaft on which the pinions are mounted, and to drive several or more parallel shafts from the same drive shaft as when helical gears or worms and worm wheels are employed.

The pitch surface of a spur gear rack is a plane. In bevel gearing the pitch surface of the crown gear, or rack, is a circular plane. The pitch surface of a rack for hyperboloidal gears is a helicoidal surface.
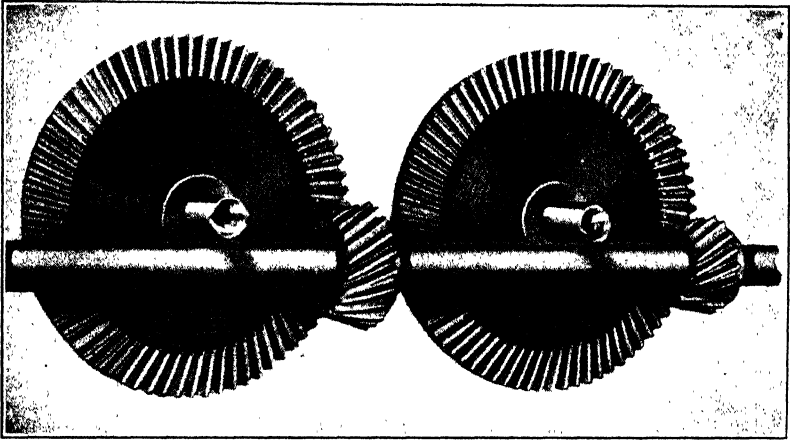
FIG. 374

For this pitch surface the helix through P in Fig. 373 is a straight line normal to the contact pitch element EE, and the curve through F is a helix. The element of this pitch surface is the pitch element PF of the hyperboloids. It is evident from the nature of the pitch surface of a rack which would act with either hyperboloid that correct teeth for hyperboloidal gears are very complex in form, and their generation an intricate problem.

CHAPTER XI

LINKWORK AND MISCELLANEOUS MECHANISMS

180. General Considerations. In the preceding chapters various direct contact mechanisms, simple or basic four-link mechanisms, and compound mechanisms have been treated. It has been shown how to find the linear displacement, velocity, and acceleration of any point in a mechanism and the angular velocity and angular acceleration of any link. Except for direct contact mechanisms, the effect of the proportions of the links of a mechanism on the motion transmitted from the driver to the follower has not, however, been discussed to any extent. A study of how the proportions of the links of a mechanism affect the motions transmitted is called *linkwork*. The term applies to those mechanisms in which the motion of the driver is transmitted to the follower by means of a rigid link and not by means of a flexible connector or by direct contact between the driver and follower. Mechanisms, or linkages, can be used to convert:

(*a*) Continuous rotation into continuous rotation with a constant or variable angular velocity ratio;

(*b*) Continuous rotation into reciprocation, rectilinear or circular, or the reverse;

(*c*) Reciprocation into reciprocation with a constant or a variable ratio.

With the above conversions in mind, some mechanisms of fundamental importance and some of common occurrence will be studied.

181. Four-Link Mechanism, Reciprocation of Driver and Follower. As pointed out in Chapter II, the simplest complete kinematic, or constrained, chain has four links. A four-link mechanism having four binary links and four turning pairs is shown in Fig. 375. In this and the figures to follow, O is the fixed link, *1* the driver, *2* the connecting link, and *3* the follower, the respective lengths of the links being l_0, l_1, l_2, and l_3. The proportions of the links in Figs. 375 and 376 are such that neither the driver *1* nor the follower *3* can make a complete rotation. Hence, proportioned as they are, these mechanisms can only convert circular reciprocation into circular reciprocation.

The mechanism in Fig. 375 is shown dotted in three limiting phases. With C at C', center B is at the dead-point position B', and the follower 3 is in its extreme position to the right. Point C' is the intersection of an arc of radius $(l_1 + l_2)$ from O_{10} as a center with an arc of radius l_3 from O_{30} as a center, B' being the intersection of a line from C' to O_{10} with an arc of radius l_1 from O_{10} as a center. With B at B'' and B''', center C is at the respective dead-point positions C'' and C''', and the driver 1 is in its extreme left positions. Points B'' and B''' are the intersections of an arc of radius $(l_2 + l_3)$ from O_{30} as a center with an arc of radius l_1 from O_{10} as a center. While center B reciprocates through the arc $B''B'B'''$ and back to B'', center C of the follower reciprocates through the arc $C''C'C'''$ and back to C''. Within these limits either member might act as the driver and its circular reciprocation cause circular reciprocation of the other. However, in the neighborhood of the dead-point positions of the follower, the action of an
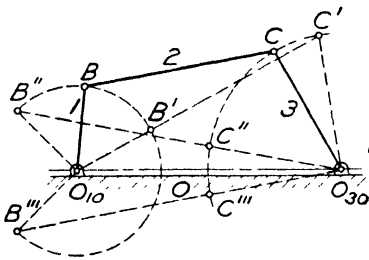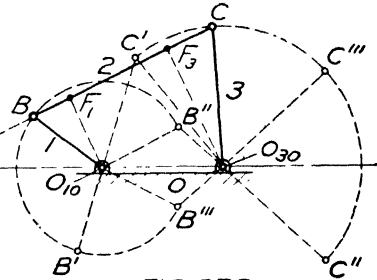
FIG. 375   FIG. 376

actual mechanism would not be smooth and satisfactory. For this reason, 1, if it is assumed to be the driver, should reciprocate through a smaller arc than $B''B'B'''$. Referring to the lengths of the links in the figure, it is to be noted that $(l_2 + l_3) < (l_0 + l_1)$, or $(l_2 - l_1) < (l_0 - l_3)$. For it to be possible for 1 to make a complete rotation, it is evident, by inspection of one of the limiting positions, that $(l_2 + l_3)$ must be equal to or greater than $(l_0 + l_1)$, or $(l_2 - l_1) \gtreqless (l_0 - l_3)$.

Fig. 376 has been lettered in the same manner as Fig. 375 and to it the same analysis applies. Theoretically, while B reciprocates through the arc $B''B'B'''$ and back to B'', center C of the follower reciprocates through the arc $C''C'C'''$ and back to C''. To be practical, as explained immediately above, the driver should reciprocate through a smaller arc than $B''B'B'''$. In Fig. 376, $l_0 < (l_2 - l_3 + l_1)$ or $(l_2 + l_1) > (l_0 + l_3)$. For it to be possible for 1 to make a complete rotation, it is evident, by inspection of one of the limiting positions, that l_0 must be equal to or greater than $(l_2 - l_3 + l_1)$, or $(l_2 + l_1) \gtreqless (l_0 + l_3)$.

Four-link mechanisms are much used to convert circular reciprocation into circular reciprocation at a constant or variable angular velocity ratio. An example from a four-valve engine using a Corliss type of valve is shown in Fig. 377. Such engines employ two oscillating valves at each end of the cylinder, one to control the admission and cut-off of steam and the other to control the exhaust and compression. The figure shows a steam valve for controlling the admission and cut-off of steam at one end of the cylinder. The valve S is oscillated about its axis O_{30} by link 3 which is driven by link 1 through link 2. The steam edge of the valve is shown coincident with the steam edge of the port. To admit steam to the cylinder the valve must be rotated clockwise and rotated counter-clockwise to stop, or cut-off, the admission of steam. The proportions of the links are such as to give a rapid opening and closing of the valve with a slower and much reduced movement of the valve after it is closed. As 1 reciprocates through the arc BB' and back to B, 3 reciprocates through the arc CC' and back to C, and the valve uncovers and then closes the steam port to the admission of steam. As the motion of 1 continues from B to B'' and back to B, 3 moves from C to C'' and back to C. It is to be noted that 1 oscillates through angle $BO_{10}B'$ during the opening and closing of the valve and through the larger angle $BO_{10}B''$ after the valve is closed, while the valve oscillates through the respectively larger and smaller angles $CO_{30}C'$ and $CO_{30}C''$. Hence, by properly proportioning the links, a quick comparatively large travel of the valve in opening and closing is secured with a slower and much reduced travel after closing.
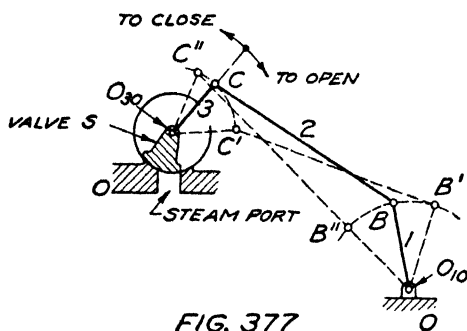
FIG. 377

For the proportions shown in Figs. 375, 376, and 377, motion is transmitted from driver to follower at a variable angular velocity ratio. This ratio may be found for any phase by the method of Art. 43 or by applying the angular velocity theorem of Art. 67. According to Art. 43 the angular velocities of driver and follower are inversely as the perpendiculars let fall from their fixed centers upon the line of transmission, or inversely as the segments into which the line of centers is cut by the line of transmission. Applying this to Fig. 376, the

ratio of the angular velocities of driver and follower for the phase
shown is,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_{30}F_3}{O_{10}F_1} = \frac{O_{13}O_{30}}{O_{13}O_{10}}$$

It is thus evident that the angular velocity ratio of driver to follower
in Figs. 375, 376, and 377 is variable.

The linear velocities of B and C for any phase may be found by the
instant center method of Art. 64 or by the velocity vector diagram
method of Art. 74.

**182. Four-Link Mechanism, Rotation of Driver and Reciprocation
of Follower.** It was shown in the preceding article that complete
rotation of the driver requires that $(l_2 - l_1) \gtreqqless (l_0 - l_3)$ or $(l_2 + l_1) \lesseqqgtr$
$(l_0 + l_3)$. In the respective Figs. 378 and 379 $(l_2 - l_1) = (l_0 - l_3)$
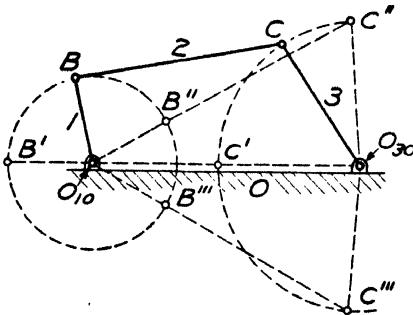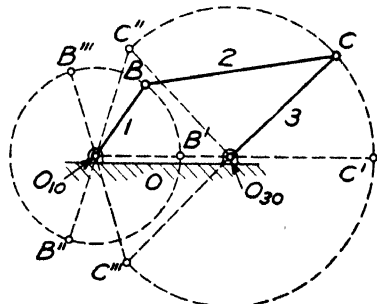
FIG. 378   FIG. 379

and $(l_2 + l_1) = (l_0 + l_3)$. While these proportions make it possible
for the driver *1* to make a complete rotation, the movement of the
follower *3* would not be completely constrained. As before, three
limiting positions are shown dotted for each mechanism. It is to be
noted that B' and C' are simultaneous dead-point positions of B and C
making it possible for the follower *3* to move either toward C'' or C'''
as the motion of *1* the driver is continued. Hence, the motion of the
follower is incompletely constrained.

For continuous rotation of the driver and complete constrainment
of the follower, $(l_2 - l_1)$ must be greater than $(l_0 - l_3)$ and $(l_2 + l_1)$
must be less than $(l_0 + l_3)$. A linkage so proportioned is shown in
Fig. 380. Continuous rotation of the driver imparts a circular recip-
rocation to the follower which is completely constrained. The propor-
tions of the mechanism are such that the angular velocity ratio of the
driver to follower is variable as can easily be shown by applying the
method of Art. 43 as was done in the preceding article.

183. Four-Link Mechanism, Rotation of Driver and Follower. If both the driver and the follower are to rotate continuously, Fig. 381, there must be no dead-points; for, if the driver *1* should reach a dead-point, the follower *3* would come to rest, and, if the follower *3* should reach a dead-point, its motion would not be fully constrained, and, generally, it would lock the driver and prevent rotation. If, in Fig. 381, either the driver or the follower were to have a dead-point at B' or C', then each would have dead-points simultaneously. It can be seen by inspection of these positions that to avoid dead-points $(l_2 + l_1)$ must be greater than $(l_0 + l_3)$ and l_2 must be less than $(l_3 - l_0 + l_1)$, or the proportions must be such that $(l_2 + l_1) > (l_3 + l_0)$ and $(l_2 - l_1) < (l_3 - l_0)$. The lengths of the links in Fig. 381 are in accordance with these relations. With the links so related there can be no dead-points, and continuous rotation of either *1* or *3* will cause the other to rotate continuously. The angular velocity ratio will, however, be variable
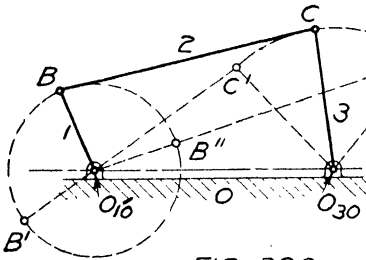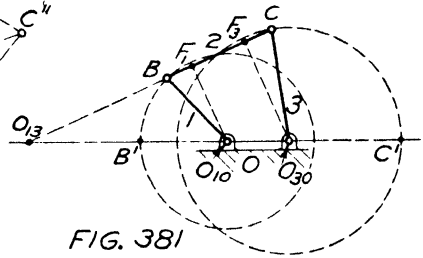
FIG. 380   FIG. 381

as can be shown by applying the method of Art. 43 or by applying the angular velocity theorem of Art. 67. Thus, for the phase shown,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_{30}F_3}{O_{10}F_1} = \frac{O_{13}O_{30}}{O_{13}O_{10}}$$

The mechanism shown in Fig. 381 is called a *drag-link* mechanism. When used to connect two cranks, the center line of whose shafts coincide, the length of the fixed link O becomes zero. In this event links *1*, *2*, and *3* act as though pinned together and cease being a kinematic chain and become a structure, the three links rotating as one piece. Hence, with l_0 equal to zero, driver and follower would rotate at the same angular velocity, and their angular velocity ratio would be unity. Two cranks of equal length are sometimes coupled up in this way in preference to having a shaft with a single center crank, the results of possible misalignment of the shafting being less objectionable for the former than for the latter arrangement. While for the former arrange-

ment a misalignment would cause a slight variation in the angular velocity ratio, the wrenching action and stresses induced would not be as severe as for the latter arrangement.

It has just been shown that if the length l_0 of the fixed link is zero, the angular velocities of driver and follower are equal whether l_1 and l_2 are equal or unequal. The angular velocities of driver and follower are also equal if $l_1 = l_3$ and $l_2 = l_0$. As shown in Fig. 382, these are the proportions that must be used in connecting the drive wheels of a locomotive. Since for these proportions the driver and follower would have simultaneous dead-point positions in the line of centers, special means must be provided to completely constrain the motion of the follower. The special means used for locomotives is to employ two side rods *2* and *2′* and two pairs of cranks *1* and *3* and *1′* and *3′* of equal length, cranks *1* and *1′* being at the same angle, usually 90 degrees, as cranks
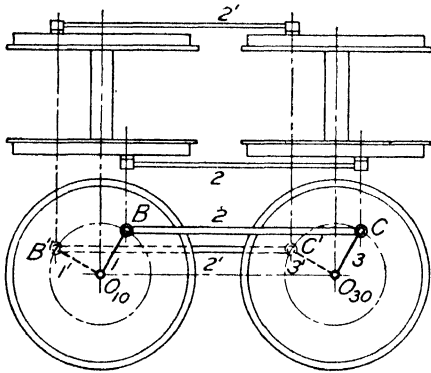
FIG. 383
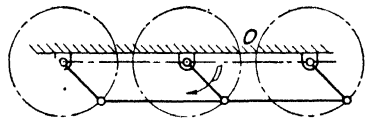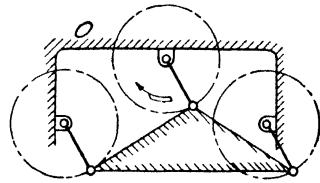
FIG. 382

FIG. 384

3 and *3′*. Then, when *1* and *3* are horizontal and on *dead center*, *1′* and *3′* will be vertical and in a favorable position to be driven by the engine on the right side of the locomotive; and *1* and *3* will be in a favorable position to be driven by the engine on the left side of the locomotive when *1′* and *3′* are on dead center. Thus the motion of the drive wheels is completely constrained and stalling of the locomotive is made impossible. That the angular velocity ratio of driver to follower is constant and equal to unity is shown by applying the method of Art. 43. Since cranks *1* and *3* are of equal length, the perpendiculars let fall from O_{10} and O_{30} on the line of transmission will always be equal, which means that the angular velocity ratio of driver to follower will always be unity. It is to be noted that this will always be true where the links, *0, 1, 2,* and *3* form a parallelogram.

Other possible arrangements for securing complete constrainment of the follower are shown in Figs. 383 and 384.   It is to be noted that in each case three equal cranks are employed whose pins are connected by a ternary link.   It is also to be noted that while the centers of rotation of the cranks may be spaced in any manner, the crank-pin centers as spaced by the ternary link must agree with the spacing of the centers of rotation of the cranks.   The middle crank, marked by an arrow, should be the driver.

184. Four-Link Mechanism with Sliding Block. In Fig. 385 is shown a four-link mechanism having four binary links and three turning and one sliding pair.   Evidently circular reciprocation or rotation of *1* will cause circular reciprocation of the sliding block *3*.   Assuming the radius of the center line of the slot to be equal to *3'*, the mechanism shown is exactly equivalent to a mechanism having four turning pairs and four links *0*, *1*, *2*, and *3'*, or equivalent to such a mechanism as is shown in Fig. 380.   If the radius *3'* of the center line of the slot were made of infinite length, the slot would be straight and the sliding block
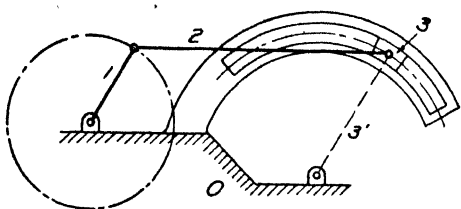
FIG. 385

would have rectilinear reciprocation.   Such a four-link chain with three turning and one rectilinear sliding pair is called the "slider crank chain," the four inversions of which are shown in Figs. 38, 39, 40, and 41.   The most widely used slider crank mechanism is the one where the fixed member is the guide on which the slide reciprocates.   It is the mechanism generally used to convert rotation into rectilinear reciprocation or rectilinear reciprocation into rotation.

185. Slider Crank Mechanism, Connecting-rod of Finite Length. In Fig. 386 is shown a slider crank mechanism with a connecting-rod *2* of finite length.   This is the typical mechanism used in engines to convert rectilinear reciprocation into rotation and in pumps and compressors for converting rotation into rectilinear reciprocation.   This mechanism has previously been treated in many articles.   In Art. 65 it was shown how to find, by the method of instant centers, the linear velocity of the cross-head and the velocity of any point in the connecting-rod for any position and velocity of the crank.   In Art. 70 it was shown how to find the linear acceleration of the cross-head from a plot of its linear velocity.   In Art. 72 a full analytical treatment of the velocity and acceleration of the cross-head and connecting-rod was given, and in Art. 76 the determination of these velocities and accelerations

by means of velocity and acceleration vector diagrams was fully treated. In view of this previous treatment, only matters relating to the effect of proportions will be treated here.

In practically all engines, pumps, compressors, etc., the center line of travel of the cross-head passes through the shaft center as shown in Fig. 386. The cross-head position corresponding to any crank position, or the crank position corresponding to any cross-head position, is easily found graphically. The cross-head position D corresponding to crank position D' is where an arc having a radius equal to the length l of the connecting-rod from D' as a center intersects the center line HC'. The crank positions corresponding to any cross-head position are where an arc of radius l from the cross-head position as a center intersects the crank circle. Thus, for cross-head position D, the crank is either at D' or D'' depending upon whether the cross-head and piston are on their forward or return stroke. Obviously, H' and C' are the respective
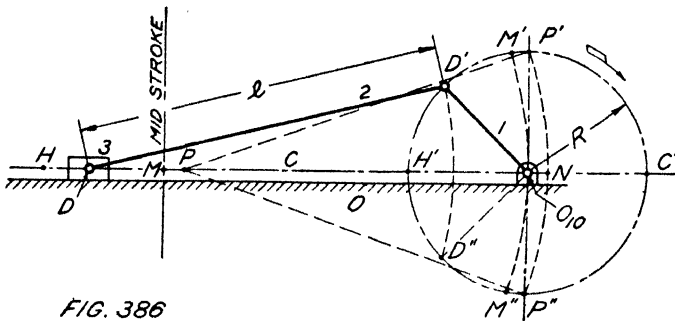
FIG. 386

dead center positions of the crank and H and C the corresponding positions of the cross-head at the head and crank ends of the stroke. With the crank vertically up or down at P' or P'', it is to be noted that the cross-head is at P, a distance MP from the middle of its stroke. This is due to the fact that the connecting-rod is of finite length. In designing engines this effect of " the angularity of the connecting-rod " on the events of the cycle must be considered. For the cross-head at the mid-stroke position M, the crank is either at M' or M''. Hence the first half of the forward stroke is covered in less time than the last half and the second half of the return stroke in less time than the first half. The greater the length of the connecting-rod relative to the crank the less MP and the smaller the difference in time between the halves of the stroke and the nearer will the motion of the cross-head approach being truly harmonic. For the crank at P' or P'',

$$MP = MO_{10} - PO_{10} = l - \sqrt{l^2 - R^2} = R(n - \sqrt{n^2 - 1}) \quad (297)$$

where $n = \dfrac{l}{R}$.   It is to be noted that MP decreases as n increases, being zero for a rod of infinite length.

In Fig. 387, B is where the connecting-rod extended intersects a line through the shaft center perpendicular to the center line DO_{10}. Making the usual assumption that the crank rotates at a uniform rate, it was shown in Art. 70 that the intercept $O_{10}B$ represents the linear velocity of the cross-head to the same scale that $O_{10}D'$ represents the constant linear velocity of the crank-pin. Hence the linear velocity of the cross-head is the same as that of the crank-pin for crank positions for which the intercept is equal to the crank length R. This is obviously true for the vertical position $O_{10}\ P'$ of the crank and is also true for the crank position $O_{10}M'$. As the crank passes through the dead center positions H' and C' the velocity of the cross-head is zero.
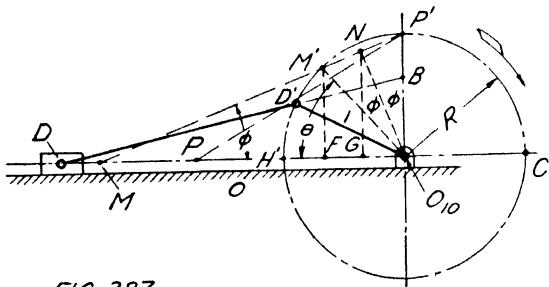
FIG. 387

Hence, the velocity of the cross-head increases from zero to a value equal to the linear velocity of the crank pin as the crank rotates from H' to M', rises above this value between M' and P', and then falls off to zero as the crank reaches C'. The position of the crank-pin M' at which the velocity of the cross-head first equals that of the crank-pin may be determined by computing the length of the perpendicular FM'. Since the triangle $P'O_{10}M'$ is isosceles, $O_{10}N$, drawn perpendicular to $MM'P'$, will equally divide the angle $P'O_{10}M'$. Also, since $P'O_{10}$ and NO_{10} are respectively perpendicular to DO_{10} and $MM'P'$, angle $P'O_{10}N$ is equal to ϕ, likewise angle $M'MO_{10}$. The displacement of the crank when the velocity of the cross-head is first equal to the linear velocity of the crank-pin, is, therefore,

$$\theta = 90 - 2\phi$$

or,

$$\sin \theta = \sin (90 - 2\phi) = \cos 2\phi = 1 - 2\sin^2 \phi \quad . \quad . \quad (298)$$

But,

$$FM' = l \sin \phi = R \sin \theta . \quad . \quad . \quad . \quad . \quad (299)$$

Hence,

$$\sin \phi = \frac{R}{l} \sin \theta = \frac{\sin \theta}{n}$$

Substituting this value of sin ϕ in (298) gives

$$\sin \theta = 1 - \frac{2}{n^2} \sin^2 \theta$$

or

$$\sin^2 \theta + \frac{n^2}{2} \sin \theta = \frac{n^2}{2}$$

Completing the square and solving gives,

$$\sin \theta = \frac{n}{4}[\pm \sqrt{8 + n^2} - n]. \quad . \quad . \quad . \quad . \quad (300)$$

Since the sine of an angle can not be numerically greater than unity, the double sign of the radical may be dropped. Taking the plus sign and substituting the value of sin θ from (300) in (299) gives,

$$FM' = \frac{Rn}{4}[\sqrt{8 + n^2} - n]. \quad . \quad . \quad . \quad . \quad (301)$$

where a line a distance FM' from and perpendicular to the center line DO_{10} intersects the crank circle locates the position M' of the crank-pin when the velocity of the cross-head is first equal to that of the crank-pin.

The crank position corresponding to the maximum velocity of the cross-head is of interest since at that instant the acceleration of the reciprocating parts is zero. Equation (94) gives the value of the acceleration of the reciprocating parts for any angular displacement θ of the crank. The velocity of the reciprocating parts is therefore a maximum when the right-hand member of this equation equals zero, or,

$$n \cos \theta + \cos 2\theta = n \cos \theta + 2 \cos^2 \theta - 1 = \text{zero}$$

Transposing and completing the square,

$$\cos^2 \theta + \frac{n}{2} \cos \theta + \left(\frac{n}{4}\right)^2 = \frac{1}{2} + \left(\frac{n}{4}\right)^2$$

or,

$$\cos \theta = \tfrac{1}{4}[\sqrt{8 + n^2} - n]. \quad . \quad . \quad . \quad . \quad (302)$$

The reciprocating parts reach their maximum velocity just after the position where the crank is perpendicular to the rod; that is, just after the position where,

$$\tan \theta = \frac{l}{R} = n$$

If, as shown in Fig. 388, the center line of the motion of the cross-head does not pass through the center of rotation O_{10} of the crank, the stroke of the cross-head will not be equal to the diameter of the crank-pin circle and the to-and-fro strokes of the cross-head will not be covered in equal intervals of time. Thus, in Fig. 388, if EF is to be the path of travel of the cross-head, one end of the stroke will be at H where an arc of radius $(l + R)$ from O_{10} as a center intersects EF, and the other end of the stroke will be at C where an arc of radius $(l - R)$ from O_{10} as a center intersects EF. The corresponding crank positions are H' and C''. Hence the stroke to the right is made while the crank-pin rotates through the arc $H'D'C'$, and the stroke to the left while the crank-pin rotates through the arc $C'D''H'$. For uniform rotation of the crank, the stroke to the right is therefore made in less time than the stroke to the left. Because of this fact the mechanism is sometimes used as a " quick return mechanism." Within certain limits the proportions of a mechanism for a given crank length R and connecting-rod length l may be determined for a given ratio of the out to the return stroke. The crank circle is first divided into arcs $C'D''H'$ and $H'D'C'$ corresponding to the given ratio thereby determining
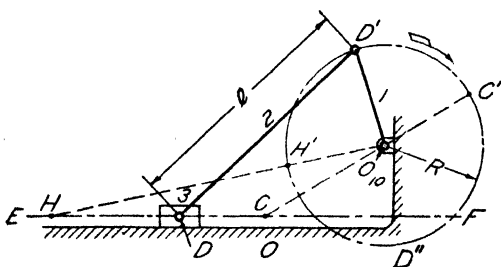
FIG. 388

the crank positions $O_{10}H'$ and $O_{10}C'$. Points C and H marking the ends the stroke of the cross-head are then determined by laying off from O_{10} on $C'O_{10}$ and $O_{10}H'$ extended the respective distances $(l - R)$ and $(l + R)$. A line through H and C gives the center line of travel of the cross-head, and the perpendicular distance from this line to O_{10} the offset of the center of rotation of the crank.

The slider crank mechanism with an offset crank has been used in single acting gas and steam engines to give a quicker return, or exhaust, stroke than the pressure, or working, stroke. To illustrate this application by means of Fig. 388, the direction of rotation of the crank must be the reverse of that shown. Thus, on the pressure, or working, stroke, the crank would rotate through the arc $H'D''C'$ and through the arc $C'D'H'$ on the return, or exhaust, stroke. By using two cylinders and cranks at 180 degrees it is possible by this means to have the piston of one cylinder complete its exhaust stroke and to start on its pressure stroke before the piston of the other cylinder

has completed its pressure stroke, thus avoiding points of zero turning moment. Furthermore, for a given ratio of connecting-rod length to crank length, an offset crank will reduce the obliquity of action during high pressure on the working stroke and increase the maximum obliquity of action on the exhaust stroke. Or, to state the case somewhat differently, the length of the connecting-rod can be reduced and the design made more compact for a given maximum obliquity of action during the period of high pressure on the working stroke.

186. Slider Crank Mechanism, Connecting-rod Length Equal to the Crank Length. When the length of the rod in a slider crank mechanism is made equal to that of the crank, the motion of the follower, cross-head, will not be definitely constrained. If, in Fig. 386, the length of the connecting-rod is made equal to that of the crank, it is evident that as the crank is rotated from H' to P' the cross-head will move to O_{10} and for any further rotation of the crank the connecting-rod will rotate with the crank. Only the inertia of the cross-head as it approaches O_{10} can insure its passing this dead-point position. Hence, for a length of connecting-rod equal to that of the crank, the sliding block or follower, is not definitely constrained. If, as in Fig. 389, the crank is attached to the middle of a ternary link fitted with two sliding blocks and having a length equal to twice that of the crank, both 3 and 4 will be constrained to reciprocate in rectilinear paths. It is to be

FIG. 389
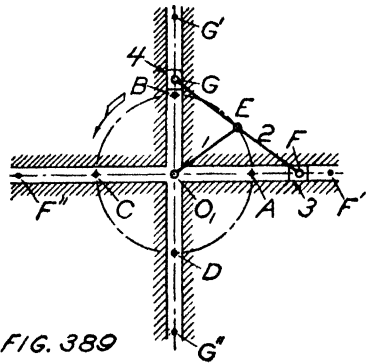
noted that this mechanism is not a simple four-link but a compound mechanism; in fact, a double four-link mechanism. The stroke of each block will be twice instead of being equal to the diameter of the crank-pin circle. When E is at A, F is at F' and G is at O_1. As the crank rotates from A to B, F moves from F' to O_1 and G moves from O_1 to G'; and as the crank moves from B to C, F moves from O_1 to F'' and G moves from G' to O_1. As the crank rotates through the second half of its revolution from C through D to A, F moves from F'' through O_1 to F' and G moves from O_1 to G'' and back to O_1. For any phase the distance of G from O_1 corresponds to the distance $O_{10}B$ of Fig. 387, and hence is proportional to the velocity of F for uniform rotation of the crank; likewise O_1F is proportional to the velocity of G at any phase. In this mechanism the rod, or ternary link, 2 is subjected to a bending action as well as to tension and compression. So far as the

motion of either *3* or *4* is concerned, the mechanism of Fig. 77 is the equivalent of that of Fig. 389.

The *elliptic trammel* which is used for drawing ellipses is an application of the mechanism of Fig. 389.   If, in Fig. 390, GF is made equal to the difference of the semi-major and semi-minor axes of the ellipse

FIG. 390
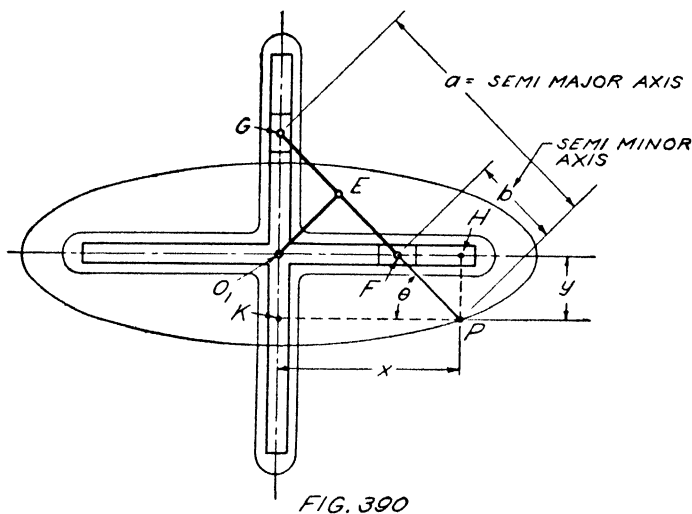
desired, and the extension FP is made equal to the semi-minor axis; then a pencil at P will trace an ellipse in one turn of the crank O_1E. If an ellipse is traced, then, for any phase such as shown, PK would represent x and PH would represent y.   Triangles KPG and HFP are similar; hence,

$$\cos \theta = \frac{x}{a} \quad \text{and} \quad \sin \theta = \frac{y}{b}$$

Squaring these equations and adding them together gives,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 \theta + \sin^2 \theta = 1$$

which is the equation of an ellipse about co-ordinate axes coincident with the major and minor axes.

An Oldham coupling is shown in Fig. 391.   This coupling is used to transmit motion between parallel shafts whose axes F and G are a short distance apart.   As shown in Fig. 392, it is an inversion of the mechanism of Fig. 389 with link *2* instead of O as the fixed member.   Evidently

with *2* fixed the locus of O_1 is a circle of diameter FG. Let O_1' and O_1'' be two positions of O_1. Then angles $O_1'GO_1''$ and $O_1'FO_1''$, being angles of the same segment of a circle, are equal. Hence, if *3* is turned through the angle $O_1'FO_1''$, *4* will be turned through an equal angle $O_1'GO_1''$. If, therefore, *3* and *4* are mounted on shafts, rotary motion may be transmitted from one to the other at an angular velocity ratio equal to unity. The form actually taken by the coupling is shown
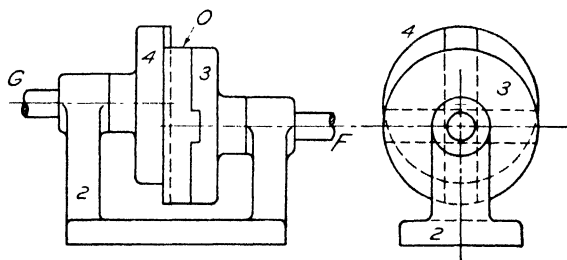
FIG 391
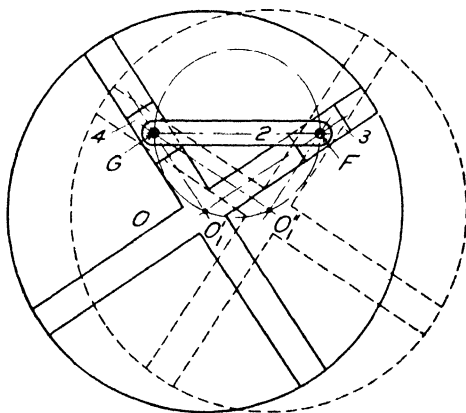
FIG. 392

in Fig. 391, which is lettered and numbered the same as Figs. 389 and 392. Disk O, instead of being shown with two slots at right angles, is shown with a slot on one side and with a guide strip on the other side at right angles to the slot.

187. Slider Crank Mechanism, Connecting-rod of Infinite Length. In treating the slider crank mechanism with a connecting-rod of finite length, it was pointed out in Art. 185 that the longer the connecting-rod relative to the crank the more nearly would the motion of the cross-

head approach being harmonic.   In the slider crank mechanism shown in Fig. 393, a slotted cross-head and sliding block have been substituted for the connecting-rod of Fig. 386.   Just as the cross-head and its guide are equivalent to a link of infinite length, so the slot in the cross-head and the sliding block are equivalent to a connecting-rod of infinite length.   As pointed out in Art. 72, the slotted cross-head is frequently called the Scotch yoke.
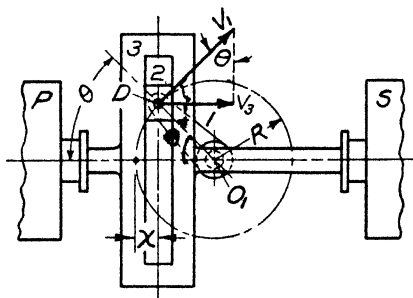
FIG. 393

It is to be noted that for any angular displacement θ of the crank the displacement x of the cross-head is equal to $R(1 - \cos\,\theta)$; in other words, as pointed out in Art. 17, the cross-head has simple harmonic motion for uniform rotation of the crank.   If V_1 is the constant linear velocity of the crank-pin, then, for any angular displacement θ of the crank, the corresponding velocity of the cross-head is, by resolution,

$$V_3 = V_1 \sin\,\theta = \omega R \sin\,\theta$$

and the acceleration of the cross-head is,

$$A_3 = \frac{d}{dt}\,(V_3) = \omega^2 R \cos\,\theta$$

These equations are the same as the equations in Art. 17 applying to simple harmonic motion.

The application of this mechanism to fire engines and other steam pumps is indicated in Fig. 393, where S is the steam cylinder and P the pump cylinder.   The function of the crank and shaft in this case is to carry a flywheel which, by absorbing and redistributing energy, keeps the crank rotating at approximately a constant speed, thus regulating the motion of the cross-head and pistons of the steam and pump cylinders.   The practical advantage of using this equivalent of a rod of infinite length is that it results in a more compact mechanism than the usual slider crank mechanism with a rod of finite length.   The distance between the stuffing-box glands of the steam and pump cylinders only needs to be equal to the stroke plus the outside width of the yoke plus a suitable allowance for clearance at each end of the stroke.

The crank-pin is fitted with the sliding block *2* to avoid line contact

and undue wear. Kinematically it is not essential. Even when the pin is fitted with a block, the slotted cross-head and sliding block can not be as satisfactory as a connecting-rod with its pin connections owing to the fact that sliding pairs can not be lubricated as effectively as turning pairs. The slotted cross-head is only used where the utmost compactness is essential.

188. Eccentric and Rod. An eccentric is merely a crank-pin which is larger in diameter than the shaft to which it is fastened. It is a modified crank. Thus, in Fig. 394, O_{10} is the center of a shaft of radius r, and E the center of an eccentric of radius R. The eccentricity, or equivalent crank length, is $O_{10}E$. The eccentric, eccentric-rod, sliding block, and frame are, kinematically, the same as the slider crank mechanism composed of the links O, 1, 2, and 3, for which the stroke of the sliding block is seen to be twice $O_{10}E$ or equal to the diameter of the dotted circle. With E in its left dead center position, F on the
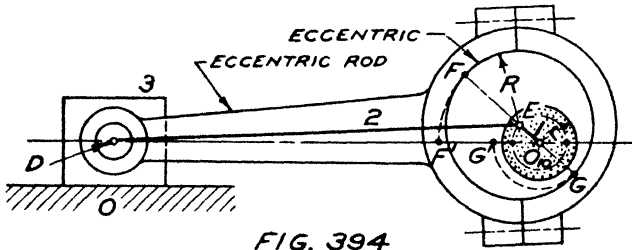
FIG. 394

eccentric is at F'; and with E in its right dead center position, G is at G'. · It is to be noted that

$$F'G' = O_{10}F - O_{10}G = (O_{10}E + R) - (R - O_{10}E) = 2O_{10}E$$

or $F'G'$ is equal to the diameter of the dotted circle. Hence the eccentric, eccentric-rod, sliding block, and frame are equivalent to the slider crank mechanism composed of the links O, 1, 2, and 3. Radius $O_{10}E$ is the eccentricity, or throw, of the eccentric, and DE is the length of the eccentric-rod.

An eccentric is used to obtain a reciprocating motion from a rotating shaft. Since it can be attached anywhere along a shaft, it avoids the necessity of forming any part of the shaft into a crank, which might be awkward and expensive and might weaken the shaft. Eccentrics are very commonly used to drive the valve gears of engines. They are not used to transmit any considerable amount of power or used in place of large throw cranks as the friction loss due to their size would be objectionable if not prohibitive.

189. Oscillating Beam.   An oscillating beam, as shown in **Fig. 395** at *B*, is sometimes used in engines, pumps, and other machinery.   In the figure, *CC* is the center line of the piston motion, *L* is the distance from the center of oscillation to this center line, and *2S* is the stroke. In order to reduce the angularity of the connecting-rod *R*, the beam should oscillate through the same angle on each side of center line *OF*, which is at right angles to *CC*, and the distances *FE* and *ED* should be
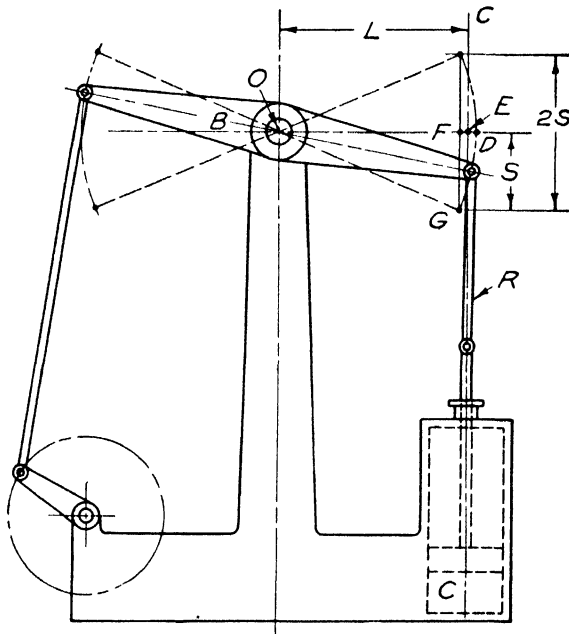
FIG. 395

equal.   The length of *OG* of the beam to fulfill these requirements may be found thus:

$$OG = OD = L + ED \quad . \quad . \quad . \quad . \quad . \quad . \quad (303)$$

or,

$$\overline{OG}^2 = (L + ED)^2$$

But,

$$\overline{OG}^2 = S^2 + \overline{OF}^2 = S^2 + (L - ED)^2$$

Hence,

$$(L + ED)^2 = S^2 + (L - ED)^2$$

From which,

$$ED = \frac{S^2}{4L}$$

Substituting this value of ED in (303) gives the length OG of the beam as,

$$OG = L + ED = L + \frac{S^2}{4L} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (304)$$

190. Bell-Crank. In Fig. 396 is shown a method of proportioning a bent lever, or bell-crank, to receive and to transmit motion along the lines OA and OB in the ratio of m to n. Lay off on OB a distance m equal to the desired motion along OA, and on OA lay off a distance n equal to the desired motion along OB. Through the points a and b thus determined draw lines parallel respectively to OA and OB intersecting at q. Any point as Q on the line drawn through O and q that will give suitable angular displacements may be taken as the center of the bell-crank. From Q drop perpendiculars QP_a and QP_b on OA and OB. To reduce the distortion of the desired motion to a minimum, the angular displacement of the arms of the bell-crank on each side of these perpendiculars should be equal, and
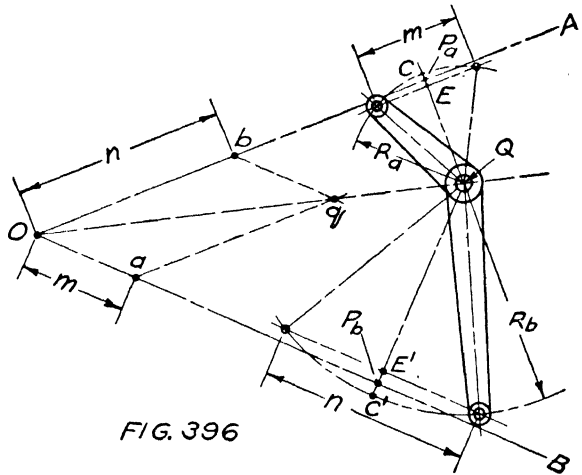
FIG. 396

the respective radii R_a and R_b should be such as to make $CP_a = EP_a$ and $C'P_b = E'P_b$. Lengths m and n being known the radii R_a and R_b may be computed by the method of the preceding article.

191. Quick Return Mechanisms. Various forms of quick return mechanisms are to be found in machine tools and other classes of machines either as auxiliary or as primary mechanisms. The primary mechanism in shapers and slotters is a quick return mechanism. In the operation of these machine tools a cutting tool makes a cutting, or working, stroke and an idle return stroke for each cut that is taken. By making the idle return stroke at a higher speed than the working stroke, time is conserved and the capacity of the machine thereby increased.

One of the simplest quick return mechanisms is the slider crank mechanism with an offset crank. This mechanism was used in some of

the early shapers and is shown in diagrammatic form in Fig. 388. The application of the mechanism and how it is proportioned for a given time ratio of working to return stroke were discussed in Art. 185. When used as a shaper mechanism, the sliding block *3* is made longer and the cutting tool attached to its outer end, the slide carrying the tool head being known as the ram. Also, when used as a shaper mechanism, the crank *1* is slotted and fitted with an adjustable crank-pin. By moving the crank-pin toward or away from the center of rotation of the crank, the length, or throw, of the crank, and hence the stroke of the ram, is decreased or increased. It is to be noted that as the length of the stroke is decreased, the time ratio of the working to the return stroke is decreased.

The drag link, or MacCord, quick return mechanism is shown diagrammatically in Fig. 397. This compound mechanism is formed
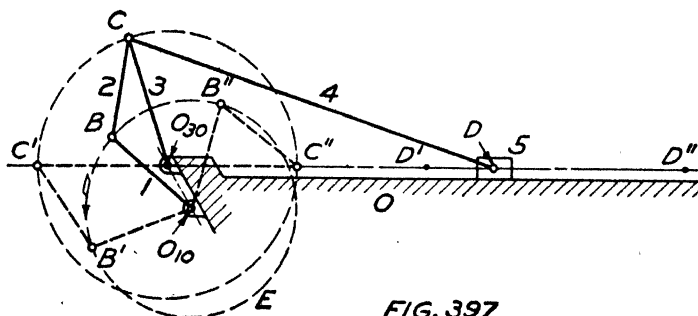
FIG. 397

by adding links 4 and 5 to the drag-link mechanism shown in Fig. 381. In designing the basic four-link mechanism of the linkage shown in Fig. 397, the proportional limits given in Art. 183 should be followed. When used as a shaper mechanism, link *3* is slotted and fitted with an adjustable crank-pin for changing the length of the stroke. As the driver *1* rotates counter-clockwise from $O_{10}B'$ to $O_{10}B''$, it is to be noted that $O_{30}C$ rotates counter-clockwise from $O_{30}C'$ to $O_{30}C''$ and D moves from D' to D'' on the working stroke. The return stroke, D'' to D', is completed as $O_{30}C$ rotates from $O_{30}C''$ to $O_{30}C'$, or as $O_{10}B$ rotates counter-clockwise from $O_{10}B''$ to $O_{10}B'$. Assuming the driver *1* to rotate at a uniform rate, the time ratio of the working to the return stroke is the ratio of arc $B'EB''$ to arc $B''BB'$. It is one of the advantages of this mechanism that this time ratio does not change with a change in the length of the stroke.

If the angular velocity ratio of *3* to *1* is desired for the MacCord

mechanism, it may be found for any phase by applying the angular velocity theorem of Art. 67.   Thus,

$$\frac{\omega_{30}}{\omega_{10}} = \frac{O_{10}O_{13}}{O_{30}O_{13}} \quad \text{or} \quad \omega_{30} = \omega_{10}\frac{O_{10}O_{13}}{O_{30}O_{13}}$$

The linear velocity of 5, the ram, may be found by the method of resolution, the method of instant centers, or by the velocity vector diagram method; and the acceleration of the ram may be found by the acceleration vector diagram method.   If interested in both the velocity and acceleration of the ram, the velocity should be found by the vector diagram method; if interested only in the velocity of the ram, either the vector diagram method or the method of instant centers should be used.   Having the angular velocity of 1, the driver, the linear velocity of any point in 1 becomes known.   Having the linear velocity of a point in 1, the linear velocity of 5, the ram, is, for any phase, equal to the linear velocity of the instant center O_{15}, which is at the instant a point common to both 1 and 5.   Thus,

$$\frac{V_{O_{15}}}{V_{O_{12}}} = \frac{O_{10}O_{15}}{O_{10}O_{12}} \quad \text{or} \quad V_{DO} = V_{O_{15}} = V_{O_{12}}\left(\frac{O_{10}O_{15}}{O_{10}O_{12}}\right)$$

The methods just stated may be applied to any of the quick return mechanisms yet to be described.

A diagrammatic representation of the Whitworth quick return mechanism is shown in Fig. 398. This compound mechansim is formed by adding links 4 and 5 to the inversion of the slider crank chain shown in Fig. 39, where the crank is the fixed member. As the driver 1 rotates clockwise from $O_{10}B'$ to $O_{10}B''$, $O_{30}C$ rotates clockwise from $O_{30}C'$ to $O_{30}C''$ and D moves from D' to D'' on its working stroke. The return stroke, D'' to D', is completed as $O_{30}C$ rotates from $O_{30}C''$ to $O_{30}C'$ or as $O_{10}B$ rotates clockwise from $O_{10}B''$ to $O_{10}B'$. Assuming the driver 1 to rotate at a uniform rate, the time ratio of the working to the return stroke is equal to the ratio of arc $B'BB''$ to arc $B''EB'$. By moving the crank-pin C along 3 toward or away from O_{30}, the length of the crank and the stroke of the ram is decreased or increased.   It is to be noted that the time ratio of the working to the return stroke is not affected by changing the length of the stroke of the ram.

Shaper mechanisms are usually designed for a given time ratio of the working to the return stroke for a given maximum length of stroke. Usually the minimum length of stroke is also fixed.   For the Whitworth mechanism shown in Fig. 398 a suitable ratio of connecting-rod length

CD to the crank length $O_{30}C$ for the maximum length of stroke would be selected, also a suitable length $O_{10}B$ for the driver crank. Points B' and B'' would then be determined by dividing a circle of radius $O_{10}B$ into arcs $B'BB''$ and $B''EB'$ in the given time ratio of the working to the return stroke. The center O_{30} and the center line of travel of the ram is then determined by passing a line through B' and B''.

The practical form of the Whitworth quick return mechanism is indicated in Fig. 399. It has been lettered and numbered the same as the skeleton diagram in Fig. 398 and is shown for the same phase. The double slotted member 3 rotates about the axis O_{30} of its supporting spindle S. This spindle has its supporting bearings in the fixed stud
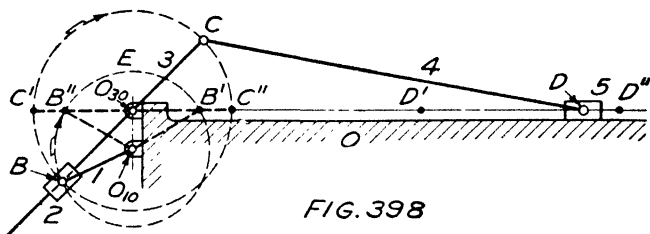
FIG. 398
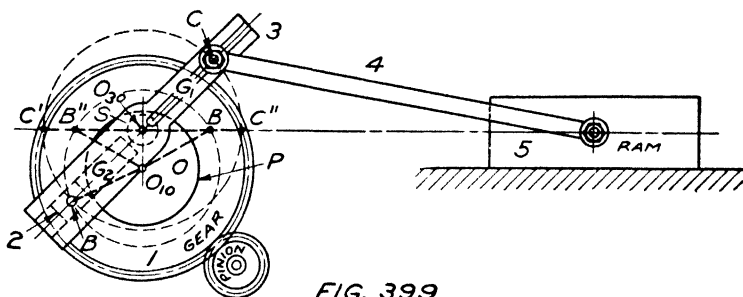
FIG. 399

P. The gear carrying the crank-pin B and sliding block 2 rotates on this stud, thus causing the double slotted member 3 to rotate about its axis O_{30}. In changing the length of the stroke, the adjustable crank-pin C is moved further in or further out in the slot G_1.

In Fig. 400 is shown a so-called " crank shaper " mechanism. As crank 1 rotates clockwise from $O_{10}B'$ to $O_{10}B''$, the vibrator 3 rotates through the angle θ and the ram moves from D' to D'' on its working stroke. The return stroke, D'' to D', is completed as the crank rotates clockwise from $O_{10}B''$ to $O_{10}B'$. Assuming crank 1 to rotate at a uniform rate, the time ratio of the working to the return stroke is seen to be equal to the ratio of the respective angles $(180 + \theta)$ and $(180 - \theta)$. Obviously, since θ decreases with the stroke, the time ratio decreases as

the stroke is shortened. In spite of this defect, this type of mechanism for shapers is favored over others because it places the driving mechanism below the ram. The stroke is shortened by moving crank-pin B toward its center of rotation O_{10}. In the practical forms of this mechanism, this can be done while the shaper is running.

In Fig. 401 is shown the velocity diagram for the crank-shaper
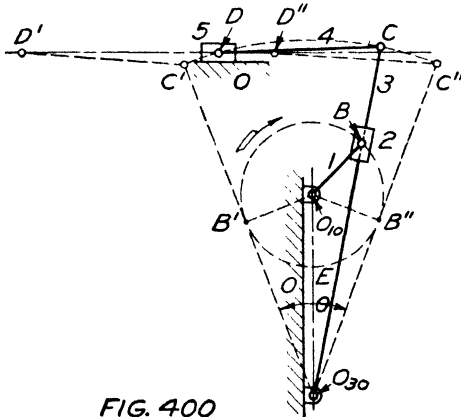
FIG. 400
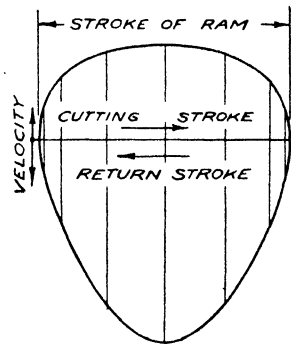
FIG 401

mechanism of Fig. 400 for the maximum stroke of the ram. As before pointed out, the velocity of the ram for any phase may be found by the method of resolution, by the method of instant centers, or by the velocity vector diagram method.

It is to be noted that the type of mechanism shown in Fig. 400 may be varied somewhat by changing the form of the connection at the lower
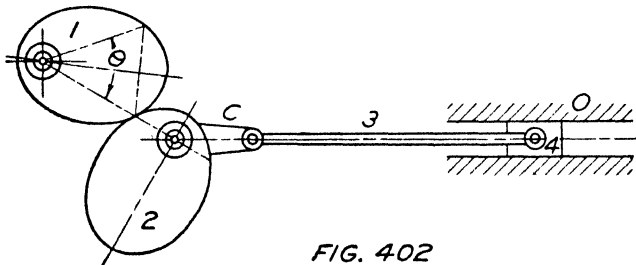
FIG. 402

and at the upper end of the vibrator. One variation is to interpose a link between the lower end of the vibrator and the frame and to connect the upper end of the vibrator directly to the ram by means of a turning pair. Other variations are to connect the vibrator either to the ram or the frame by a turning pair and to use a sliding pair at the other end.

In Fig. 402 the quick return of the ram is obtained by using a pair

of elliptical gears.   Gear *1*, which rotates at a uniform rate, drives gear *2* to which crank *C* is attached.   The length of the stroke is varied by changing the length, or throw, of the crank.   With the gears, crank, and ram related as shown, the return stroke of ram *4* is made while gear *1* rotates clockwise through the angle *θ*, and the cutting, or working, stroke is made while gear *1* rotates through an angle of (360 − *θ*). The time ratio of the working to the return stroke is therefore equal to the ratio of the respective angles (360 − *θ*) and *θ*.   It is to be noted that this ratio is a maximum for the gears, crank, and ram related as shown; that is, this ratio is a maximum when, for the ram at the end of its working stroke, gear *1* makes contact with gear *2* at the point
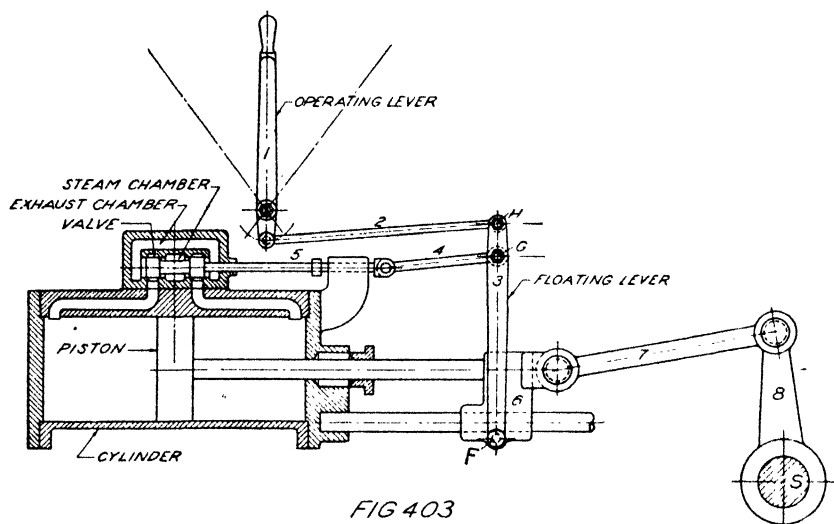
FIG 403

where a line through the axis of rotation of gear *2* at right angles to its major axis intersects its pitch line.

192. Floating Lever.   The floating lever is, as its name indicates, a lever whose fulcrum is not fixed.   The application shown in Fig. 403 is from the field of marine engineering.   The figure shows a small engine, called a reversing engine, whose function is to position the valve gear of the main propelling engine for part or full steam ahead or astern.   The valve gear for each cylinder of the main engine is attached by a connecting-rod to a crank keyed to shaft *S*.   The operating crank *8* is also keyed to this shaft and is connected to cross-head *6* by connecting-rod *7*.   Hence the position of the valve gear of the main engine depends on the position of the piston of the reversing engine, which, in turn, depends on the position of the hand-controlled operating

lever *1*.   In the figure, the operating lever, valve, and piston are shown in their mid positions which correspond to the mid, or neutral, positions of the valve gear of the main engine.   It is to be noted that moving the operating lever to the left moves the valve to the right and admits steam to the right end of the cylinder, thus causing the piston and cross-head to move to the left.   Point *F*, which was the fulcrum of the floating lever, now moves to the left with the cross-head while *H* acts as the fulcrum thus causing *G* and the valve to move to the left.   As the valve moves to the left, the flow of steam to the right end of the cylinder is cut off, the piston of the reversing engine is brought to rest, and the valve gear of the main engine has been moved to a new operating position.   How far the piston of the reversing engine moves to the left
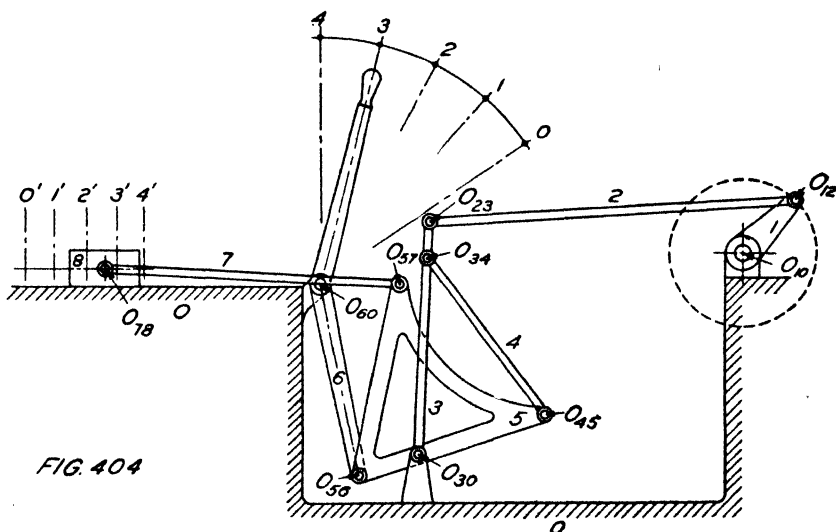
FIG. 404

or right as the operating lever is moved to the left or right, is thus controlled by the floating lever *3*.

The principle of the floating, or differential, lever is also applied to the operating gear of direct-acting steam pumps, steam-hydraulic forging presses, and other machines.

193. Variable Stroke Mechanism.   In Fig. 404 is shown a variable stroke mechanism in which the rotating crank *1* is the driver and sliding block *8* the follower.   With the control lever *6* at *0*, the slide *8* will be at *0'* and its stroke will be zero.   With the control lever at *1* slide *8* will reciprocate from *0'* to *1'*, from *0'* to *2'* for the control lever at *2*, from *0'* to *3'* for the control lever at *3*, etc.   With the control lever to the right of *0*, slide *8* will reciprocate to the left instead of the right of *0'*.   These

changes in the length and direction of the stroke of *8* may be made, if desired, while the crank continues to rotate.

It is to be noted that the length of link *4* is equal to the distance on link *3* between the centers O_{34} and O_{30}. Hence, with the control lever at *0*, center O_{45} will coincide with O_{30} and no motion can be transmitted to *8* by the rotating driver. Link *5* is isosceles.

194. Ackermann Automobile Steering Mechanism. While the old turn-table steering mechanism as used on horse-drawn vehicles is theo-

FIG. 405   FIG. 406

retically correct, it imposes too many limitations and difficulties of design and operation to be applied to automobiles. Automobiles use side pivot steering, the requirements for which, for perfect action, are shown in Fig. 405. For absence of skidding when a turn is made, the front wheels should turn, within their limits of action, through such angles α and β that their axes will intersect at a common point *O* on the axis of the rear wheels. In this event all four wheels will follow circular paths about *O* as a center, and the only tendency to skid will be due to

centrifugal action. The proper relation of the angles α and β in terms of the pivot distance P and the length of the wheel base L, may be derived thus,

$$A = L \cot \alpha \quad \text{and} \quad B = L \cot \beta$$

Hence,

$$P = A - B = L \cot \alpha - L \cot \beta$$

or

$$\cot \alpha - \cot \beta = \frac{P}{L}. \quad \ldots \ldots \quad (305)$$

The Ackermann side pivot steering mechanism, shown in Fig. 406, is the most widely used mechanism for automobile steering. In this mechanism, links *1* and *2* are compound links in the form of bell-cranks. The phase of the mechanism for straight-ahead running is shown in heavy dotted lines, and the phase of the mechanism for turning in the shortest permissible radius is shown in heavy full lines. For the angle θ and proportions of the links chosen, it is to be noted that the axes of the front wheels *1* and *2* intersect the axis of the rear wheels at points O_1 and O_2 and not at a common point as required for perfect action. Hence the wheels will tend to follow paths that do not have a common center, and there will be some skidding. While with the Ackermann mechanism skidding can not be wholly eliminated, angle θ and the proportions of the mechanism should be so chosen as to reduce the inherent tendency to skid to a minimum, especially for turning radii that could be taken at high speeds where centrifugal action would tend to increase the skidding.

195. Straight-Line Mechanisms. A large number of linkages have been devised to make a point move in a straight line independently of any planed guides. Although commonly called *parallel motions*, such linkages may more appropriately be called *straight-line mechanisms*.

Watt's straight-line mechanism is shown in Fig. 407. Links *1* and *3* oscillate about their fixed centers O_{10} and O_{30}. Any angular motion of one link is transmitted through link *2* to the other link. If either *1* or *3* is moved from its central position, A will be drawn to the left and B to the right. If P is properly located on link *2*, it will trace a path very closely approximating a straight line, provided *1* and *3* are not oscillated through too great an angle. The complete path of P, as shown dotted in the figure, is shaped somewhat like the figure 8. The length of the straight portion of this path will be a maximum when centers O_{10} and O_{30} are so located that link *2* will be at right angles to *1* and *3* when they are parallel. For *1*, *2*, and *3* so related, point P, if properly located on *2*, will travel in a path very closely approximating

a straight line for a distance equal to the length of *2*. Whether or not *1*, *2*, and *3* are so related, point *P* should be so located that the segments *AP* and *BP* of link *2* will be inversely as the lengths l_1 and l_3 of links *1* and *3*. In Fig. 407 the fixed centers O_{10} and O_{30} are on opposite sides of *2*. When these centers are on the same side of *2*, point *P* will be on *2* extended.

Instead of using a cross-head and guides as is now common, Watt used the mechanism of Fig. 407 to guide the piston-rod in many of his

FIG. 407   FIG. 408
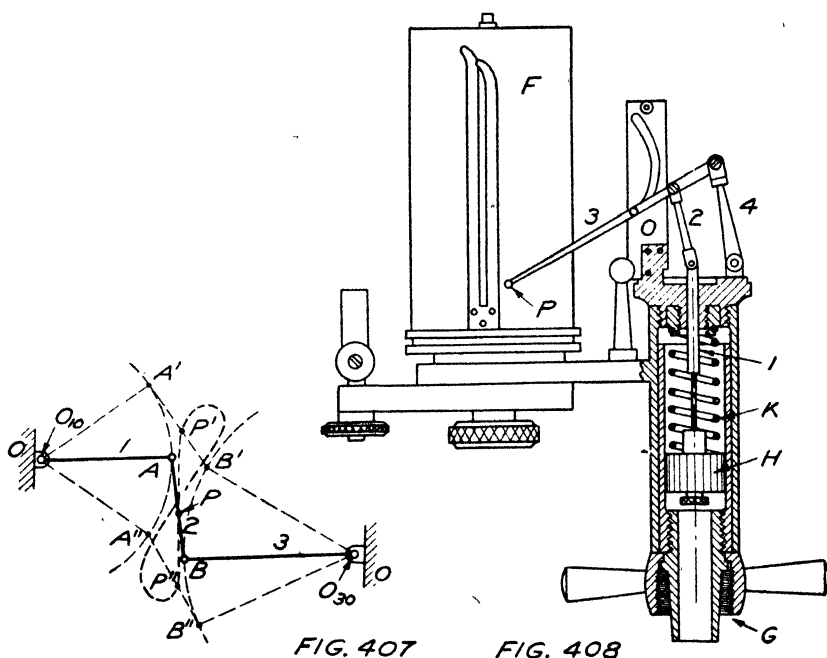
engines. The earliest of the modern type of steam-engine indicator, the Richard's Indicator, used the Watt mechanism.

In Fig. 408 is shown a Tabor engine indicator in part section. The purpose of this instrument is to automatically record how the steam or gas pressure in an engine cylinder varies with the position of the piston of the engine. The indicator is attached at *G* to a short pipe leading to the engine cylinder. The movement of the piston *H*, to which the piston rod *1* is attached, is governed by the helical spring *K*. The variation of pressure in the engine cylinder as the piston moves back and forth, causes the piston *H* and rod *1* to move up and down. This movement in being transmitted to the pencil point *P* is multiplied by links *0*, *2*, *3*, and *4*, the pencil point moving vertically

up and down parallel to the axis of the drum F. By a cord attached to a reducing motion this drum is oscillated through an arc of about 4 inches in unison with the motion of the engine piston. The simultaneous motion of the pencil and drum causes a figure to be drawn on a strip of paper placed on the drum. This figure is called an indicator card. By scaling the card the pressure in the engine cylinder for any position of the piston may be determined. The length of the card is proportional to the stroke of the engine, and its area is proportional to the power developed in that end of the cylinder to which the indicator is attached.

The pencil mechanisms of indicators are interesting examples of the application of straight-line mechanisms. The Tabor indicator mechanism is illustrated in Fig. 408 and Fig. 409. In this mechanism the pencil P is constrained to move in a straight line mm by a pin D moving in a slot of proper shape in the fixed link O. It is evident that the
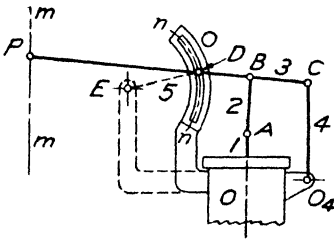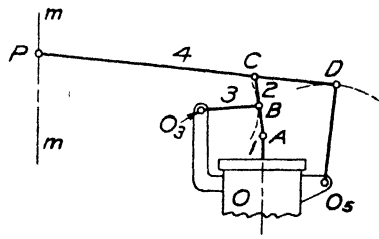
FIG. 409   FIG. 410

center line nn of this slot may be determined by plotting the curve traced by the center of pin D as P is moved in a straight line mm. If a circular arc can be found which will agree closely with nn, a link 5 having a length equal to the radius of the arc could be substituted for the slotted link O. This link is indicated by a dotted line in Fig. 409. Making this substitution results in a mechanism similar to that used on the Thompson indicator.

Fig. 410 represents the pencil mechanism of the Crosby indicator. If P is moved in a straight line mm, point B in link 2 will trace a curve. The length of link 3 is made equal to the radius of a circular arc most nearly approaching the curve traced by B.

Two other approximate straight-line mechanisms are shown in Figs. 411 and 412. Fig. 411 represents Roberts' straight-line mechanism. With 1 and 3 of equal length and with point P of link 2 in the middle of line AB when CD is parallel to AB, point P will, within limits, trace an approximately straight line. For the limiting positions of P to coincide with A and B, the length CD must be equal to $0.50(AB)$ and the

length of links *1* and *3* must not be less than $0.593(AB)$. With *1* and *3* equal to or longer than $0.593(AB)$, *P* will coincide with *B* when *AC* and *CD* are in line and will coincide with *A* when *DB* and *CD* are in line. The longer *1* and *3* the more accurately will *P* trace a straight line between *A* and *B* when $CD = 0.50(AB)$.
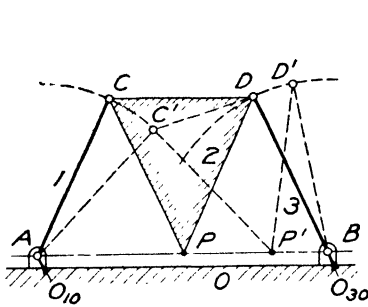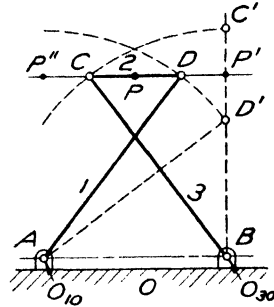
FIG. 411   FIG. 412

Fig. 412 represents Tchebicheff's straight-line mechanism. In this mechanism links *1* and *3* are of equal length, and the tracing point *P* is in the middle of *CD* and is in mid position when *CD* is parallel to *AB*. Within limits *P* will trace an approximately straight line parallel to *AB*. If *CD* is made $0.50(AB)$, and the length of links *1* and *3* is made equal to $1.25(AB)$, then *P* will lie in the extension of the mid position of *CD* when either *1* or *3* is vertical. When *3* is vertical, *P* will be at *P'* and at *P''* when *1* is vertical. For the links so proportioned, *P* will trace an approximately straight line from *P'* to *P''* parallel of *AB*.
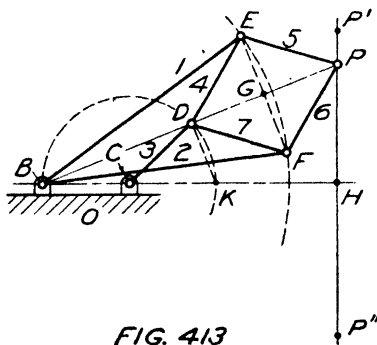
FIG. 413

Paucellier's straight-line mechanism, shown in Fig. 413, is exact. It contains no sliding pairs. Links *1* and *2* have a common fixed center at *B*; the length of link *3* is equal to *BC*; and links *4, 5, 6,* and *7* are of equal length. One limit of its motion is when links *3* and *4* are in line, the other limit is when *3* and *7* are in line. In accordance with these limits, *P'* is the upper and *P''* the lower limiting position of *P*. Between these limits *P* will move in a straight line. This can be shown by proving that the distance *BH* to the foot of a perpendicular dropped from *P* is the same for all phases of the mechanism between the limits mentioned. Take any phase as shown, drop a perpendicular *PH* on the

extension of *BC*, and draw the line *DK*. *PHB* is a right triangle by construction, and *KDB* is a right triangle since its hypotenuse is the diameter of a circle. Hence,

$$\frac{BP}{BH} = \frac{BK}{BD}$$

or,

$$BH = \frac{(BD)(BP)}{BK} \quad . \quad . \quad . \quad . \quad . \quad (306)$$

But since triangles *EBF*, *EDF*, and *EPF* are always isosceles, a line joining *B* and *P* will always contain *D* and bisect *EF* at right angles at *G*. Hence,

$$(BD)(BP) = (BG - DG)(BG + DG)$$

$$= \overline{BG}^2 - \overline{DG}^2 = (\overline{BE}^2 - \overline{EG}^2) - (\overline{DE}^2 - \overline{EG}^2) = \overline{BE}^2 - \overline{DE}^2$$

Substituting this value of $(BD)(BP)$ in (306) gives the distance from *B* to the foot of a perpendicular dropped from any position of *P* as

$$BH = \frac{\overline{BE}^2 - \overline{DE}^2}{BK} \quad . \quad . \quad . \quad . \quad . \quad (307)$$

Since *BE*, *DE*, and *BK* = 2*BC* are constants, the distance *BH* is constant, which means that *P* travels in a straight line.

An exact straight-line mechanism can be made by slightly modifying the mechanism of Fig. 389. If block *4* is omitted, point *G* is definitely constrained by block *3* to move in a straight line for an oscillation of link *1* of less than 90 degrees either side of center line *CF'*. This modification of Fig. 389 is known as the Scott Russell straight-line mechanism.

196. Pantographs. A pantograph is a mechanism in which if **one** point is made to travel in a certain path, some other point in the mechanism will trace a similar enlarged or reduced path. Such a mechanism is shown in Fig. 414 with its fixed center at *A*. Links *1*, *3* and *5* are equal and parallel, also links *2* and *4*. If point *p* is on a line joining *A* and *P*, it can be shown that when *P* is moved to *P'* along any path, *p* will move to *p'* along a similar path. From similar triangles,

$$\frac{AP}{Ap} = \frac{AE}{AC} \quad \text{and} \quad \frac{AP'}{Ap'} = \frac{AE'}{AC'}$$

But *AE'* = *AE* and *AC'* = *AC*. Therefore,

$$\frac{AP}{Ap} = \frac{AP'}{Ap'}$$

Hence, for any movement of P, the distances of p from A is proportional to the distance of P from A. Since triangles AEP and ACp are always similar, it follows that p will always lie on the line joining A and P. The angular motion of p about A will therefore always equal the angular motion of P about A. Since the angular motions of p and P about A are always equal and their radii vectors always proportional, p and P must trace similar paths.

In Fig. 415 is shown a form of pantograph called the "lazy-tongs." It is shown, as sometimes used, as a mechanism to reduce the motion of the cross-head of an engine when using an indicator. The pantograph is attached to the cross-head at P with a fixed center at A.
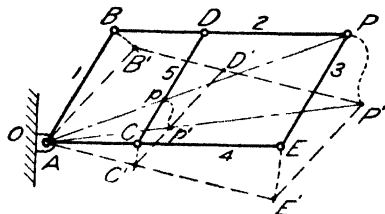
FIG. 414
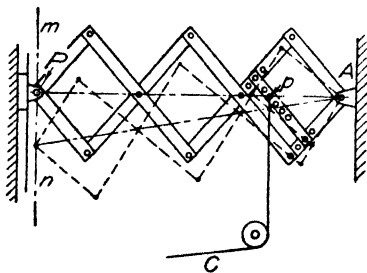
FIG. 415
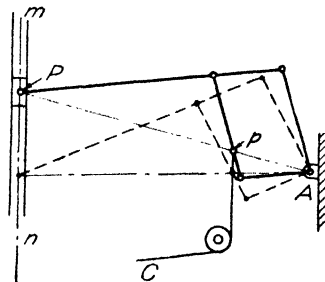
FIG. 416

The indicator cord C is attached at p in the line AP and is run parallel to the path mn of P and then around a pulley to the indicator. The practical objection to the lazy-tongs is the large number of joints and consequent liability to lost motion from wear. The pantograph reducing motion shown in Fig. 416 has fewer joints.

Pantographs are used for reducing or enlarging drawings and are used in engraving and other machines to cause a point or a tool or an oxygen jet to follow a given pattern.

197. Hooke's Coupling. Hooke's coupling, or *universal joint*, is used to connect two shafts which intersect. Fig. 417 is a picture of such a coupling, the requirements of whose construction are clearly shown in Fig. 418. Each shaft is fitted with a forked end, and these

forks are connected by a rigid piece having four pins at right angles whose axes intersect at the point of intersection O of the axes of the shafts. As the shafts rotate, each point in the pivot piece moves in the surface of a sphere. The coupling is equivalent to a four-link chain the axes of whose pivots lie in the radii of a sphere instead of normal to a plane. It is equivalent to what Reuleaux calls a four-link conic chain. Such a conic chain may be produced by eliminating the half aOf of the cross and the halves am and fn of the forks and adding the spherical link bg, which is shown dotted. Kinematically, this mechanism would be the exact equivalent of the original mechanism.

While for a complete revolution of one shaft the other makes a complete revolution, it can be shown that the angular velocity ratio of one to the other is not constant during the revolution. The plane of pro-
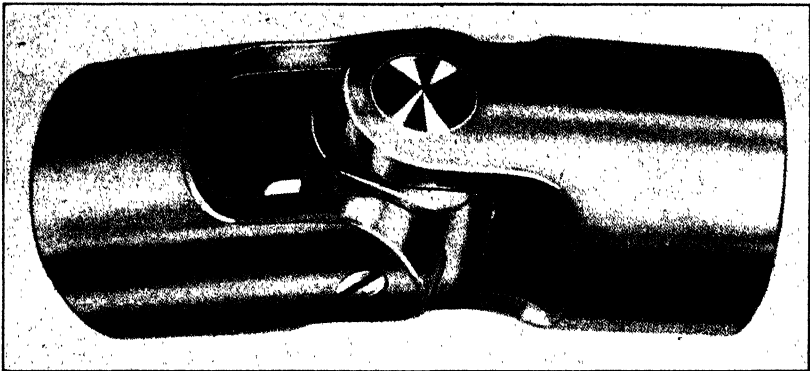
Fig. 417.

jection of the plan shown in Fig. 419(a) is parallel to the plane of the axes, MO and NO, of two shafts which intersect at an angle β. The projection of the plane of rotation of f and g of Fig. 418 is normal to NO and is represented in Fig. 419(a) by the line $f\bar{g}$, the projection of the plane of rotation of a and b being the line PL normal to MO. Fig. 419(b) is the elevation projected on a plane normal to MO. In this elevation the path of a and b of Fig. 418 is represented by the circle $ACBD$. The circular path of f and g of Fig. 418 appears in Fig. 419(b) as the ellipse $AFBG$, in which $OF = OG = Of \cos \beta = OA \cos \beta$. When the pivot a, taken as the driver, is at A in Fig. 419(b), the follower pivot f is at F. For any angular movement θ of the driver arm, A moves to A_1, and F of the follower moves to F_1. Since the driver and follower arms are at right angles, the projected angle A_1OF_1 will be a right angle. This follows since the projected angle of two lines at right angles on a plane

parallel to one of them is always a right angle.   Hence, angle FOF_1 is equal to θ.   But FOF_1 is only the projection of the angle described by the rotation of the follower.   To obtain the magnitude of the actual angle of rotation, plane $AFBG$ must be rotated about AB into the plane $ACBD$.   Before making this rotation draw F_1E perpendicular to GF. On making the rotation, as can be seen by reference to Figs. 419(a) and 419(b), F will coincide with C, F_1 with $F_1{}'$, and E with E', giving angle
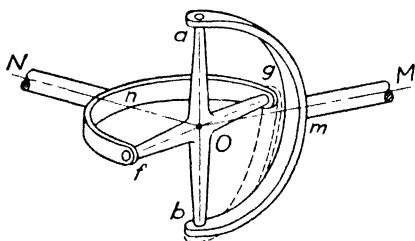
FIG. 419(a)
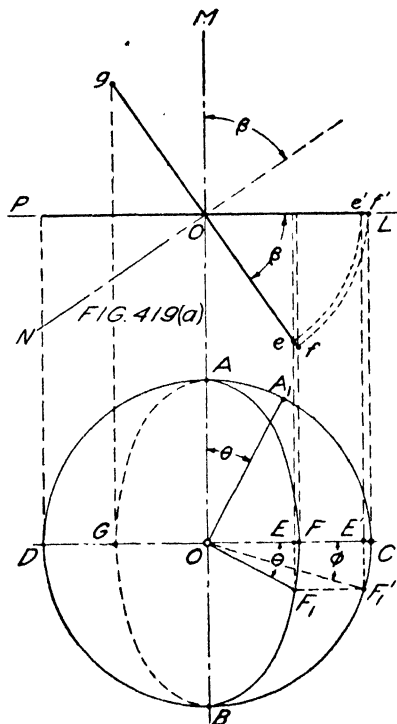
FIG. 419(b)

FIG. 418

$F_1{}'OC$, or ϕ, as the angle rotated through by the arm f as arm a rotates through an angle θ.   Hence,

$$\tan \phi = \frac{E'F'_1}{OE'} \quad \text{and} \quad \tan \theta = \frac{EF_1}{OE}$$

Since $EF_1 = E'F'_1$ and $OE = OE' \cos \beta$,

$$\frac{\tan \phi}{\tan \theta} = \frac{OE}{OE'} = \frac{OE' \cos \beta}{OE'} = \cos \beta \quad \ldots \ldots \quad (308)$$

or

$$\phi = \tan^{-1} (\tan \theta \cos \beta) \quad \ldots \ldots \quad (309)$$

in which $\cos \beta$ is a constant. The angular velocity ratio of the follower to the driver may be obtained by differentiating (309) with respect to θ. Hence,

$$\frac{\omega_n}{\omega_m} = \frac{d\phi}{d\theta} = \frac{d}{d\theta}\left[\tan^{-1}(\tan\theta\cos\beta)\right] = \frac{\cos\beta\sec^2\theta}{1 + \cos^2\beta\tan^2\theta}$$

$$= \frac{\dfrac{\cos\beta}{\cos^2\theta}}{\dfrac{\cos^2\theta + \cos^2\beta\sin^2\theta}{\cos^2\theta}} = \frac{\cos\beta}{1 - \sin^2\theta\sin^2\beta} \quad \cdot \quad \cdot \quad \cdot \quad (310)$$

The angular velocity ratio may be expressed also in terms of ϕ and β by obtaining the value of $\sin^2\theta$ from (308) and substituting the same in (310).

$$\frac{\omega_n}{\omega_m} = \frac{1 - \cos^2\phi\sin^2\beta}{\cos\beta} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (310a)$$

Equations (310) and (310a) show that the angular velocity ratio, $\dfrac{\omega_n}{\omega_m}$, of follower to driver is a minimum when $\sin^2\theta$ is a minimum and $\cos^2\phi$ is a maximum, or when either θ or ϕ equals $0°$, $180°$, etc. These equations also show that $\dfrac{\omega_n}{\omega_m}$ is a maximum when $\sin^2\theta$ is a maximum and $\cos^2\phi$ is a minimum, or when either θ or ϕ equals $90°$, $270°$, etc. Hence, for uniform angular velocity of the driver, the follower has its minimum angular velocity when the driver is at A or B and the follower at F or G. The follower has its maximum angular velocity when the driver is at C or D and the follower at B or A.

The angular acceleration α_n of the follower for uniform angular velocity of the driver and for any given angle β between shafts, may be obtained from equation (310) for any angular displacement θ of the driver. Thus,

$$\alpha_n = \frac{d\omega_n}{dt} = \frac{d\theta}{dt}\left(\frac{d\omega_n}{d\theta}\right) = \omega_m^2\frac{d}{d\theta}\left(\frac{\cos\beta}{1 - \sin^2\theta\sin^2\beta}\right)$$

$$= \omega_m^2\left[\frac{\cos\beta\sin^2\beta\sin 2\theta}{(1 - \sin^2\beta\sin^2\theta)^2}\right] \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (311)$$

By using two couplings and an intermediate shaft K, as shown in Fig. 420, variation of angular velocity between the driver and follower can be eliminated. To accomplish this, the driver and follower shafts M and N must make the same angle β with the intermediate shaft K

and the forks on K must be so related that when one lies in the plane of shafts K and M the other will lie in the plane of K and N. With the universal joints so related, uniform rotation of either M or N will impart, in accordance with the law deduced above, exactly the same
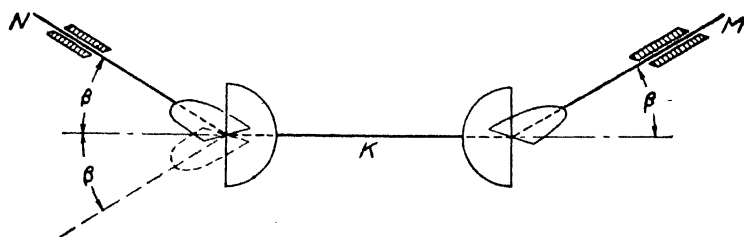
FIG. 420

motion to K. Hence, whatever the motion of M or N, the intermediate shaft K will impart to the other exactly the same motion.

198. Intermittent Gears. The purpose of intermittent gears is to convert continuous into intermittent rotation. The angular motion of
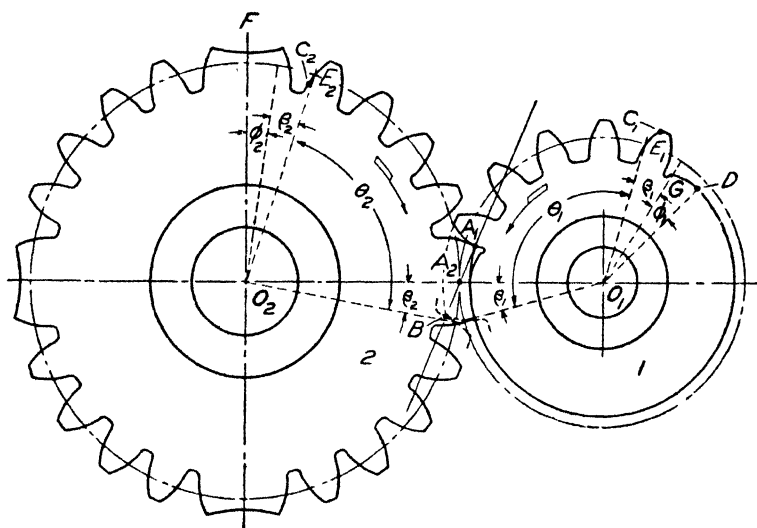
FIG. 421

the driving gear is continuous while that of the driven gear, or follower, is intermittent, the follower being locked in position during its periods of dwell. In Fig. 421, 1 is the driver and 2 the follower. They have $22\frac{1}{2}$-degree involute teeth and their respective pitch diameters correspond

to **16** and **24** teeth. They are designed for an intermittent rotation of the follower of one-fourth of a revolution for every complete revolution of the driver. Although the driver may rotate clockwise or counter-clockwise, it is represented in the figure as rotating counter-clockwise, the follower being driven clockwise. Tooth A_1, acting on the involute profile of stop A_2 of the follower, drives the follower until it leaves contact at B, after which, the driver and follower act as ordinary gears until points C_1 and C_2 are going out of contact at B. During this period, the driver and follower have rotated through the respective angles θ_1 and θ_2 which are inversely as the pitch diameters of the gears, or as 3 to 2. After C_1 and C_2 have reached B, the follower has still to be moved through the angle ϕ_2 before the center line O_2F of the second stop will coincide with the line of centers O_1O_2. This movement is caused by curve G as the driver rotates through the angle ϕ_1, bringing D to the line of centers. If G is so formed as to keep continuously in contact with the end of the stop, the angular velocity ratio from the beginning to the end of driving will be inversely as the pitch diameters. Driving ceases as D reaches the line of centers and will not begin again until the driver has rotated through an angle equal to $(360 - \theta_1 - \phi_1)$, or until tooth A_1 again comes into the position shown. Hence, as the driver rotates through an angle of $(\theta_1 + \phi_1)$, the follower is turned through 90 degrees and remains at rest while the driver rotates through an angle of $(360 - \theta_1 - \phi_1)$.

Intermittent gears of the general type just described are frequently used in counting mechanisms. It is to be noted that the operation of such gears is accompanied by shock at the beginning and at the end of action. At the beginning of action, the continuously rotating driver comes suddenly into contact with the stationary follower; and, at the end of action, the rotating follower is suddenly brought to rest by the cylindrical portion of the driver. Hence, such gears are unsatisfactory at high speeds and also unsatisfactory if the gears are heavy or if they are to be used to overcome any appreciable resistance.

199. Geneva Wheels. Geneva wheels are better adapted than intermittent gears for converting continuous into intermittent rotation. As will be shown, they are superior because the follower is started gradually and gradually brought to rest without shock. The principle of their construction is identical with that of the *Geneva Stop*. In the Geneva stop, which was used to prevent the over-winding of a spring, one of the slots of the follower was blanked in order to bring the driver and follower to rest after a certain fixed number of revolutions of the driver. In Geneva wheels, the follower has a complete complement of slots, thus permitting unlimited rotation of the driver. Geneva

wheels are used in moving picture machines, as indexing devices in machine tools, and have many other applications.

Two pairs of Geneva wheels are shown in Figs. 422 and 423 for exactly the same intermittent rotation of the follower for continuous rotation of the driver. They differ only in the method of locking the follower during its periods of dwell. In Fig. 422, 1 is the driver and 2 the follower. It is to be noted that the driving pin P enters the slot of the follower tangentially, thus starting the follower without shock. Assuming the driver to rotate uniformly, as is usually the case, the follower is accelerated in its motion until the driving pin is in the line of centers O_1O_2, after which, the follower is decelerated and comes to rest

FIG. 422   FIG. 423

with the pin leaving the slot tangentially without shock. In order that the follower may rotate while the pin is in action, the driver is hollowed out at A to clear the follower. The follower is rotated through an angle ϕ while the driver rotates through an angle θ. In the example shown $\phi = 90$ degrees. Hence, the driver and follower make a quarter turn together after which the driver completes its rotation while the follower is locked and held from rotating by the cylindrical portion C of the driver. If a follower is desired for one-sixth of a turn for eac revolution of the driver, ϕ would be equal to 60 degrees; θ would I equal to $(180 - \phi)$, or 120 degrees; and the follower would be lock while the driver turned through $(360 - 120)$ or 240 degrees.

In Fig. 423 the driving pin is fitted with a roller, P, which, as for the example in Fig. 422, works in slots at 90 degrees. The follower 2 is fitted with four locking rollers, A, B, C, and D, the driver 1 being grooved for these rollers. It is to be noted that the driver is just releasing roller D as the driving roller P enters a driving slot. As P is leaving the slot after driving the follower through 90 degrees, roller C has moved to B and is entering the locking slot at end E. This locking slot prevents the follower from moving as the driver rotates through $(360 - \theta)$, or 270 degrees. Hence, as for the example in Fig. 422, the driver makes three-quarters of a revolution during the dwell of the follower.

200. Ratchets. Ratchet and pawl mechanisms are used to convert rectilinear or circular reciprocation into intermittent rectilinear or cir-
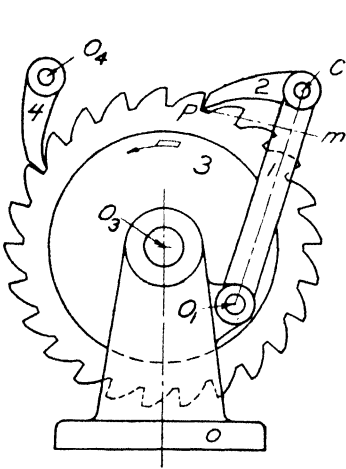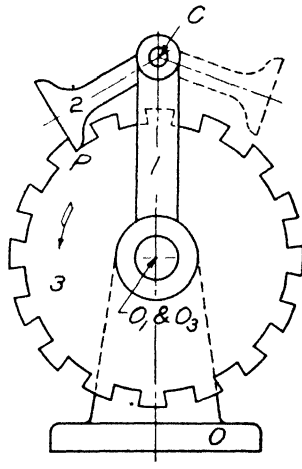
FIG. 424   FIG. 425

cular motion. In Fig. 424, O is the fixed member, 1 the vibrating driver, 2 the driving pawl, and 3 the ratchet wheel. As 1 moves to the left on its working stroke, the ratchet wheel 3 is rotated counter-clockwise by the direct action of pawl 2. Since 1 and 3 have independent fixed centers, the mechanism on the working stroke acts as a four-link mechanism, O, 1, 2, and 3 being the four links. On the return, or idle, stroke of the driver, the ratchet wheel either remains or is held stationary. If center O_1 of the driver coincided with center O_3 of the ratchet wheel, there would be no relative motion between links 1, 2, and 3 as there is for the example shown. Ordinarily the fixed centers of driver and follower coincide.

In order that the resistance to rotation of the ratchet wheel may not

tend to force the driving pawl out of action, the line of action Pm of the force between the pawl and tooth must pass between the centers O_1 and C. Obviously, if this line of action were to pass outside of C, the force on the pawl would throw it out of contact unless the friction of contact were sufficient to keep the pawl seated. Hence, for ratchet mechanisms, the normal to the face of the tooth through its point of contact with the pawl, must pass between the centers O_1 and C, or between the center of rotation of the driver and the center of rotation of the pawl. This principle must be observed both for driving and for locking pawls.

In situations where the ratchet wheel would rotate backward, *or overhaul,* during the return stroke of the driver, a locking pawl as shown at 4 is introduced. This pawl is free to rotate about its fixed center O_4. It prevents backward rotation without interfering with the forward rotation of the ratchet wheel. It is impossible, however, even with the best design and adjustment of the mechanism to entirely eliminate overhaul. Whether the amount is slight or practically equal to the pitch of the ratchet teeth will depend upon where the driver ends its working stroke. The amount of the overhaul may be reduced by reducing the pitch of the teeth or by employing more than one locking pawl. The pitch must not be made so small, however, as to make the teeth too weak. By using two or more pawls of different length, the maximum overhaul may be reduced to a little less than the pitch divided by the number of pawls.

For feed motions, it is sometimes desirable to be able to reverse the direction of rotation of the ratchet wheel. Such a ratchet mechanism is shown in Fig. 425. The pawl, being symmetrical about its center line, will work on either side of the driver. When on the left, the ratchet wheel will be rotated counter-clockwise and will be rotated clockwise when the pawl is on the right side of the driver. It is to be noted that the centers of rotation of driver and follower coincide.

A double acting ratchet is shown in Fig. 426. The ratchet wheel is driven on the clockwise as well as the counter-clockwise stroke of the driver. In this case no locking pawl is needed to prevent overhaul.

In Fig. 427 is shown a ratchet wheel 3 without teeth driven by a friction pawl 2. The driving is noiseless and by proper design can be made almost positive. The magnitude of the angular motion of the ratchet wheel is not, as for toothed ratchet wheels, restricted to a multiple of a given pitch. When necessary another friction pawl 4 may be introduced to prevent overhaul.

In the type of frictional ratchet shown in Fig. 428, the pawls are either balls or rollers. In this form of ratchet either the ratchet wheel

or the driver is fitted with teeth. Whether the one or the other is sup-
plied with teeth, it is to be noted that the magnitude of the angular
motion of the ratchet wheel is in no way governed by the pitch of
the teeth. Like the ratchet shown in Fig. 427, the drive is by friction
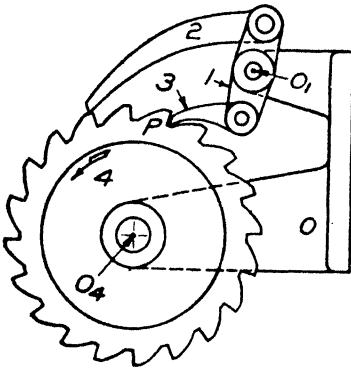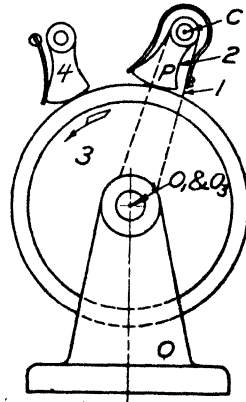
FIG. 426   FIG. 427

and is almost noiseless and can, by proper design, be made almost
positive.

The ratchets thus far described and their numerous modifications
are suitable for many purposes requiring the conversion of oscillation
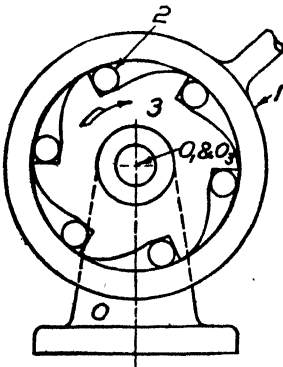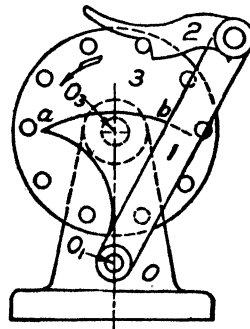
FIG. 428   FIG. 429

into intermittent rotation. They are suitable for feed and other mech-
anisms wherever the required oscillation of the driver is not too rapid.
Where the oscillation is rapid, there is too much shock when the driving
pawl makes contact with the stationary ratchet wheel or an unduly

heavy spring is required to keep the pawl from flying out of position on the return stroke of the driver, thus increasing the wear and noise. Also, if the oscillation of the driver is too rapid, the rotation of the ratchet wheel may, due to its inertia, exceed that corresponding to the stroke of the driver. This latter defect makes the ordinary ratchet unsuitable for revolution and other counting devices. A ratchet suited to such a purpose is shown in Fig. 429.

In Fig. 429, the driver *1* carries a tooth, or beak, so formed and placed as to make overtravel of the ratchet wheel impossible. The outline *ab* of this beak should be a circular arc with O_1 as a center so that the pin it stops will rest against it without being moved on the return stroke of the driver.

In Fig. 430 is shown another device much used in counters. On the upward stroke, the driver tooth *b*, acting on tooth *a* of the ratchet wheel, causes a rotation equal to one-half the pitch; and on the return stroke, the driver tooth *b'*, acting on the tooth *a'*, completes the rotation of the ratchet, or star, wheel through an arc equal to the pitch measured at the bottom of the tooth spaces. Hence, one rotation of the 10-tooth star wheel *2* requires 10 double strokes of the driver *1*.

201. Escapements. Mechanisms that are used to cause a wheel to rotate intermittently that would otherwise rotate continuously, are called escapements. With slight modifications the mechanism shown in Fig. 430 could be made into an escapement mechanism suitable for controlling the train work of a clock. If, under the action of a spring or weight, wheel *2* should tend to rotate continuously in a counterclockwise direction, it would tend to cause *1* to reciprocate. If *1* were attached to a pendulum, the pendulum in swinging would cause the wheel to rotate intermittently instead of continuously as it otherwise would. In order that the wheel may be released as the pendulum swings back and forth, the frictional resistance tending to reduce the amplitude of the swing of the pendulum must be overcome. This is done by shaping the teeth of *1* and *2* so that each time *2* is released it will give to the pendulum a slight impulse which will maintain the amplitude of its swing.

The Graham escapement is shown in Fig. 431. Under the action of a spring or weight, the escape wheel *1* tends to rotate continuously in a clockwise direction. The forked lever *F* and anchor *2* are attached to the verge, and the oscillation of the pendulum is transmitted to the anchor by means of the forked lever *F*. In the phase shown, a tooth of the escape wheel is acting on the right-hand pallet of anchor *2* causing the anchor to swing to the right and to transmit to the pendulum a

slight impulse through the forked lever F. As before explained, this impulse is necessary to maintain the amplitude of the swing of the pendulum by overcoming the frictional resistance which otherwise would bring the pendulum to rest. As the pendulum and anchor continue to swing to the right, the right pallet clears the wheel and the left pallet enters a tooth space of the escape wheel and brings it to a stop. As the pendulum swings back, the escape wheel, just as it is released, gives a slight impulse to the left pallet before it swings free of the wheel. Just before the left pallet clears the wheel, the right pallet has started
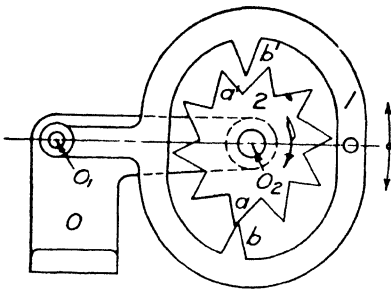
FIG 430
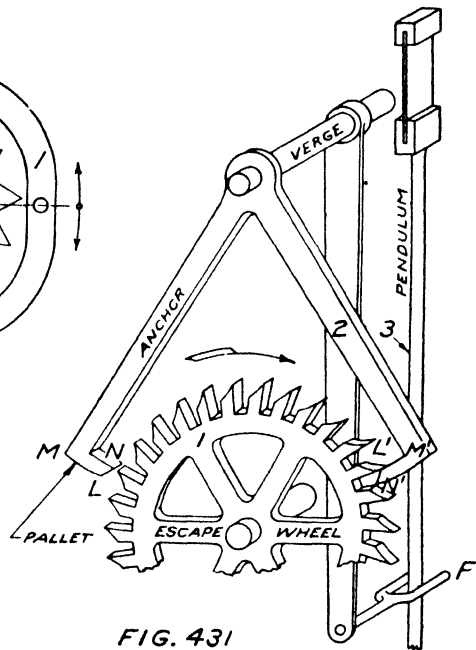
FIG. 431

to enter a tooth space preparatory to bringing the escape wheel again to rest. Hence, for every swing of the pendulum, the escape wheel is permitted a rotation corresponding to one tooth. In portable time-pieces, a balance wheel instead of a pendulum is used to regulate the period of the vibrating member.

The escapements just described permit equal fractions of a revolution of the escape wheel in equal periods of time. They are in consequence called *uniform escapements*. Those permitting unequal portions of a revolution in equal periods of time are called *periodical escapements*. The mechanism in clocks controlling the striking of the hours is an example of this form of escapement. Those permitting

both the intermittent rotation and the time intervals to vary are called *adjustable escapements*.

For a greater variety of ratchets and escapements than can be shown here, the reader is referred to the Constructor by F. Reuleaux.[1] Sufficient has been given to establish the general principles governing the action of such mechanisms.

[1] English translation made in 1893 by H. H. Suplee.

CHAPTER XII

BELT, ROPE, AND CHAIN TRANSMISSION

202. Belts. Belts are flexible connectors used to transmit motion from a member on one shaft to a member on another shaft. Although in recent years rubber belts of trapezoidal section to run in V grooves have come into quite extended use, belts are, as a rule, thin flat bands of leather, rubber, cotton, camel's hair, or steel.

The best leather belting is made from the center back portions of the hides of steers. Single-ply belting may run from $\frac{1}{8}$ in. to $\frac{7}{32}$ in. thick, the average being about $\frac{5}{32}$ in. thick. Double-ply, triple-ply, and thicker belts are made by cementing together two, three, or more thicknesses of leather. Belts thicker than double-ply are, as a rule, not carried in stock. Double-ply belting is designated as light, regular, and heavy double, and may run from $\frac{3}{16}$ in. to $\frac{3}{8}$ in. thick, the average thickness for ordinary widths being about $\frac{1}{4}$ in. Commercial sizes of belts range from $\frac{1}{2}$ in. to 72 in. wide.

Rubber belting is made by impregnating canvas with rubber, folding the same into a belt, and then vulcanizing it. The number of thicknesses of canvas in the belt designates the ply of the belt which may be 3, 4, 5, 6, 7, 8, or 10 ply. Rubber belting is especially favored in damp places. It is somewhat cheaper than leather belting. Besides being used to transmit power, it is much used in belt conveyors for transporting materials.

Textile belts of cotton may be woven to the desired width and thickness, or they may be built up to the desired thickness by stitching together layers of cotton duck. Such belts are impregnated with certain compounds to make them practically impervious to moisture and to the action of mineral oils and to increase their ability to transmit power. Cotton belting that has been impregnated with balata gum is known to the trade as *balata belting*. In weaving camel's hair belting the lengthwise strands are of camel's hair yarn.

The use of thin bands of steel, while one of the most recent developments in belting, has made little progress.

In making leather belting the ends of the strips of leather are scarfed and cemented together. The joints of endless leather belts are made

in a similar manner.   Where leather belts are not endless, the ends are joined by means of rawhide or soft wire lacing, or some patented form of metal joint is used.   Endless rubber, cotton, and balata belts are furnished on order by the manufacturers.   Where these varieties are not endless, they are usually joined by some patented form of metal fastener.

203. Belt Transmission and Angular Velocity Ratio.   Flexible connectors in the form of thin flat bands are occasionally used, as shown in Fig. 432, to transmit oscillating motion from one shaft member to another.   One end of the band *2* is wrapped part way around and fastened to *1*, and the other end is wrapped part way around and fastened to *3*. Since the band is flexible, it is only by a pull in the band that one member can drive the other. Hence, *1*, acting as an oscillating driver, can positively drive the follower *3* on the clockwise, or working, stroke, but will not drive the follower on the counter-clockwise, or return, stroke.   To have the mechanism function, *3*, the follower, must be rotated counter-clockwise by a spring or counterweight on the return stroke of the driver.
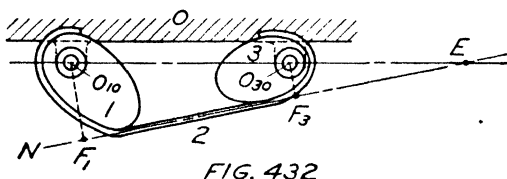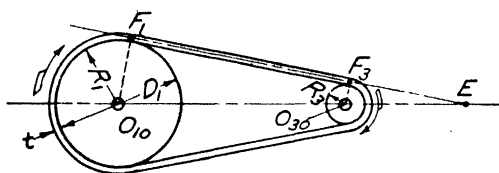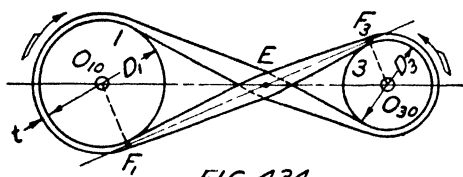
FIG. 432

FIG. 433

FIG. 434

In accordance with Art. 43, the angular velocities of driver and follower are inversely as the perpendiculars let fall from their fixed centers on the line of transmission, or inversely as the segments into which the line of centers is cut by the line of transmission.   The line of transmission NE in Fig. 432 lies in the mid, or neutral, plane of the band.   Hence,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_{30}F_3}{O_{10}F_1} = \frac{O_{30}E}{O_{10}E}$$

If a constant angular velocity ratio is desired, it therefore follows that E must be a fixed point in the line of centers; that is, members *1* and

3 must be cylinders of circular section or segments of such cylinders. If segments are used, only an oscillating motion may be transmitted, and the ends of the band must be fastened to the segments. By using an endless band tightly fitted over two cylinders, as shown in Figs. 433 and 434, either oscillation or continuous rotation may be transmitted from one cylinder to the other. This is possible for two cylinders a given distance apart since the necessary length of the belt does not change with the angular displacement of the cylinders. If the band is fastened to the cylinders, oscillation of either cylinder will cause positive oscillation of the other. If the band is not fastened to the cylinders, then, due to the friction between the band and the surfaces of the cylinders, continuous rotation may be transmitted from one cylinder to the other. The use of belts is practically confined to the continuous driving of one pulley by another in this manner.

Obviously, for the open belt illustrated in Fig. 433, the follower rotates in the same direction as the driver. If the follower is to be rotated in a direction opposite to that of the driver, the belt must be crossed as shown in Fig. 434. These directional relations of the rotation of driver and follower are indicated by the fact that the intersection E of the line of transmission with the line of centers is between the fixed centers for the crossed belt and for the open belt is on the line of centers extended. In accordance with the deductions stated above from Art. 43, the relation of the angular velocities of driver and follower for open and closed belts is,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{O_{30}F_3}{O_{10}F_1} = \frac{O_{30}E}{O_{10}E}$$

It is to be noted that

$$O_{30}F_3 = R_3 + \frac{t}{2} = \frac{1}{2}(D_3 + t) \quad \text{and} \quad O_{10}F_1 = \tfrac{1}{2}(D_1 + t)$$

Hence,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{(D_3 + t)}{(D_1 + t)} \quad \text{. . . . . . .} \quad (312)$$

Since the thickness of a belt is generally small as compared to the diameters of the pulleys, the effect of the thickness on the angular velocity ratio is often neglected. Hence,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{D_3}{D_1} \quad \text{. . . . .} \quad (313)$$

that is, the angular velocities of belt driven pulleys are inversely as their diameters.

As stated above, the friction between the belt and the faces of the pulleys makes it possible for the driving pulley to overcome a resisting moment at the follower pulley and to cause it to rotate. It is in this way that power is transmitted from one shaft to another by the use of belts and pulleys. Since the transmission is by friction there is more or less slip, usually not over two per cent, between the belt and the pulleys. Due to this slip and to the elasticity of the belt material, the transmission, while very efficient, is not positive. Since, however, there are many situations where positive driving is not essential, transmission of motion and power by means of belts is widely used. Generally speaking, belts are used where the distance between the shafts is too great for gears but not so great as to make the use of belting uneconomical. For the usual belt widths and pulley diameters, the maximum distance between shafts will be found to run from 15 to 30 feet. The minimum distance between shaft centers is usually given as either 3.5 times the diameter of the largest pulley or as 10 times the diameter of the smallest pulley. Due to the slip and elasticity of belting, belt drives adjust themselves without shock to sudden changes in load. Because of this, belt instead of chain drives are often used for center distances that are suitable to the use of chains. This is particularly true of the use of multiple V-belts with grooved pulleys.

As shown in Fig. 434, crossed belts rub together where they cross between the pulleys. For this reason they should not be used at high speeds as the wear would be excessive. Since at low speeds the wear would not be serious, advantage may be taken of the fact that crossed belts give a greater arc of contact than open belts.

204. Belt Tension and Power. If a belt is fitted to a pair of pulleys with a certain initial tension, then, due to friction and to the elasticity of the belt material, the tension will increase on the driving or tight side of the belt and will decrease on the slack side as a driving torque is applied to overcome the resisting torque at the follower pulley. If, in Fig. 435, T_1 represents the pull on the tight side and T_2 that on the slack side, then the summation of moments about the center O of the pulley will give the turning moment on the pulley; or, neglecting the thickness of the belt,

$$\text{Turning moment} = T_1 R - T_2 R = (T_1 - T_2)R \quad . \quad (314)$$

$(T_1 - T_2)$ is called the effective pull of the belt. The peripheral velocity in feet per minute of a pulley having a radius of R feet and rotating at N revolutions per minute is

$$V = 2\pi RN$$

Assuming the velocity of the belt to be the same as that of the pulley, the horse-power that may be transmitted by the belt is

$$\text{H.P.} = \frac{(T_1 - T_2)V}{33,000}. \quad \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \quad (315)$$

where $(T_1 - T_2)$ is the effective belt pull in pounds.

To use a belt most effectively, $(T_1 - T_2)$ or the ratio $\dfrac{(T_1 - T_2)}{T_1}$ should be as large as possible. For leather belts on cast-iron pulleys, this ratio may be as high as 0.80 and as high as 0.90 for paper pulleys. Besides the possible effective pull depending on the kind of belt and the kind of pulley, it depends also on the diameter of the pulley, the arc of contact, the slip, the velocity of the belt, and other factors. The
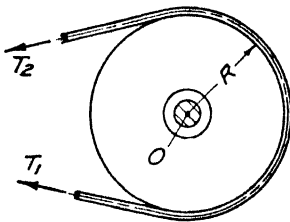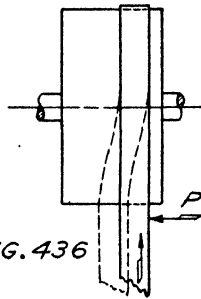
FIG. 435   FIG. 436

effective pull $(T_1 - T_2)$ increases with the slip, with the diameter of the pulley, and with the velocity of the belt up to velocities of from 6000 to 7000 feet per minute. For average conditions of power and speed these factors are often not considered in calculating the width of a belt. In rough calculations the width of the belt is based on an allowable effective pull per inch of width. For leather belting the following average values are sometimes used:

> 50 pounds per inch of width for single-ply belts;
> 80 pounds per inch of width for double-ply belts;
> 100 pounds per inch of width for triple-ply belts.

Thus, according to equation (315), the total effective belt pull to transmit 15 horse-power at 2500 feet per minute would be,

$$(T_1 - T_2) = \frac{(\text{H.P.})\,33,000}{V} = \frac{15 \times 33,000}{2500} = 198 \text{ pounds}$$

Assuming a single-ply leather-belt and an allowable effective pull of 50 pounds per inch of width would call for a belt width B of

$$B = \frac{T_1 - T_2}{t_1 - t_2} = \frac{198}{50} = 3.96 \text{ in. or 4 in.}$$

205. Shifting of Belts. If, as shown in Fig. 436, a force P is applied on an edge of the advancing side of a belt, the belt will be deflected in the direction of the applied force as shown by the dotted lines. This causes each successive portion of the belt as it passes on the pulley to take up a new position, thus shifting the belt in the direction in which the force acts. A much smaller force is required to shift a belt when applied to the advancing side than when applied to the receding side. When applied to the receding side, the force must be sufficiently large
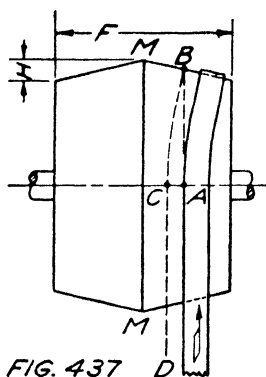
FIG. 438
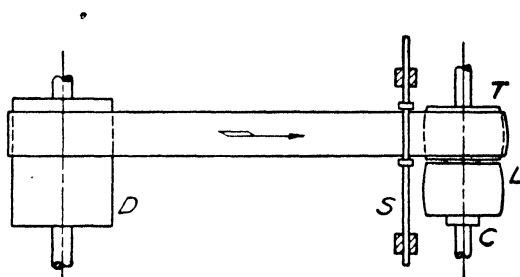
FIG. 437

to actually overcome the frictional resistance between the belt and the pulley. It must be remembered, however, that the receding side of the belt relative to one pulley is the advancing side relative to the other pulley.

In belt shifting mechanisms a fork spanning the belt and suitably mounted on a slide is shifted to the right or to the left as required.

206. Crowning of Pulleys. If a belt is placed under some tension on a conical pulley, it will tend to lie flat on the conical surface of the pulley. The belt will, due to its lateral stiffness, assume some such position as shown by the full lines in Fig. 437. It is to be noted that the edges of the belt are curved with respect to the plane of rotation of the pulley. Hence, as rotation of the pulley moves the belt in the direction indicated by the arrow, point A on the edge of the belt will move to B, and the belt will be advanced to the dotted position BCD. As rotation continues, the belt will continue to climb toward the base

of the cone. If, as shown, the pulley is made up of two equal frusta whose bases coincide, the belt will continue to shift its position until its center line lies in the base plane MM of the pulley. Having arrived, the belt will continue to run in this position since any tendency to displace it is countered by an equal tendency to return it.

For convenience of manufacture, pulleys are usually crowned by giving the rim a straight taper both ways from the mid plane of the pulley face as shown in Fig. 437. To curve the face is better but more expensive. Excessive crowning is detrimental to the life of the belt. Where the pulley shafts are truly parallel only a slight amount of crowning is necessary. It is quite usual practice to make the height of crown H correspond to $\frac{1}{8}$ in. per foot of width of pulley face F. It is recognized, however, that the crown for leather should be greater than for fabric belts, and greater for low than for high speeds. In view of this the crown for leather belts is sometimes given as $\frac{1}{4}$ in. per foot for low speeds and $\frac{1}{8}$ in. or less per foot for high speeds.

207. Tight and Loose Pulleys. It is frequently found desirable in machines and in the transmission of power to stop the driven shaft without stopping the driving shaft. One of the advantages of belt transmission is that this may be done by mounting a tight and a loose pulley on the driven shaft as shown in Fig. 438. The loose pulley L is free to rotate on the driven shaft, while the tight pulley T is keyed to the shaft. Generally the hub of the tight pulley is utilized to keep the loose pulley from sliding in one direction while a collar C, fastened to the driven shaft, prevents it from sliding in the opposite direction. The tight and loose pulleys are each made somewhat greater in width than the belt which is to run on them. The width of the driving pulley D, which is keyed to the driving shaft, is made equal to the sum of the widths of the tight and loose pulleys plus the small clearance between their rims. When it is desired to stop the driven shaft, the belt is shifted from the tight to the loose pulley, and from the loose to the tight pulley when it is desired to start the driven shaft. This is done by applying a pressure to one edge or the other of the advancing side of the belt as explained in Art. 205. This is conveniently done by means of the belt shifter S, which, as shown, is merely a sliding bar to which is attached a fork which spans the belt.

That the belt may not have a tendency to run off the pulley to which it is shifted, both the tight and loose pulleys are crowned. To facilitate the shifting of the belt, the wide driving pulley D is not crowned.

208. Length of Belt. The length of belt required for a drive may be found by measuring a scale drawing of the drive or by direct measure-

ment after the drive is installed. The length may also be computed from the pulley diameters and distance between centers. It is only when the length of belt is essential to the design of a drive, as for cone pulleys, that a careful calculation of the length is made.

The length of the belt for a given pair of pulleys and distance
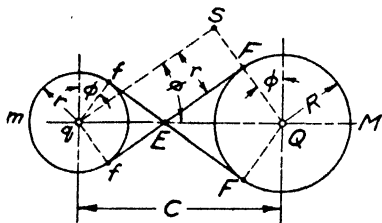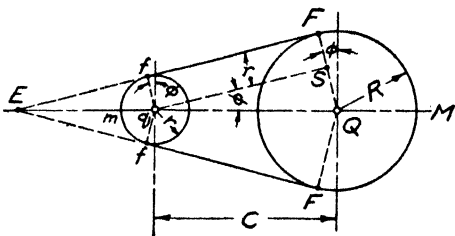
FIG. 439   FIG. 440

between centers depends upon whether the belt is crossed or open. If the belt is crossed, as shown in Fig. 439, its length L is evidently,

$$L = 2(Ff) + \text{arc } (FMF) + \text{arc } (fmf) \quad . \quad . \quad . \quad (316)$$

where,

$$2(Ff) = 2(Sq) = 2\sqrt{C^2 - (R + r)^2}$$

$$\text{arc } (FMF) + \text{arc } (fmf) = R(\pi + 2\phi) + r(\pi + 2\phi) = (R + r)(\pi + 2\phi)$$

But,

$$\sin \phi = \frac{R + r}{C} \quad \text{or} \quad \phi = \sin^{-1}\left(\frac{R + r}{C}\right)$$

Hence,

$$\text{arc } (FMF) + \text{arc } (fmf) = (R + r)\pi + 2(R + r) \sin^{-1}\left(\frac{R + r}{C}\right)$$

Substituting in (316) the values found for $2(Ff)$ and the sum of the arcs gives the length of a *crossed* belt as

$$L = 2\sqrt{C^2 - (R + r)^2} + (R + r)\pi + 2(R + r) \sin^{-1}\left(\frac{R + r}{C}\right) \quad (317)$$

In a similar manner, using Fig. 440, the length of an *open* belt is found to be

$$L = 2\sqrt{C^2 - (R - r)^2} + (R + r)\pi + 2(R - r) \sin^{-1}\left(\frac{R - r}{C}\right) \quad (318)$$

To be strictly accurate in using equations (317) and (318) the radii R and r should be to the neutral plane of the belt. Assuming the neutral plane to correspond to the mid plane of the belt, the values of R and r would be equal to the respective radii of the pulleys plus one-half

the thickness of the belt. Whether it is necessary or not to consider the effect of the thickness of the belt will depend on the distance between centers. The greater the distance between centers the smaller will be the effect of the belt thickness.

209. Cone Pulleys. It is often found desirable to be able to change the speed of a shaft which is to be driven by another shaft rotating at a constant speed. For this purpose a pair of stepped pulleys, commonly called *cone pulleys*, is often employed. Such a pair of pulleys is shown in Fig. 441. The driving cone rotates at a constant speed, and the speed of rotation of the driven cone is changed by shifting the belt from one step to another. Such pulleys are ordinarily designed so that the possible speeds of the driven pulley will be in geometric progression. The progression ratio, the distance between centers, and the minimum and maximum angular velocities of the driven

FIG. 441

pulley are fixed by the conditions of service. Having these and the belt velocity, power to be transmitted, and the space available, it is possible to fix on reasonable values for the respective radii of the first step of the driving and driven pulleys. In Fig. 441, let,
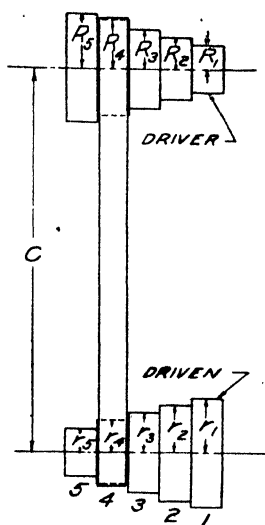
ω = the constant angular velocity of the driving pulley;

R_1 and r_1 = the respective radii of the first step of the driving and driven pulleys for the minimum angular velocity ω_1 of the driven pulley;

R_n and r_n = the respective radii of the nth step of the driving and driven pulleys;

ω_n = the angular velocity of the driven pulley for the belt on the nth step;

k = the progression ratio.

In accordance with Fig. 441, and the above notation, the following relations may be written:

$$\frac{\omega_1}{\omega} = \frac{R_1}{r_1}. \qquad \qquad \qquad \qquad (319)$$

$$\frac{\omega_2}{\omega} = \frac{R_2}{r_2} \text{ or, since } \omega_2 = k\omega_1, \quad \frac{\omega_2}{\omega} = k\frac{\omega_1}{\omega} = k\left(\frac{R_1}{r_1}\right)$$

$$\frac{\omega_3}{\omega} = \frac{R_3}{r_3} = k^2\left(\frac{R_1}{r_1}\right)$$

$$\frac{\omega_n}{\omega} = \frac{R_n}{r_n} = k^{n-1}\left(\frac{R_1}{r_1}\right). \quad \ldots \ldots \ldots \ldots \quad (320)$$

Dividing (320) by (319) gives,

$$\frac{\omega_n}{\omega_1} = k^{n-1}. \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (321)$$

This relation enables the necessary number of steps n to be computed for any progression ratio k and any maximum and minimum angular velocities ω_n and ω_1 of the driven pulley; or, the progression ratio k may be computed for any given number of steps and angular velocities ω_n and ω_1.

If, as is usual, the driving and driven cones are to be equal, then $\left(\dfrac{R_n}{r_n}\right)$ must be equal to $\left(\dfrac{r_1}{R_1}\right)$. Making this substitution in (320) and solving gives,

$$\frac{r_1}{R_1} = \sqrt{k^{n-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (322)$$

As an illustration, let it be assumed that the maximum and minimum angular velocities, ω_n and ω_1, of the driven pulley are to be 110 and 50 respectively and that the progression ratio k, as for machine-tool cone-pulley drives, is not to exceed 1.25. Then, according to (321), the necessary number of steps is,

$$n = \frac{\log\left(\dfrac{\omega_n}{\omega_1}\right)}{\log k} + 1 = \frac{\log\left(\dfrac{110}{50}\right)}{\log 1.25} + 1 = 4.533, \text{ say } 5$$

Since the computed value of n had to be changed to a whole number, it becomes necessary to recompute k. For 5 steps equation (321) gives the recomputed value of k as $\sqrt[4]{2.2} = 1.218$. Assuming, as is usual, that the pulleys are to be of the same size, then, by (322),

$$\left(\frac{r_1}{R_1}\right) = \sqrt{k^{n-1}} = \sqrt{1.218^4} = 1.483, \text{ say as } 4\tfrac{1}{8} \text{ in. is to } 2\tfrac{25}{32} \text{ in.}$$

The radii of the remaining steps of the cone pulleys must be such as to be in the proper ratio and to require the same length of belt. For

a crossed belt, the relation of the radii is simple. For any given distance C between centers, it is obvious from inspection of (317) that the length of the belt for any step x will be the same as required by the first step if

$$(R_x + r_x) = (R_1 + r_1) \quad . \quad . \quad . \quad . \quad . \quad (323)$$

That is, the radii of the steps of a pair of cones for a crossed belt are independent of the distance C between centers. From (320)

$$\frac{\omega_x}{\omega} = \frac{R_x}{r_x} = \left(\frac{R_1}{r_1}\right)k^{x-1} \quad \text{or} \quad R_x = r_x\left(\frac{R_1}{r_1}\right)k^{x-1}$$

Substituting this value of R_x in (323) and solving, gives

$$r_x = \frac{R_1 + r_1}{1 + \left(\dfrac{R_1}{r_1}\right)k^{x-1}}. \quad . \quad . \quad . \quad . \quad (324)$$

Having the radius r_x of step x of the driven cone, the radius R_x of the corresponding step of the driving cone becomes, from (323),

$$R_x = (R_1 + r_1) - r_x \quad . \quad . \quad . \quad . \quad . \quad (325)$$

Because of the greater ease with which the belt can be shifted, cone pulleys are usually designed for open belts. The radii of the steps of a pair of cones for an open belt are not, as for a crossed belt, independent of the distance C between centers. While for any given value of C the radii of the steps must be such as to require the same belt length for each position of the belt, it is very difficult to obtain from equation (318) an expression for R_x or r_x in terms of the known values R_1, r_1, and C. The most practical solution is a graphical one derived by Burmester.* This method is illustrated in Fig. 442. A 45-degree line AB, equal in length to the distance C between the shaft centers, is first laid off; and a line BE, equal in length to $\dfrac{C}{2}$, is then drawn at right angles to AB.

Through E is drawn an arc from A as a center. On a vertical line through any convenient point on AB the known radii R_1 and r_1 of the first step of the respective driving and driven pulleys are laid off as shown. A line through N parallel to AB intersects the arc at F; and a line through M parallel to AB intersects a vertical through F at G. The intersection with AB of a horizontal through G determines the origin O. For any step x for an angular velocity ratio of $\dfrac{\omega_x}{\omega}$, the cor-

* Lehrbuch der Kinematik, Vol. 1, by Dr. L. Burmester, page 395, published by Verlag Von Arthur Felix, Leipzig, Germany.

responding radii R_x and r_x of the driving and driven pulleys may be found by drawing a line OK at an angle θ_x with OH such that

$$\cot \theta_x = \frac{\omega_x}{\omega} = \frac{R_x}{r_x} \qquad \ldots \ldots \ldots \quad (326)$$

If the speeds are in geometric progression, then, in accordance with (320),

$$\cot \theta_x = \frac{\omega_x}{\omega} = \frac{R_x}{r_x} = \left(\frac{R_1}{r_1}\right)k^{x-1}$$

While the proof of the above construction is too long and intricate to be given here, it may be said that the method is, in theory, almost

FIG. 442

absolutely accurate. The arc EFJ is the only approximation in the construction, and it does not deviate from the theoretically correct curve as much as 0.02 per cent for values of ϕ, see Fig. 440, from zero to 75 degrees. It can therefore be said, that the only errors involved in the use of the method are those of drafting and measurement.

210. Speed Cones. To change the speed of a driven shaft by the shifting of a belt, speed cones, or conical drums, are occasionally used instead of cone pulleys. The shape of the cones will depend upon how the speed of the driven shaft is to vary for equal increments of shift of the belt. If, for equal increments of shift, the speed of the driven cone is to change in geometric progression, the shape of the cones for a crossed belt will be as shown in Fig. 443 and as shown in Fig. 444 for an open belt. In these illustrations short cones and a high progression ratio were used to make the variation in shape more pronounced and obvious. For the pair of equal cones shown in Fig. 443 the dotted diameters at equal intervals along the axes were computed by equations (324) and (325). In Fig. 444 the diameters at equal intervals were determined by the Burmester method. Since the relation $(R_x + r_x) = (R_1 + r_1)$ is true for the frusta of a pair of equal cones having straight sides, a crossed belt would be equally tight in all positions on such a pair of cones. Since any speed from the lowest to the highest can be obtained by shifting the belt the proper amount, frusta of equal cones having straight

FIG. 443   FIG. 444

sides are generally used for crossed belts. For such a pair of cones the speed of the driven cone would not, however, change in geometric progression for equal intervals of shift of the belt. On such a pair of cones an open belt would not be equally tight in all positions. Where, however, for a reasonable distance between centers, the cones are long as compared to their diameters, the variation in tightness is often not sufficient to defeat the transmission of small or moderate amounts of power.
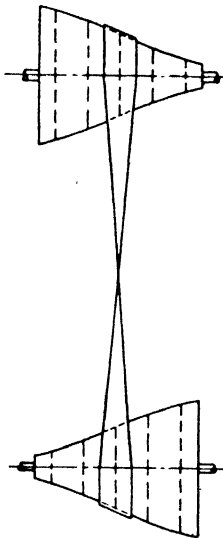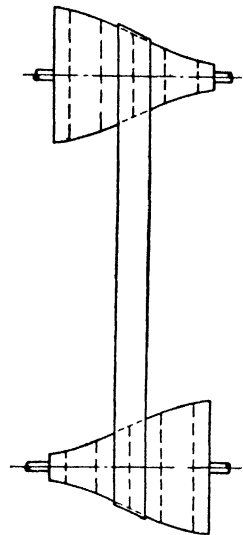
Speed cones have the advantage of permitting every possible change of speed from the minimum to the maximum but are open to the objection that the belt tends to climb toward the large end of each cone of the pair thus tightening the belt and increasing the pressure on the

bearings. To prevent this action, the belt must be guided on the approaching side of both the driving and driven cones. In shifting the belt both these guides must be shifted an equal amount simultaneously. It is because of the difficulty of shifting the belt and of the tendency of the belt to climb that speed cones are so little used. When used they

are made longer in proportion to their diameters than those shown in Figs. 443 and 444.

211. Variable Speed Belt Transmission. A vertical unit of a Reeves variable speed transmission is shown in Fig. 445. In this type of transmission, variation of speed is secured, in effect, by changing the diameters of the pulleys. Rotation is transmitted from one double cone to the other by means of a special type of belt B. As shown, the belt consists of an endless rubber band to each side of which are bolted hardwood blocks.
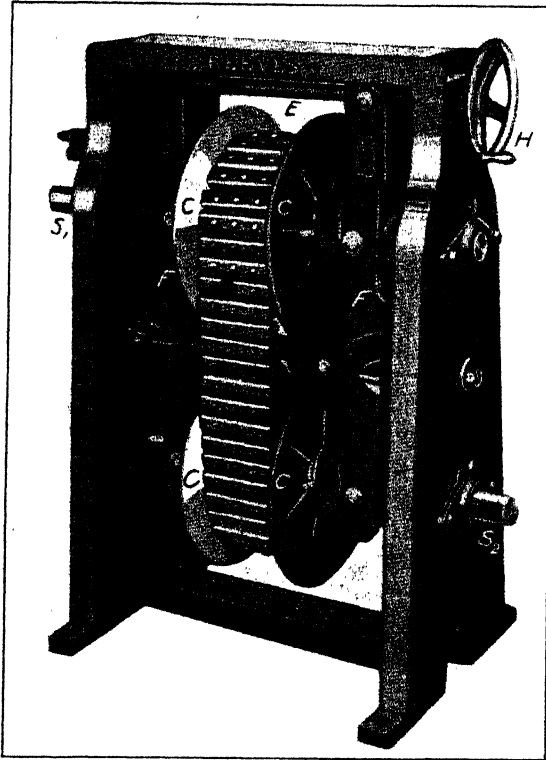
Fig. 445.

The ends of these blocks are beveled, faced with leather, and formed to fit the sloping sides of the cones C. It is therefore the edges and not the under face of the belt that transmit the motion from one double cone to the other. To change the angular velocity ratio, the pair of cones on one shaft are pulled apart while the pair on the other shaft are pushed together in such proportion as to keep the belt equally tight. This is effected by turning the screw E by means of handwheel H. The screw, having right-hand threads on one end and left-hand threads on the other, operates the levers F about their fulcrums thus spreading the pair of cones on one shaft and pushing the pair on the other shaft closer together. By this means every pos-

sible change of speed between the minimum and maximum limits can be secured.

Horizontal units are more generally used than vertical. Either shaft, S_1 or S_2, may act as the driver and be connected by gearing or by a belt or transmission chain to an electric motor or other source of power. Units are made in capacities to transmit from a fraction of to 150 horse-power and for ratios of the maximum to the minimum revolutions of the driven shaft from 2 up to 16.

212. Belt Transmission between Non-Parallel Shafts. It is sometimes desired to connect two shafts which are not parallel by a belt. To do this, the mid line of the advancing side of the belt must lie in the mid plane of the pulley upon which it advances. The application of this fundamental principle, which applies to all belt drives, may be illustrated by Fig. 446. In this figure let the full circle *1* and the dotted circle *2* represent two pulleys in the same plane on parallel shafts, and let the line T_1T_2 be drawn through the points where the mid line of the belt is tangent to the receding side of the pulleys. If the lower pulley and its shaft are turned about T_1T_2 as an axis to the position shown by the full lines, the line T_1T_2 will represent the intersection of the mid planes of the pulleys. Hence the line A_1A_2 in which the belt advances on the lower pulley will lie in the plane of the lower pulley; and the line B_1B_2 in which the belt advances on the upper pulley will lie in the plane of the upper pulley. It was shown in Art. 205 that a belt will run on a pulley without shifting its position if its line of travel lies in the plane of the pulley. Hence, for the direction of motion indicated, one shaft can drive the other without the belt running off the pulleys. If, however, the direction of motion of the belt be reversed, the belt will not advance upon each pulley in the plane of rotation and, therefore, would at once run off. For a belt to drive non-parallel shafts in either direction necessitates the use of guide pulleys.

If on rotating pulley *2* the shafts are to remain in parallel planes, one or both pulleys must be adjusted until T_1T_2 is perpendicular to a plane through the axis of *1* parallel to the axis of *2*. Where a belt connects two non-intersecting shafts at 90 degrees, the drive is called a *quarter-turn* drive.
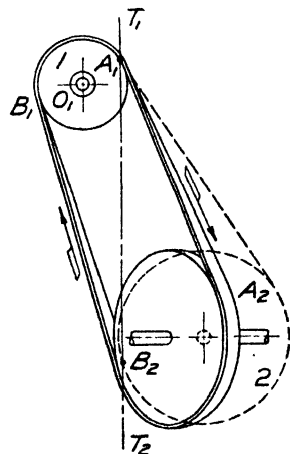
FIG. 446

213. Guide Pulleys. The principle just developed above that the mid line of the advancing side of a belt must lie in the mid plane of the pulley, is generally applicable to belt drives. By means of guide pulleys, or idlers, two intersecting or non-intersecting shafts can be connected by a belt. If desired the belts can be made to run in either direction by so placing the guide pulleys that both sides of the belt will lie in the planes of the pulleys. A few arrangements of belt-connected shafts are shown in Figs. 447, 448, and 449. For the drives shown in Figs. 447
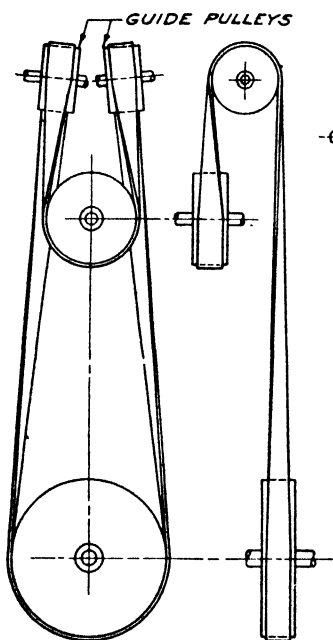
FIG. 448
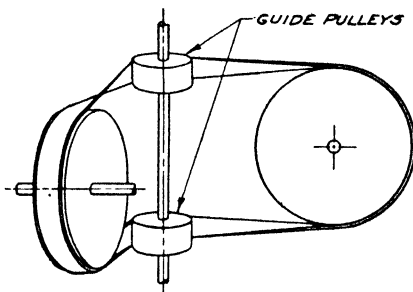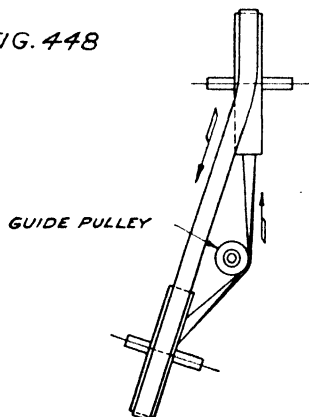
FIG. 447

FIG. 449

and 448 the belt may run in either direction; for the drive shown in Fig. 449 the belt will only remain on the pulleys when run in the direction indicated by the arrows.

214. Belt Tighteners. Besides being used as guide pulleys or as supporting pulleys for long runs of belt, idle pulleys, when suitably mounted, are used as belt tighteners. Their function when so employed is to increase the efficiency of the drive by maintaining the belt tension and increasing the arc of contact between the belt and the pulleys as shown in Fig. 450. In the example shown, the frame carrying the idler is in the form of a lever pivoted about the axis of the motor shaft.

The position of the idler is automatically maintained by the adjustable weight W at the opposite end of the lever. Springs whose tension or compression can be adjusted by means of a screw, are sometimes used instead of adjustable weights. By employing belt tighteners the distance between the shafts of high ratio belt drives can be reduced
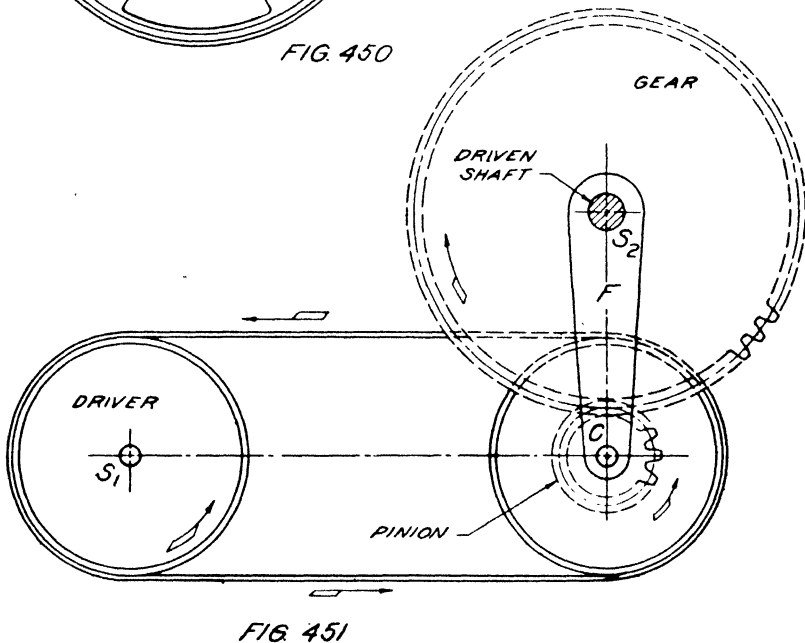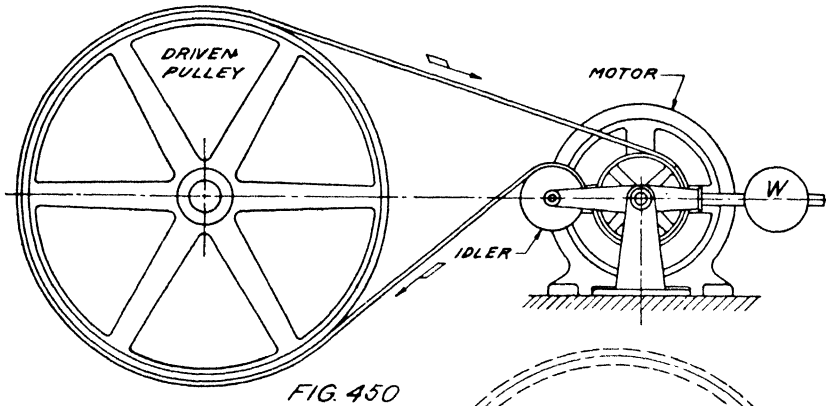
FIG. 450

FIG. 451

considerably below the minimum limits given in Art. 203 without sacrifice of efficiency. Such drives are generally known as short-center-distance drives.

In Fig. 451 the counter shaft C to which the pinion and driven pulley are keyed is carried by a suitable frame F pivoted about the axis of

the driven shaft S_2.   With the belt running in the direction shown, the reaction on the pinion teeth tends to rotate the frame F counter-clockwise about S_2 thus automatically maintaining the tension in the driving belt.

215. V-Belt Drives.   In recent years endless bands of vulcanized rubber reinforced by cotton cord and fabric have come into extensive use for short-center-distance transmissions.   As shown in Fig. 452, the bands are trapezoidal in section and run in the V grooves of the pulleys. They are referred to by some as V-belts and by others as V-ropes.   For transmitting any considerable amount of power they are used in multiple as shown in the figure.   Their great advantage over flat belts is due to
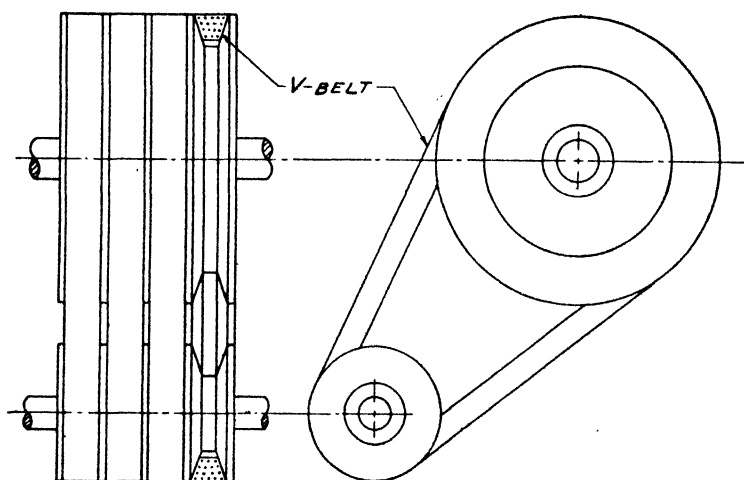
FIG. 452

their wedging action, which greatly increases the amount of power that may be transmitted for a given arc of contact.   Because of the greater area of contact with the sides of the pulley grooves, V-belts are more efficient than endless loops of fibrous rope used in a similar manner. They are silent, shock absorbing, and almost positive and slipless in action.   Hence, for many short-center-distance machine drives, they have displaced flat belt, chain, and gear drives.   Plain idler pulleys pressing against the outside of the V-belts on the slack side are used as tighteners for such drives where provision can not be made for taking up the slack by increasing the distance between centers.

216. Rope Drives.   Ropes running on grooved sheaves are often used to transmit power from one shaft to another.   Drives transmitting from one to 2000 horse-power are quite common and larger powers are

not unusual. In the multiple, or English, system, two or more endless loops of rope are used in the same manner as shown in Fig. 452 for V-belts. In the continuous, or American, system, a single loop of rope is used as shown in Fig. 453. The multiple system dates from 1863, and the continuous system from 1883. The first multiple rope power transmission was designed by James Combe of Belfast, Ireland, in 1856 and was installed in 1863. This drive was, in 1924, and probably still is, in operation.

Each system has its advantages. For large powers and for parallel shafts in a horizontal plane, or in a plane not too far removed from the
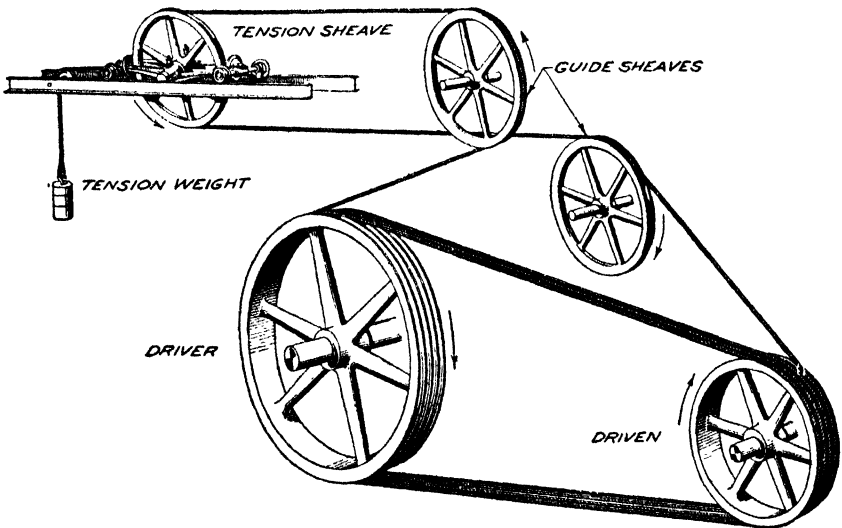
FIG. 453.

horizontal, the multiple is superior to the continuous system. It is simpler and cheaper, the ropes have longer life, and there is more security against interruption of service, since, if a rope breaks, the other ropes will carry the load until repair can be made. The continuous system has wider adaptability and is more suitable for vertical, quarter turn, and complex drives. In this system a uniform tension is maintained by the use of a tension sheave. As shown in Fig. 453, the rope should be guided from the slack side of the driving pulley to the tension sheave and from there to the slack side of the driven sheave. Instead of doing this as shown in the figure, the rope could be passed from the slack side of the driver around a loose sheave on the driven shaft from which it could be passed around a tension sheave to the slack side of the driven sheave. It is apparent that, with either system, several shafts may be

driven from the same driving sheave, and that either system of drive may be used over short or long distances. Shafts 175 feet apart have been successfully driven by ropes without using idlers. Where supporting idlers are used, power can be efficiently transmitted over longer distances. For very long distances wire rope is used in preference to cotton or manila, especially if the drive is exposed to the elements.

Fibrous rope drives are almost noiseless, are shock absorbing, and are almost positive and slipless in action. Where the power to be transmitted is over 200 horse-power and the distance between centers is over 25 or 30 feet, rope drives are cheaper to install and to maintain than rubber or leather belt drives.

The larger the ratio of the diameter of the sheave to the diameter of the rope the less will be the bending of the rope as it runs over the sheave and, in consequence, the less will be the wear of the rope due to
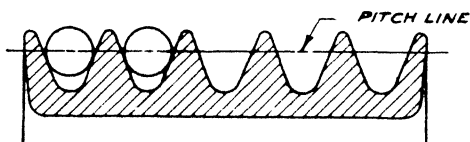
FIG. 454
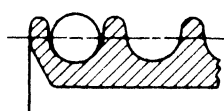
FIG. 456
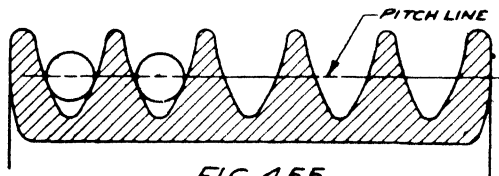
FIG. 455
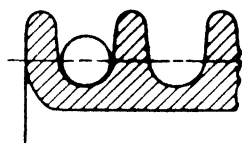
FIG. 457

internal chafing. For power transmission this ratio should be at least 36. In order that each run of rope may take its share of the load, the pitch diameter of each sheave groove should be the same; and, to reduce the wear of the rope, the grooves should be smooth.

Since in the continuous system the rope is kept taut by a tension sheave, the sheave grooves may be shallower than for the multiple system without danger of the rope leaving the grooves. The type of sheave groove for the continuous system is shown in Fig. 454 and for the multiple system in Fig. 455. It is to be noted that the deeper groove for the latter system requires a wider sheave. The groove angle should not be made less than 45 degrees nor more than 60 degrees, 45 and 50 being common angles. Since guide and tension sheaves do not transmit power, wedging action is unnecessary, and the grooves may be made as shown in Figs. 456 and 457.

The grooves of wire rope hoisting sheaves are usually made as shown

in Fig. 458. Since for the transmission of power the velocity of the rope is usually much greater than in hoisting, the bottom of the grooves of such sheaves are usually filled with rubber or leather as shown in Fig. 459. This practice increases the grip of the rope and decreases the wear very materially.

217. Hoisting Chain. The kind of chain generally used in this country for hoisting purposes is shown in Fig. 460, the links being made of wrought-iron or steel. The chain is either wound on a grooved drum or is propelled by a pocket wheel, or *sprocket*, as shown in the figure. When the chain links are sufficiently uniform as to size and form to use with grooved sheaves and drums, the chain is generally referred to as *coil* chain. If the links are sufficiently uniform as to size and form to work well in sprockets, the chain is generally called *pitch* chain.
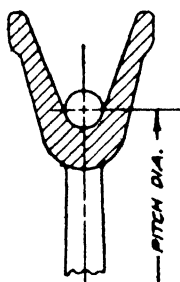
FIG. 458
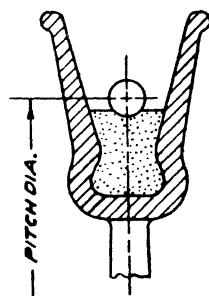
FIG. 459

218. Transmission Chains. Power may be transmitted from one shaft to another by means of chains and toothed wheels, or *sprockets*, as shown in Fig. 461. Except for a limitation that will be mentioned later, transmission chains may be used instead of gears to secure positive driving where the distance between shafts is too great for gears or where the use of gears would be less convenient. Chains are used instead of belts either because positive driving is required or because the center distance is so short that a flat belt would be less convenient and satisfactory.
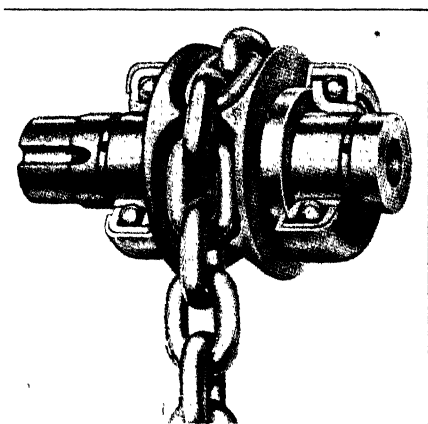
FIG. 460.

The Ewart detachable link chain, shown in Fig. 462, was invented in 1873. The method of inserting or detaching a link is clearly shown in the figure. The links are of high grade malleable iron and are available in some two dozen stock sizes. The chain is used with cast sprockets for the transmission of motion and power at velocities not exceeding

400 feet per minute. The ratio of the sprockets should not be greater than 5/1, and the smaller sprocket of a pair, where possible, should have not less than 10 teeth. These remarks apply also to the interlock-
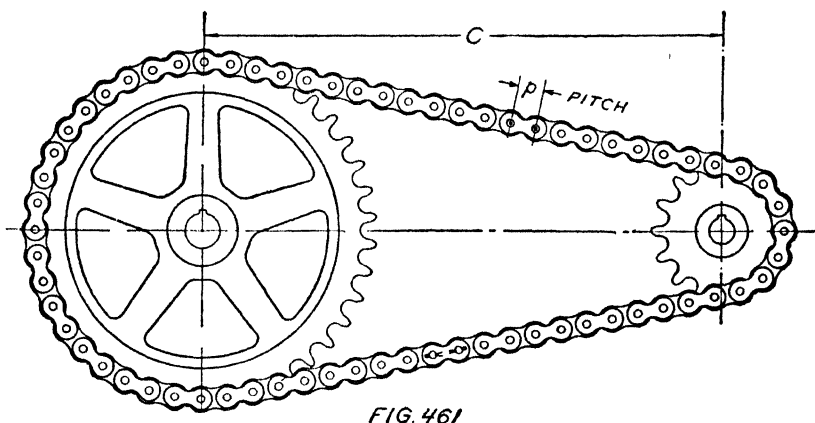
FIG. 461

ing pintle chain with malleable iron links shown in Fig. 463. When used as conveyor chain, the links of these chains are cast with various kinds of projections for attaching the conveying elements.
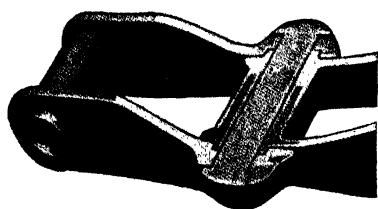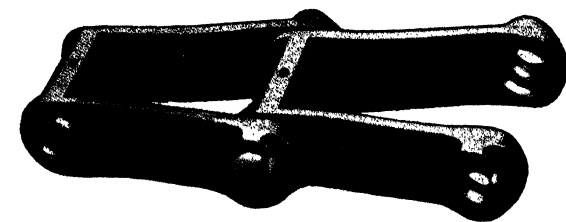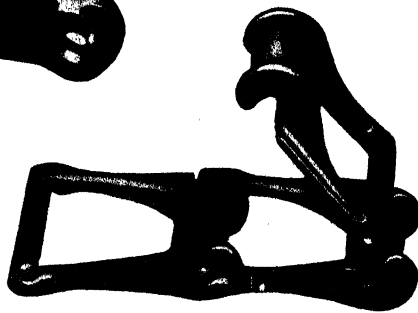
FIG. 463.   FIG. 462.

The block chain shown in Fig. 464 is a finished steel chain made up of steel blocks connected by steel side links. Since this type of chain is only suitable for the transmission of small amounts of power at relatively low velocities, it is only stocked in a few small sizes. The roller

chain shown in Fig. 465 is much more suitable for the transmission of power and is available in many different sizes ranging in pitch from $\frac{1}{2}$ to 4 inches. The flat links are made from cold rolled high carbon steel, the pins are made of alloy steel and heat treated, and the bushings and rollers are case hardened. When made with two or more rows of rollers, the chain is referred to as multiple strand roller chain. By this means the power capacity for any given pitch is increased.
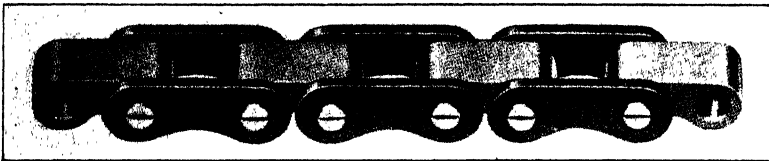
Fig. 464.

With proper lubrication, velocities as high as 2000 feet per minute have been successfully used with multiple strand roller chain, and as high as 1400 feet per minute with single strand chain. For general purposes, however, the velocity of single strand chain is usually limited to 700 to 900 feet per minute. The ratio of the sprockets should not exceed 7/1 to 10/1, and the smaller sprocket of a pair, where possible, should have not less than 11 teeth. The smaller the pitch the higher the rotative speed permissible for a given diameter of sprocket. Hence, for a given
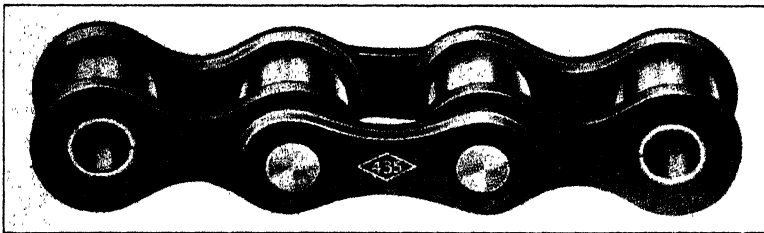
Fig. 465.

power and diameter of sprocket, a desirable reduction in pitch may be secured by using a multiple instead of a single strand chain. Finished block and roller chains are run on finished sprockets. With roller chains a mechanical efficiency of transmission as high as 0.96 is possible. When roller chain is used as conveyor chain, the side links are so formed that the conveying elements may be attached.

The type of toothed, or silent, chain shown in Fig. 466 was invented by Hans Renold of England. In Fig. 467 is shown the Morse chain of

this type. The chains differ only in the type of joint used. In the Renold chain, a round pin works in a segmental bushing, while in the Morse chain two pins are used to form a rocker joint. With these chains a sustained efficiency of transmission of over 0.98 is possible. The links are punched from cold rolled alloy steel and then hardened; the pins are cut to length from alloy steel rods which have been drawn
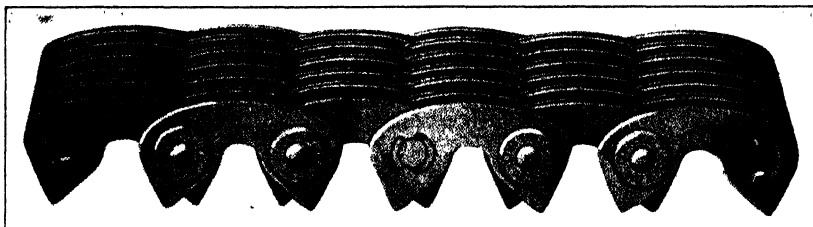
Fig. 466.

to size and are then hardened. Power has been successfully transmitted at velocities as high as 2500 feet per minute. For general purposes, however, the velocity is usually limited to 1200 to 1400 feet per minute. The ratio of the sprockets should not exceed 7/1 to 10/1, and the smaller sprocket of a pair, when possible, should have not less than 17 teeth, an odd number of teeth being better than an even number. The larger sprocket also, where the ratio permits, should have an odd number
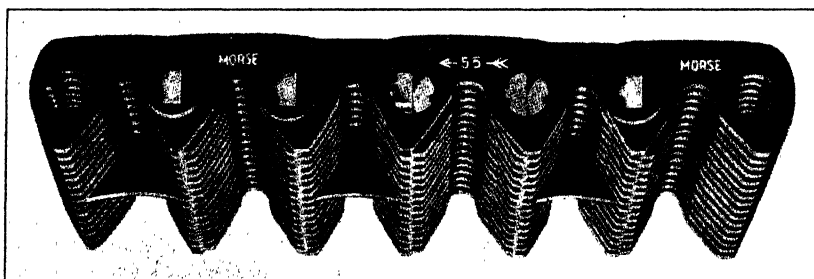
Fig. 467.

of teeth if the use of a special link to join the chain is to be avoided. In other words, the chain, if possible, should have an even number of links. As for roller chain, the smaller the pitch the higher the rotative speed permissible for any given diameter of sprocket.

219. Velocity Ratio. With respect to chain sprockets, the ratio of the revolutions is, as for gears, inversely as the numbers of teeth in the sprockets. That is to say, if the driving sprocket has 24 teeth and the

driven sprocket 12 teeth, one revolution of the driver will cause the driven sprocket to make 2 revolutions. But, unlike gears, the angular velocity ratio is not constant for all phases of the sprockets. In demonstrating this, a roller chain will be used. This choice is merely a matter of convenience since the relative motion of adjoining links for each of the chains shown in the previous article is, as for the roller chain, about a pin or axis.

In Fig. 468 the chain has 12 links, and the sprockets *1* and *2* have 8 and 4 teeth respectively. To obtain a pronounced variation in the angular velocity ratio, a short center distance and a pinion having
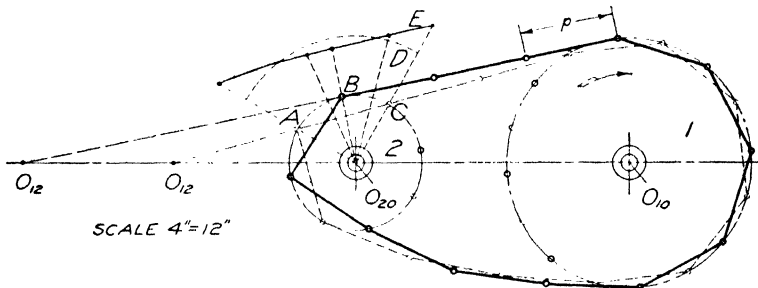
FIG. 468

very few teeth were purposely selected. It was proved in Art. 43 that the angular velocities of driver and follower are inversely as the perpendiculars let fall from their fixed centers on the line of transmission, or inversely as the segments into which the line of transmission cuts the line of centers. This was again proved by the method of instant centers in Art. 67. Hence,

$$\frac{\omega_{20}}{\omega_{10}} = \frac{O_{10}O_{12}}{O_{20}O_{12}} \quad \text{or} \quad \omega_{20} = \left(\frac{O_{10}O_{12}}{O_{20}O_{12}}\right)\omega_{10}$$

Assuming the driver *1* to rotate at a constant rate, the angular velocity of *2* will be a maximum when the intersection O_{12} of the line of transmission with the line of centers is nearest to O_{10} and O_{20}, and the angular velocity of *2* will be a minimum when O_{12} is farthest from O_{10} and O_{20}. For the proportions chosen, ω_{20} is a maximum for the phase shown dotted when the pitch line of a link on the follower lies in the line of transmission, or when roller *A* is just coming into action and roller *C* is just going out of action. The angular velocity of the follower will be a minimum when the line of transmission is tangent to the pitch circle of the follower, as shown by the full lines in the figure. If the sprockets

acted like gears, the angular velocity ratio would be constant and inversely as the numbers of teeth, or

$$\omega_{20} = \left(\frac{N_1}{N_2}\right)\omega_{10} = 2.0\omega_{10}$$

Because sprockets do not act like gears, the angular velocity of the follower from the time a roller goes into action at A until it leaves action at C decreases from a maximum of $2.51\omega_{10}$ at A to a minimum of $1.828\omega_{10}$ at B and then increases to the maximum of $2.51\omega_{10}$ at C. There is, therefore, a variation in the angular velocity of the follower of 37.3 per cent above the minimum, or a variation from constancy of 8.6 per cent below to 25.5 per cent above the value of $2.0\omega_{10}$. A polar diagram of the variation of the angular velocity of the follower from the instant a roller comes into action at A until it leaves action at C is represented by the full curve E, the dotted curve D representing the constant angular velocity ratio corresponding to the numbers of teeth in the sprockets.

While it is impossible by the use of chains to transmit motion from one shaft to another at a constant angular velocity ratio, it is possible to make the variation so slight as to make chain transmission applicable to many purposes. This is done by giving due consideration to the pitch of the chain, the number of teeth on the smaller sprocket, the ratio of the sprockets, and the distance between centers. The smaller the pitch and the greater the number of teeth on the pinion for a given sprocket ratio and distance between centers, the smaller will be the variation of the angular velocity ratio.

The limitation on the substitution of chain for gear transmission mentioned at the beginning of Art. 218 arises from the fact that the angular velocity ratio of chain sprockets is not constant and inversely as the numbers of teeth as for gears. Hence, chain can not be substituted for gear drives where it is absolutely essential that the angular velocity ratio shall remain constant throughout each revolution of the driver and driven shafts. Thus gear and not chain transmission is used to drive the lead screw of a lathe, the succession of feed rolls in a printing press, many machine tool indexing mechanisms, etc.

It is found most convenient in chain transmission to express the disance between centers as a multiple of the pitch of the chain and to determine this distance in terms of the numbers of teeth in the sprockets and the desired number of links or pitches in the chain. It should be noted that the distance between centers for sprockets and chain depends upon the phase of rotation of the sprockets. In obtaining the center distance, the phase desired is, of course, the one giving the shortest center distance.

220. Sprockets for Roller Chain. As a roller chain runs on or off a sprocket, the roller which is about to seat and the roller which is unseating swings, relative to the sprocket, about the axis of the adjoining roller as shown in Fig. 469. Hence, the outlines of the sprocket teeth are made up of tangent arcs of two different radii drawn from the pitch points A, B, C, etc., as centers. Thus, as link D rotates about center A, center E describes an arc BF of a radius equal to the pitch, and the roller describes an arc G of a radius equal to the pitch minus the roller radius. This arc G is tangent to an arc from B as a center having a radius equal to that of the roller. A sprocket with teeth so formed will work properly with the chain so long as the pitch of the chain remains exactly equal to that of the sprocket teeth. However, as the

pitch of the chain increases because of wear at the joints, the action of the chain becomes imperfect. One of the first methods to be applied to correct this defect was to decrease the thickness of the sprocket teeth, thus increasing the tooth spaces as shown at H. This permitted the chain to stretch but caused the pull in the chain to be taken by the first tooth of the driver and the last tooth of the follower, resulting in the chain slipping forward relative to the sprockets as each incoming roller seated on the driver. If the pitch of the sprocket is made more

FIG. 469

than that of the chain, the last tooth of the driver and the first tooth of the follower will take the load, resulting in the chain slipping backward relative to the sprockets as each successive roller comes into action. An expedient that would permit the chain to stretch without resulting in defective action was first applied by Hans Renold. His expedient was to so modify the sprocket teeth that the chain would automatically move radially outward as it stretched.
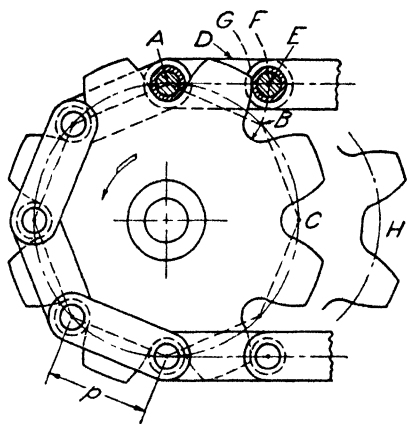
The Renold roller chain sprocket tooth is shown in Fig. 470. Line CF is drawn tangent to the roller circle at an angle of 30 degrees with the radial line through center A of the roller. The point of the tooth is finished off by drawing the clearance arc EG from P as a center, BE being perpendicular to DE. If it were practicable to hold to the same clearance arc for all numbers of teeth, the tooth spaces would be the same for all sprockets for a given diameter of roller. The tooth outline

DEG falls within the arc *H* drawn from *B* as a center as for the tooth outlines in Fig. 469.   It is to be noted that the line of pressure between the roller and such an outline as *H* would coincide with the pitch chord
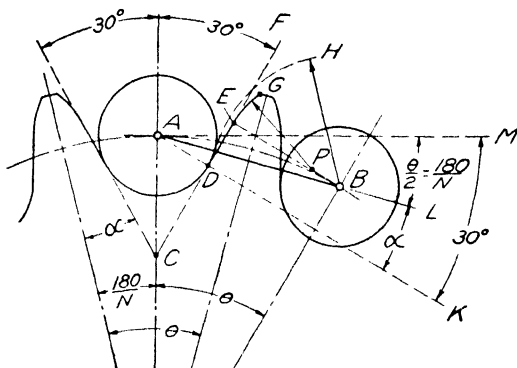
FIG. 470

AB, or line of pull *AL*, while, for the Renold tooth, the line of pressure *AK* is at an angle α with the pull.   It is this obliquity of action which causes the chain to automatically move radially outward in contact
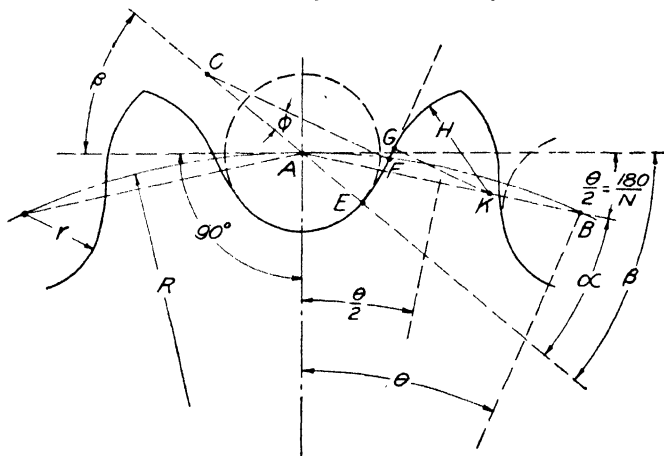
FIG 471

with the sprocket teeth as the chain stretches.   This angle of obliquity in terms of the pitch angle θ and number of teeth N is, in degrees,

$$\alpha = 30 - \frac{\theta}{2} = 30 - \frac{180}{N} \quad \cdots \cdots \quad (327)$$

For $N = 6$ it is to be noted that α becomes zero, DE is reduced to zero, P is moved to B, and the tooth outline would coincide with H and be in agreement with the teeth of Fig. 469.

The form of sprocket tooth developed by the Diamond Chain and Manufacturing Company and approved by the A.S.M.E., the S.A.E., and the A.G.M.A. is shown in Fig. 471. With reference to the figure, let

$p = $ the pitch of the chain;

$N = $ the number of teeth in the sprocket;

$\theta = $ the pitch angle in degrees $= \dfrac{360°}{N}$;

$R = $ the radius of the pitch circle of the sprocket $= \dfrac{p}{2 \sin \dfrac{\theta}{2}}$;

$d = $ the nominal diameter of the chain roller;

$r = $ the radius of the bottom of the tooth space $= 0.5025d + 0.0015$;

$\beta = 35° + \dfrac{60°}{N}$;

$\phi = 18° - \dfrac{56°}{N}$;

$\alpha = $ the pressure angle $= \left(35° - \dfrac{120°}{N}\right)$ maximum;

$\qquad = \left(17° - \dfrac{64°}{N}\right)$ minimum;

$AC = 0.80d$;

$AK = 1.24d$.

To determine the outline of a tooth, draw the pitch circle, locate the pitch points A and B, draw a circle of radius r from A as a center, locate centers C and K, draw tangent arc EF from C as a center, draw the tangent line FG, and complete the outline of one side of the tooth by a tangent arc from K as a center. While the principle of the tooth is the same as for the Renold tooth, it is to be noted that the pressure angle changes as the chain stretches and moves radially outward in contact with the teeth.

221. Sprockets for Toothed, or Silent, Chain. The Renold toothed chain as made by the Link-Belt Company is shown in Fig. 466. The riveting and washer of the mid joint has been cut away to show the

construction of the joint. It is to be noted that the bushing segment
on the left of the pin is carried by the set of links to the right, and that
the segment of bushing on the right of the pin is carried by the set of
links to the left. Hence relative motion of the sets of links to the
right and left of the pin causes the bushing segments to rotate on the
pin. The amount one set of links can swing up or down with respect
to the other set of links is indicated by the clearance spaces which are
shown black.

The Morse chain as made by the Morse Chain Company is shown
in Fig. 467. The third and fourth washers from the left have been cut
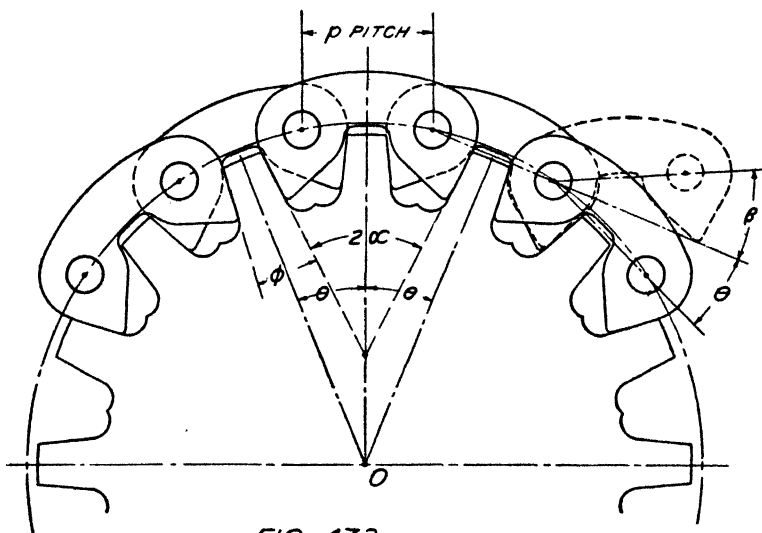
FIG. 472

away to show the construction at the joint. It is to be noted that the
pin is in two segments, the set of links to the right of the joint carrying
the flat-faced segment and the set of links to the left of the joint carrying
the rocker segment. As one set of links swings relative to the next set,
the pin segments rock one upon the other. For highest efficiency the
chain should be run in the direction indicated by the arrow. The mid
link of each alternate set of links is, as shown, a special link. By run-
ning in a groove cut in the teeth of the sprocket wheels, these special
links keep the chain from shifting in an axial direction on the sprockets.
Other means are employed for both the Renold and Morse chains to
accomplish the same thing.

The principle of the toothed chain link is shown in Fig. 472. The
ends of each link are straight and at an included angle 2α. This angle

is usually 55 or 60 degrees.   Each link enters two tooth spaces of the sprocket and makes surface contact with the sprocket teeth.   The contact surfaces being flat, the pressure acts normal to the tooth surfaces and at a constant angle with the pitch cord and the line of transmission. This angle is evidently α and equal to 30 degrees when 2α equals 60 degrees.   This obliquity of action causes the chain to move radially outward as the chain stretches with wear at the joints.

Since each link spans a sprocket tooth, it must be notched to clear the tooth.   The inside as well as the outside of the chain teeth are made straight and at such an angle as will permit the chain to wrap on the sprockets or to sag between sprockets.   As shown to the right in the figure, one set of links with respect to the next set swings through the
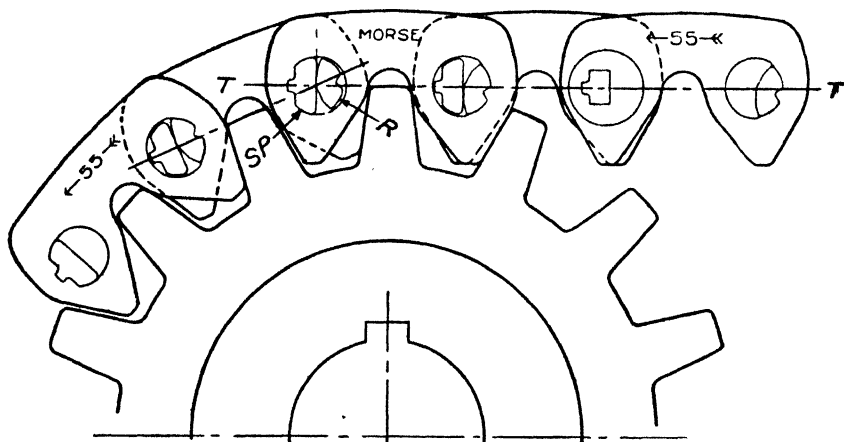
FIG. 473

pitch angle θ in wrapping on the sprocket and can swing through the angle β before the inside of the chain tooth comes in contact with a sprocket tooth.

The tooth angle ϕ in terms of α and the pitch angle θ, or in terms of α and the number of sprocket teeth N, is

$$\phi = 2(\alpha - \theta) = 2\alpha - \frac{720°}{N} \quad . \quad . \quad . \quad . \quad (328)$$

Evidently, for 2α equal to 60 degrees, the sides of the teeth of a 12-tooth sprocket will be parallel.   To avoid this and to keep down the variation in the angular velocity ratio, it is specified that the smaller of a pair of sprockets shall never have less than 15 teeth, and not less than 17 or 19 is recommended for large pitches and high speeds.

As shown in Fig. 473, the action of the teeth of a Morse chain with the teeth of the sprocket is the same as just described. The change in the relative positions of the segments of the pins as the chain wraps on the sprocket is clearly shown in the figure. It is to be noted that the seat pins SP and rocker pins R make surface contact when in the line of transmission TT and make line contact when the chain is wrapped on the sprocket. In the Renold chain there is rotational sliding between the pin and bushing segments, while in the Morse chain the pin segments rock or roll upon each other. As before mentioned, this difference in action at the joints constitutes the primary difference in the two chains.

Toothed chains seat more quietly than roller chains, adjust themselves more perfectly to the sprockets as the chain stretches and, for a given width, will transmit more power.

CHAPTER XIII

TRAINS OF MECHANISM

222. Trains of Mechanism. Machines that receive energy from some natural source, such as gas, gasoline, oil, coal, falling water, currents of air, and that transform it into mechanical energy are call prime movers. Thus internal combustion engines, steam engines, st turbines, water wheels, water turbines, and windmills are prime mo The mechanical energy from prime movers may be transmit to machines for the performance of useful work, or it may b before being transmitted to such machines. Thus r pressors, or electric generators may be directly or i prime movers, and the transformed energy be trans pneumatically, or electrically to tools and machi of useful work. Energy so transmitted is mechanical energy of reciprocation or rot and consumed. Thus the mechanical en mover may be converted by a generato energy may be transmitted to and reco mechanical energy of rotation, and tl mitted to and consumed by such mac machines, etc., in doing useful work.

In accordance with the abov mechanism or combination of purpose of generating, transf To do this, such moving part friction wheels, pulleys and chains, etc., are necessary. are combined in various changes and transformati trains of mechanism, an o be familiar, will be descri

In Fig. 474 is shown The head stock is fitted driving belt on the small lathe spindle, to which t

ead ough carriage nut to close ..eed of the car- as per inch, or the the carriage, is con- end of the lead screw and gear *11*. For this marked advantage since e center distances to trans- ty ratio. ion that trains of mechanism ning gears and other wheels to may contain such elements as ars, and all manner of linkages. anisms are to be found in machines able mechanical energy is utilized in e motion of the driving element of a d wheel trains figure very largely in the d through the machine. For this and arent, this chapter will be confined to the yclic wheel trains. In an ordinary wheel the axes of the wheels relative to the fixed heel trains such relative motion is involved. d pulleys, ropes and sheaves, chains and f gears may be variously combined to form heel train composed of gears only is called a el trains, the following common elements of

rotative speed; with the belt on the largest step of the cone and the back gears in, the lathe spindle is driven at its lowest rotative speed. The piece of metal to be formed rotates with the face plate. If the piece is quite short compared to its diameter, it is fastened to the face plate; if long compared to its diameter, it is mounted between the live and dead centers and coupled to the face plate by a dog.

The cutting tool is held in the tool post which is attached to the cross-slide. This slide is part of the tool carriage which is mounted on the ways of the lathe bed. In facing off a piece of work mounted on the

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a given w
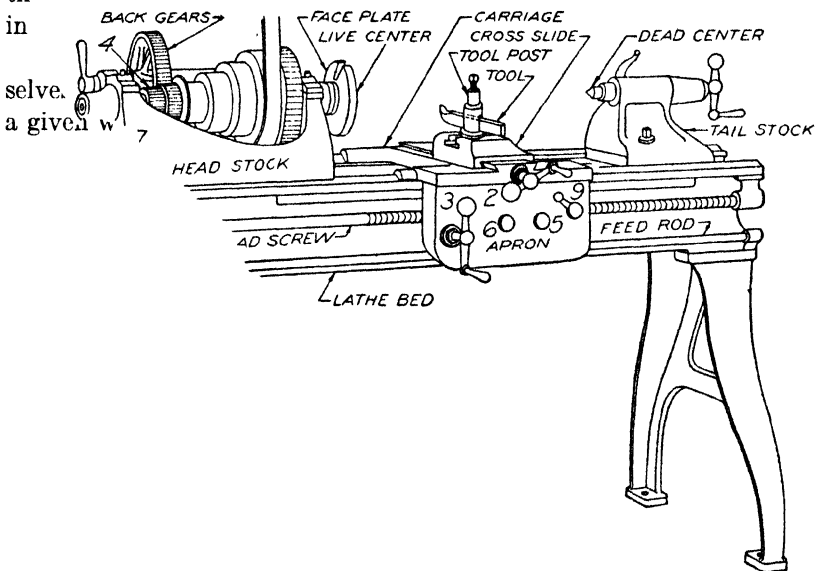
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ce mounted between centers,
to the axis of rotation of the
ylindrical form, the cutting
ition of the work. These
ial feed, may be obtained
utting tool is obtained by
r the longitudinal feed.
aken from the rotating
elt train from gear 4 on
ross feed of the tool is
itating feed rod through.

a suitable train of mechanism in the apron of the tool carriage. The longitudinal feed is controlled by knob 6 and is also obtained from the feed rod through a suitable train of mechanism in the apron. At the end of this train is a pinion which meshes with the feed rack attached to the lathe bed and causes the tool carriage to move longitudinally on its guides, or ways. It is to be noted that the longitudinal feed train, besides containing gears, cone pulleys, and a belt, contains a rack and a sliding member on the feed rod, and that the cross-feed train contains the same sliding member on the feed rod and a screw and a slide and its guides. The cross and longitudinal feeds are reversed by operating the feed control lever 7.

When cutting screw threads, the tool carriage is driven by the lead screw instead of the feed rod. This screw is driven by gear 4 through the gear train shown. The longitudinal feed of the tool and carriage is thrown in or out by operating lever 9 which causes a split nut to close over or to release the lead screw; and the direction of feed of the carriage is controlled by lever 7. The number of threads per inch, or the number of turns of the work per inch of advance of the carriage, is controlled by changing the size of the gear on the left end of the lead screw and adjusting the idler gear 10 to mesh with it and gear 11. For this train of change gears, involute gears have a marked advantage since they do not require precise adjustment of the center distances to transmit motion at an unvarying angular velocity ratio.

It is evident from the above description that trains of mechanism to be found in machines, besides containing gears and other wheels to transmit continuous rotary motion, may contain such elements as screws, cams, levers, intermittent gears, and all manner of linkages. Generally these latter parts and mechanisms are to be found in machines at or near the point where the available mechanical energy is utilized in doing useful work. Usually the motion of the driving element of a machine is that of rotation, and wheel trains figure very largely in the transmission of motion to and through the machine. For this and other reasons that will be apparent, this chapter will be confined to the treatment of ordinary and cyclic wheel trains. In an ordinary wheel train there is no motion of the axes of the wheels relative to the fixed member whereas in cyclic wheel trains such relative motion is involved. Friction wheels, belts and pulleys, ropes and sheaves, chains and sprockets, and all kinds of gears may be variously combined to form mixed wheel trains. A wheel train composed of gears only is called a gear train.

Before discussing wheel trains, the following common elements of trains will be discussed:

Sliding gears;   Reversing mechanisms;
Clutches;   Change gear mechanism;
Idler gears;   Gear cones.

223. Sliding Gears. A sliding gear is an element frequently used in gear trains to change the angular velocity ratio of adjacent shafts. It is a common element in automobile transmissions and the gear trains of many other machines. In Fig. 475 the sliding compound gear *1*, composed of gears *a* and *b*, is free to slide but not to rotate on its shaft. Gears *c* and *d* are keyed to shaft *B* forming the compound gear *2*. The forked member *F*, which operates in a groove in *1*, is fastened to a sliding rod. By shifting *F* to the left, gear *a* is thrown in mesh with gear *c*; and gear *b* is thrown in mesh with gear *d* by shifting *F* to the right.
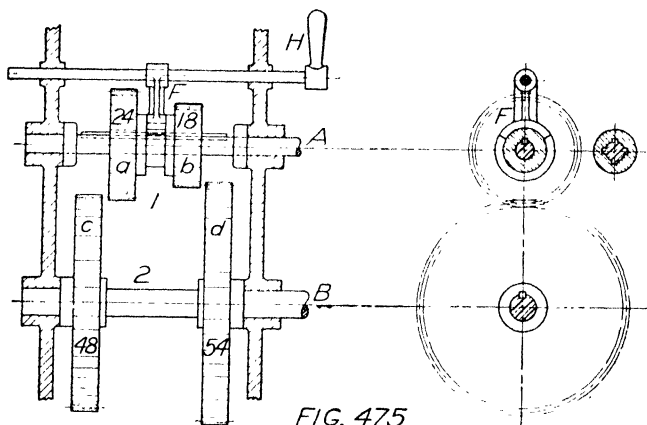
FIG. 475

By this means the ratio of the angular velocities of the two shafts *A* and *B* is changed.

In the example shown, the compound gear *1* slides on a *feather*, or *spline*. A feather, or spline, is a long key fitted and fastened into a keyway cut in the shaft on which the gear slides; or it is a shorter key fitted and fastened into the keyseat of the hub of the sliding gear or other sliding member. Generally one or two feathers are used. Where, as in automobile transmissions, the service is severe, splined shafts are used, the splines being integral with shaft as shown in the right-hand view of the figure. For such shafts the hub of the mating member is generally broached.

224. Clutches. Clutches may in general be divided into two classes, positive jaw clutches and friction clutches. Of the many possible examples of the latter class only three will be shown.

A jaw clutch is shown in Fig. 476. The clutch *1*, while free to slide, is held from turning on shaft *A* by a spline. With the clutch disengaged as shown, shaft *A* is free to rotate in gears *a* and *b*. Gears *c* and *d* are keyed to shaft *B*, forming the compound gear *2*. For shaft *A* to drive shaft *B* through gears *a* and *c*, clutch *1* is shifted to the left to engagement with *a*; and for *A* to drive *B* through *b* and *d*, the clutch is shifted to the right to engagement with *b*. The clutch is called a positive jaw clutch

because f r o m t h e moment of engage- ment it drives posi- tively, without slip. In the e x a m p l e shown, the engaging teeth are so shaped that gears *a* and *b* may be driven in either direction by the clutch. Where the drive is to be continuously in one direction, the back

END VIEW OF CLUTCH
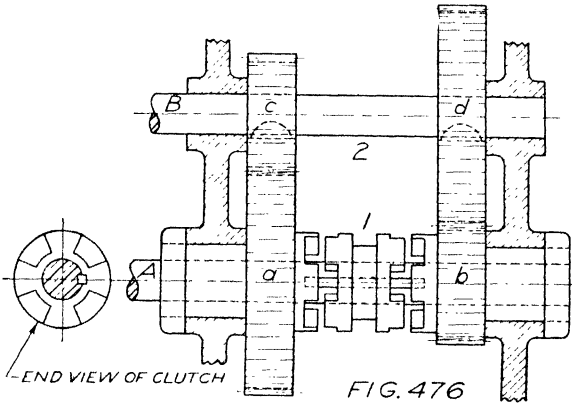
FIG. 476

faces of the teeth are often beveled to make engagement and disengagement easier. Jaw clutches should not be engaged or disengaged under load or while in motion unless the speed of rotation is low.

The double jaw clutch in Fig. 476 and the sliding gear in Fig. 475 are used for the same purpose, that is, for changing the angular velocity ratio of the two shafts *A* and *B*. It is to be noted that with the double jaw clutch both pairs of gears are always in mesh, whereas with the sliding gear one pair of gears only can be in mesh. It is also to be noted that a double friction clutch in place of the double jaw clutch could be used in Fig. 476.

Three examples of friction clutches are shown in Figs. 477, 478, and 479. The advantage of the friction clutch is that it can be engaged and disengaged at will under load and in motion. The purpose of the *cone clutch* shown in Fig. 477 is to transmit motion and power from gear *g* to shaft *A* through cone *C*, or from shaft *A* through cone *C* to gear *g*. The gear is free to rotate on the shaft; and the cone, while free to slide, is held from turning on the shaft by a spline. The cone is held out of engagement by the thrust of three helical springs, *S*, only one of which is shown; it is forced into engagement by pushing rod *R* to the right thus forcing the conical thimble *T* between the bell cranks *B* which in turn force the cone into engagement with the gear.

The purpose of the disk clutch shown in Fig. 478 is to transmit motion and power from pulley P to shaft A through member D, or from shaft A through member D to pulley P. Between the four annular disks rotating with P are three annular disks fitted over splines on D.
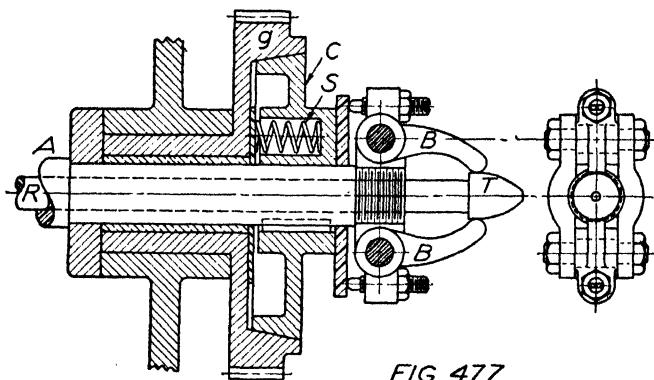
FIG 477

When the clutch is disengaged, pulley P rotates freely on the shaft. Member D, which carries three disks and the pressure element E, is keyed to shaft A. The clutch is engaged by pushing rod R to the right into the position shown. The tapered end of the rod causes the bell
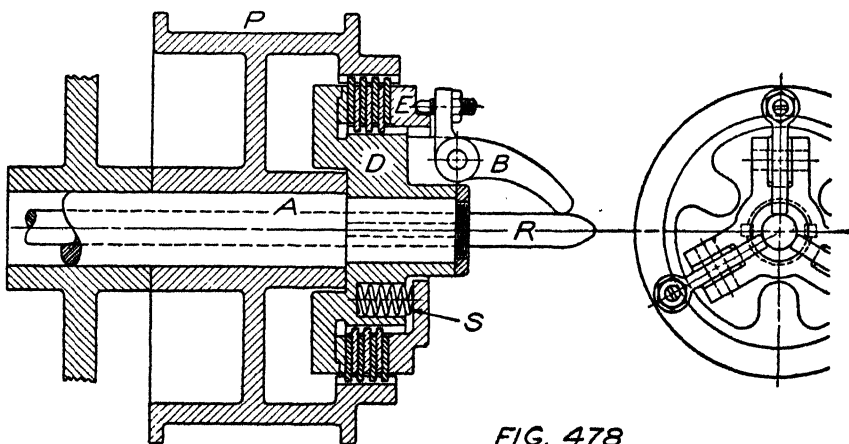
FIG. 478

cranks B to contact with E thus pressing the disks together. It is to be noted that the fulcrums of the bell cranks are in D and that their contacts with E are adjustable. When R is drawn to the left, E is pushed to the right by three helical springs S, the disks are relieved of

pressure, and those carried by P are free to rotate between those carried by D.

The purpose of the expanding ring clutch shown in Fig. 479 is to transmit motion and power from shaft A to gear g or from gear g to shaft A. The conical thimble T is splined to the shaft. When the clutch is disengaged the shaft is free to rotate in the gear. The end view of lever L and of the expansion ring R, which is keyed to the shaft, is shown to the right in the figure. When T is shifted to the left, its conical end causes lever L to turn the notched pin P to which it is attached. The action of the notched pin on the end of the split ring causes the ring to expand and to grip the gear.

225. Idler Gears. Fig. 480 shows an intermediate, or *idler*, gear *2* interposed between gears *1* and *3*. If *1* mated directly with *3* they
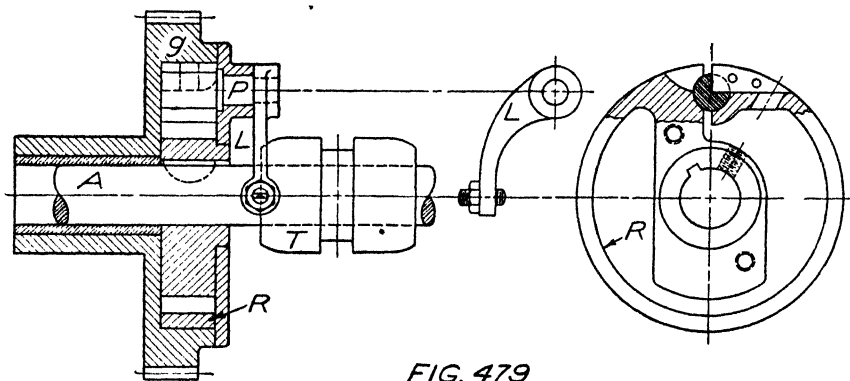
FIG. 479

would rotate in opposite directions. It is obvious by inspection that the effect of interposing the idler is to cause *1* and *3* to rotate in the same instead of in opposite directions. The angular velocity ratio of *1* and *3* has not, however, been changed by interposing the idler. Since the pitch line velocities of mating gears must be equal, the pitch line velocity of *2* is the same as *1*, that of *3* the same as *2*, and, therefore, that of *3* the same as *1* as it would be if *1* and *3* mated directly. This also is easily demonstrated by means of the angular velocity ratios. As has been shown, the angular velocities of a pair of mating gears are inversely as their pitch radii, pitch diameters, or numbers of teeth. Assuming ω_{10}, ω_{20}, and ω_{30} to represent the respective angular velocities of gears *1*, *2*, and *3*, and N_1, N_2, and N_3 the respective numbers of teeth in the gears, then

$$\frac{\omega_{10}}{\omega_{20}} = \frac{N_2}{N_1}, \text{ and } \frac{\omega_{20}}{\omega_{30}} = \frac{N_3}{N_2}$$

Multiplying these equations together gives

$$\frac{\omega_{10}}{\omega_{30}} = \frac{N_3}{N_1}$$

or, the angular velocity ratio of *1* to *3* is the same with the idler as without. Idler gears are therefore used to change the direction of rotation without changing the angular velocity ratio.
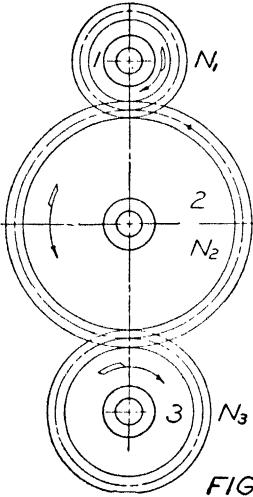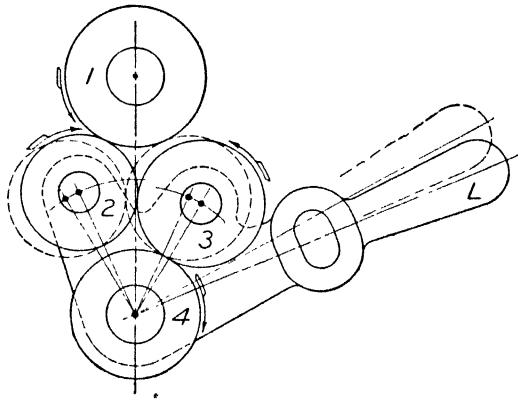
FIG. 480   FIG. 481

226. Reversing Mechanisms. In Fig. 481 a pair of *idler* gears, *2* and *3*, is used to control the relation of the directions of rotation of *1* and *4*. With the control lever *L* in the position shown, the four gears are in series, and the direction of rotation of *4* is opposed to that of *1*. To cause *1* and *4* to rotate in the same direction, gear *2* is thrown out of mesh and gear *3* into mesh with *1* and *4* by moving the control lever *L* to the position shown dotted. This is the reversing mechanism used in the head stock of the lathe shown in Fig. 474 to control the direction of rotation of the lead screw and feed rod, *7* being the control lever corresponding to *L*.
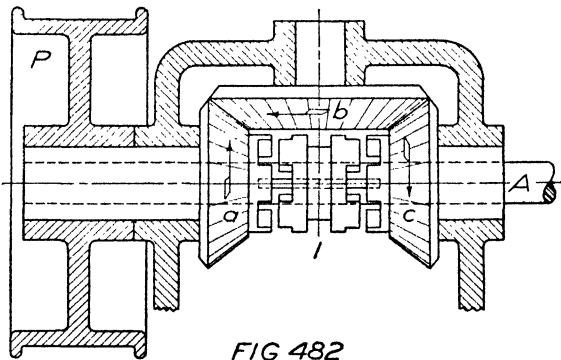
FIG 482

The reversing mechanism shown in Fig. 482 is composed of a double clutch and three bevel gears. The double jaw clutch *1* and shaft *A* are fitted with a spline. With the clutch out, the shaft is free to rotate in gears *a* and *c*. Gear *a* rotates with pulley *P* to which it is keyed, and gear *c* is driven by gear *b* in a direction opposed to that of *a*. To have the shaft rotate in the same direction as the pulley, *1* is moved to the left to engagement with *a*; to reverse the direction of rotation of the shaft, *1* is moved to the right to engagement with *c*.

227. Change Gear Mechanism. Changes in speed may be secured, as indicated in Arts. 223 and 224, by the use of sliding gears or clutches. Another way is to provide some convenient means, as shown in Fig. 483,
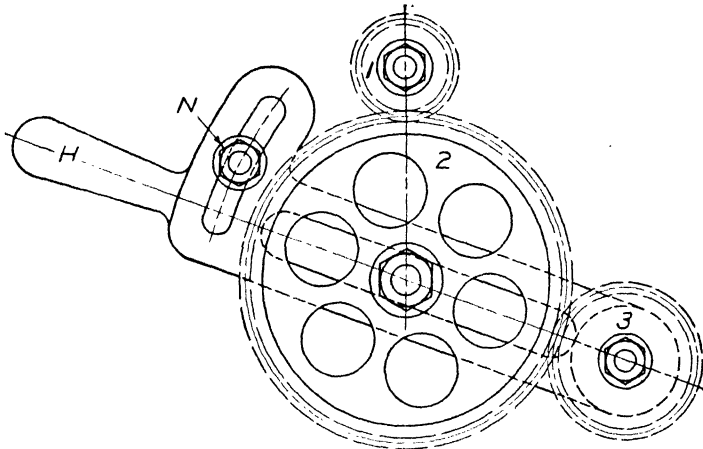
FIG. 483

of actually changing the size of gear on either or both the driven and driving shafts. The figure illustrates a change gear mechanism commonly used on lathes of the type shown in Fig. 474, where the revolutions of the lead screw per revolution of the lathe spindle are changed by changing the size of gear *12* on the lead screw. In Fig. 483, gear *3* corresponds to *12* of Fig. 474, and gear *1* to gear *11* which is driven from the lathe spindle by a reversing mechanism similar to that shown in Fig. 481. Since the shaft to which *1* is keyed and the lead screw to which *3* is keyed are a fixed distance apart, an adjustable idler gear *2* must be provided for connecting them. *H* is a slotted member which can be turned about the axis of *3* and locked in position by the washer and nut at *N*. The distance between the axes of *2* and *3* can be varied by moving the bolt on which *2* rotates along the slot in *H* and locking

it in the position desired. Member H therefore permits the ratio of gears *1* and *3* to be changed and the idler gear adjusted to connect them. The size of the idler may also be changed when desirable.

228. Gear Cones. Gear cones are often used, especially in feed trains, for making speed changes. In the feed train shown in Fig. 484, the gear cone *2* is made up by keying gears c, d, e, and f to the same spindle. In the figure, A is the driving and B the driven shaft. Through the double jaw clutch *1*, the gear cone may be driven by gear a or by gear b; and, by means of the snap key K, shaft B may be driven
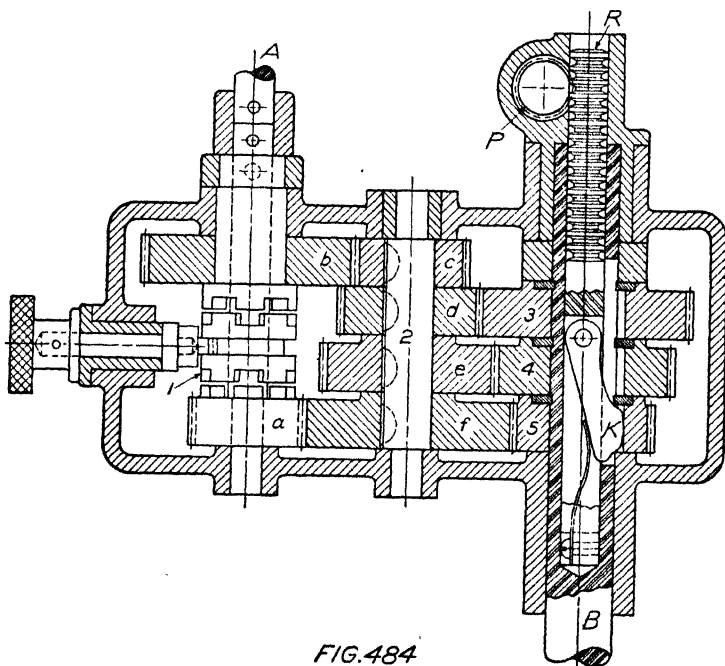
FIG. 484

through gears d and *3*, e and *4*, or f and *5*. The rod R, which carries the snap key, is fitted into the hollow portion of shaft B. The rod is shown in its lowest position with the key connecting gear *5* with shaft B. To disconnect *5* and connect *4* with shaft B, the rod and key are raised by turning the pinion P which meshes with the rack teeth turned on the upper end of the rod. With the clutch engaging gear a, the cone is driven at its lowest speed, and at its highest speed with the clutch engaging gear b. The two speeds of the cone and the three possible positions of the snap key make it possible for shaft B to be driven at six different speeds.

In Fig. 485 an idler *2* is used as a *tumbler gear* to connect gear *3* on shaft *B* with any step of the gear cone *1* on shaft *A*. This arrangement is extensively used in feed and lead screw trains and is occasionally used in the main drives of machine tools. Either shaft may be the driver. In the figure *A* is the driver. Shaft *B* may be fitted with a long spline, or the bushing to which gear *3* is keyed may be fitted with a short spline which slides in a keyway in the shaft. Member *C*, which carries the tumbler gear *2*, may be turned about the axis of shaft *B*. By sliding the tumbler mechanism along and swinging it about shaft *B*, gear *3* may be connected to any step of the cone. In the end elevation, *3* is shown connected to the largest step of the cone, the position for the highest rotative speed of shaft *B*. In the other view the tumbler
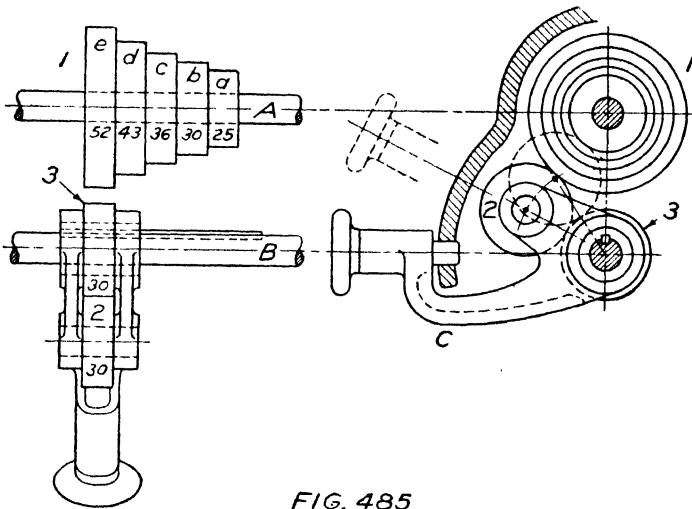
FIG. 485

mechanism is shown with the axis of *2* in the plane of *A* and *B*. The housing for the train is slotted and has a series of five holes into which the pin of the latch on the outer end of *C* may be inserted to lock the tumbler mechanism in any one of its five positions. Thus the ratio of the angular velocities of *A* and *B* may be $\frac{30}{52}$, $\frac{30}{43}$, $\frac{30}{36}$, $\frac{30}{30}$, or $\frac{30}{25}$.

229. Value of a Train. The value of a wheel train is the ratio of the angular velocities of the first and last or of the last and first wheels or shafts of the train.

It has already been shown that the angular velocities of a pair of pulleys, sheaves, or friction wheels are inversely as their radii or diameters, and that the angular velocities of a pair of gears are inversely as their pitch radii or pitch diameters or numbers of teeth. The spur

gear train shown in' Fig. 486 is composed of the single gears *1* and *4* and the compound gears *2* and *3*. For the gear train shown let ω_{10}, R_1, D_1, and N_1, respectively, represent the angular velocity, pitch radius, pitch diameter, and number of teeth of the driving gear *1*; ω_{20}, R_2 and R'_2, D_2 and D'_2, and N_2 and N'_2 the angular velocity, pitch radii, pitch diameters, and numbers of teeth of the compound
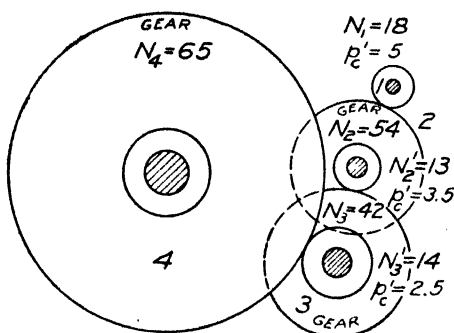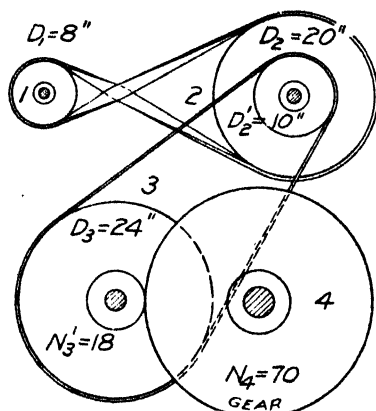
FIG. 486   FIG. 487

gear *2*; etc. Then the angular velocity ratios of the various pairs relative to the fixed member O are,

$$\frac{\omega_{10}}{\omega_{20}} = \frac{R_2}{R_1} = \frac{D_2}{D_1} = \frac{N_2}{N_1}$$

$$\frac{\omega_{20}}{\omega_{30}} = \frac{R_3}{R'_2} = \frac{D_3}{D'_2} = \frac{N_3}{N'_2}$$

$$\frac{\omega_{30}}{\omega_{40}} = \frac{R_4}{R'_3} = \frac{D_4}{D'_3} = \frac{N_4}{N'_3}$$

The product of these ratios is

$$\frac{\omega_{10}}{\omega_{40}} = \frac{\omega_{10}}{\omega_{20}}\left(\frac{\omega_{20}}{\omega_{30}}\right)\frac{\omega_{30}}{\omega_{40}} = \frac{R_2}{R_1}\left(\frac{R_3}{R'_2}\right)\frac{R_4}{R'_3} = \frac{D_2}{D_1}\left(\frac{D_3}{D'_2}\right)\frac{D_4}{D'_3} = \frac{N_2}{N_1}\left(\frac{N_3}{N'_2}\right)\frac{N_4}{N'_3} \quad (329)$$

That is, the ratio of the angular velocities of the first and last wheels or shafts of a wheel train is equal to the continued product of the radii, diameters, or numbers of teeth of the driven wheels divided by the continued product of the radii, diameters, or numbers of teeth of the driven at wheels. Since, for a given train, the angular velocity of the first

is usually known and the angular velocity of the last wheel is desired, the reciprocal of the above ratio, or the ratio of the angular velocities of the last and first wheels or shafts of a train, is often the more serviceable ratio. According to (329) this ratio is equal to the continued product of the radii, diameters, or numbers of teeth in the driving wheels divided by the continued product of the radii, diameters, or numbers of teeth in the driven wheels. Using the numbers of teeth shown in Fig. 486, the angular velocity ratio of the last to the first wheel or shaft is:

$$\frac{\omega_{40}}{\omega_{10}} = \frac{N_1}{N_2}\left(\frac{N'_2}{N_3}\right)\frac{N'_3}{N_4} = \frac{18}{54}\left(\frac{13}{42}\right)\frac{14}{65} = \frac{1}{45}$$

or gear 4 and the shaft to which it is keyed makes one revolution for every 45 revolutions of 1, the driver. Hence, if the angular velocity of the driver is 900 R.P.M., the corresponding angular velocity of 4 is

$$\omega_{40} = \omega_{10}(\tfrac{1}{45}) = 900(\tfrac{1}{45}) = 20 \text{ R.P.M.}$$

The same reduction in speed could, of course, be obtained by mating 1 with a gear having (45×18) or 810 teeth. It can easily be shown that the train has the advantage of being more compact and flexible. Assuming a diametral pitch of 5, the sum of the pitch diameters of a single pair of gears would be $\left(\dfrac{810 + 18}{5}\right)$ or 165.6 in., which corresponds to a distance between centers of 82.8 in. By using a train such as shown, the same reduction in speed is secured with smaller gears and in less space. It is to be noted, however, that the same diametral pitch cannot be used throughout the train. Since the load on the teeth of the pinion of 2 will be greater than the load on the teeth of 1, the former must have larger teeth than the latter; likewise the pinion of 3 must have larger teeth than the pinion of 2. The pitches given in the figure are for approximately the same factor of safety. For these pitches and the numbers of teeth shown, the respective distances between the centers of 1 and 2, 2 and 3, 3 and 4 are 7.20, 7.857 and 15.8 in. Assuming all the shafts of the train to be in the same plane gives a distance between the centers of the first and last shafts of 30.857 in. and an overall dimension of the pitch circles of 45.657 in. against corresponding values of 82.8 and 165.6 in. for a single pair of gears. By placing the shafts of the train in different planes, the largest overall dimension, neglecting eter addendums of the teeth, can be reduced to 34.8 in.

their pulley and gear train is shown in Fig. 487. The diameters of the

pulleys and the numbers of teeth for the gears are given in the figure. The value of the train for the proportions shown is

$$\frac{\omega_{40}}{\omega_{10}} = \frac{8}{20}\left(\frac{10}{24}\right)\frac{18}{70} = \frac{3}{70}$$

or gear 4 makes 3 revolutions for every 70 revolutions of 1, the driver. Hence, if the angular velocity of the driver is 700 R.P.M., the angular velocity of 4 is

$$\omega_{40} = \omega_{10}(\tfrac{3}{70}) = 700(\tfrac{3}{70}) = 30 \text{ R.P.M.}$$

As before pointed out, belts and pulleys are used instead of gears where positive driving is not essential and the distance between centers

FIG. 488   FIG. 489

is too great for gears; and belts and pulleys are used instead of chains and sprockets over center distances suitable to both where positive driving is not required.

In the trains shown in Figs. 486 and 487 all the shafts are parallel. In the gear train shown in Fig. 488, one of the shafts is at right angles to the other three. The train is composed of a pair of spur gears having 15 and 45 teeth, a pair of bevel gears having 25 and 30 teeth, and a single-thread worm mating with a worm wheel having 50 teeth. Since the definition of the value of a wheel train is perfectly general, the value for the train shown is

$$\frac{\omega_{40}}{\omega_{10}} = \frac{15}{45}\left(\frac{25}{30}\right)\frac{1}{50} = \frac{1}{180}$$

If the angular velocity of *1* is 1800 R.P.M., the corresponding angular velocity of *4* is

$$\omega_{40} = \frac{\omega_{10}}{180} = \frac{1800}{180} = 10 \text{ R.P.M.}$$

Pulley, sheave, and sprocket trains are used, as in the case of gear trains, to secure compactness and flexibility. Primary trains in several instead of a single step are also often desirable in order that secondary trains may be driven from one or more of the intermediate shafts of the primary train.

230. Direction of Rotation in a Wheel Train. The gears of a mating pair of external spur gears rotate in opposite directions. Hence the adjacent axes in a train of external gears rotate in opposite directions and the alternate axes in the same direction. If such a train has an even number of axes, it therefore follows that the first and last wheels or axes will rotate in opposite directions and that they will rotate in the same direction if the train has an odd number of axes. Thus *1* and *4* in Fig. 486 rotate in opposite directions.

The gears of a mating pair of internal gears rotate in the same direction. If such a pair were substituted for one of the pairs of external gears in Fig. 486, the first and last shafts would rotate in the same direction.

As was shown in Art. 225, idler gears are used to change the direction of rotation without changing the value of the angular velocity ratio.

Pulleys connected by an open belt rotate in the same direction, and pulleys connected by a crossed belt rotate in opposite directions. Although in Fig. 487 there is an even number of shafts, the first and last shafts rotate in the same direction because one pair of pulleys is connected by a crossed belt and the other pair by an open belt.

231. Back Gears. The spindle of the lathe shown in Fig. 474 is driven by a cone pulley and back gear train similar to the one shown in Fig. 489. This is a common speed change mechanism in machine tools and other machines using cone pulleys. In metal-working machines the cone pulley on the counter-shaft is usually the same size as the one on the machine. Pinion *a* and the cone pulley to which it is attached are free to rotate on the spindle when gear *d*, which is keyed to the spindle, is disconnected from the cone pulley. In the figure the compound gear *2*, which rotates freely on its spindle, is shown out of mesh with *a* and *d*. It can be put in mesh by turning handle *H* forward, the portion of the spindle on which *2* rotates being eccentric with respect to the ends which are supported in bearings in the frame. If the counter-shaft is a single speed shaft, the arrangement shown provides

eight changes in speed. With the back gears out and d connected to the cone, four different spindle speeds may be obtained by shifting the belt on the cone pulley, the highest spindle speed being obtained with the belt running from the largest step of the counter-shaft cone to the smallest step on the machine cone. By disconnecting d from the cone and placing the back gears in mesh with a and d, four more changes of speed are secured. The diameters of the pulley steps and the numbers of teeth on the gears are given in the figure. Assuming the counter-shaft to run at 200 R.P.M. and the belt progressively changed from the smallest to the largest step of the machine cone, the following four spindle speeds shown to the left are obtained:

$$200\left(\frac{10.75}{4.75}\right) = 452.63 \qquad\qquad 51.05$$

$$200\left(\frac{8.75}{6.75}\right) = 259.22 \qquad\qquad 29.22$$

$$200\left(\frac{6.75}{8.75}\right) = 154.28 \qquad\qquad 17.39$$

$$200\left(\frac{4.75}{10.75}\right) = 88.37 \qquad\qquad 9.96$$

The additional four speeds obtained by using the back gears are shown above to the right. These four speeds were obtained by multiplying the first four speeds by the reduction factor $\left(\frac{27}{85} \times \frac{22}{62}\right)$. It is to be noted that the eight speeds are approximately in geometric progression. An exact series, according to equation (321), would call for a progression ratio of

$$k = \sqrt[7]{\frac{452.63}{9.96}} = 1.725+$$

By using a two-speed counter-shaft, eight more changes in speed may be secured. If the possible sixteen speeds are to be in geometric progression, with 453 as maximum, the second speed of the counter-shaft must be

$$n = 200 \div \sqrt{1.725} = 152 \text{ R.P.M.}$$

Twelve spindle speeds in geometric progression may be obtained with 10 as the minimum and 453 as the maximum R.P.M. by changing the counter shaft speed, the diameters of the cone pulley steps, and substituting a compound sliding gear for a and a compound gear for b.

232. Geared Lathe Heads. In the earlier machine tools speed changes were secured by the use of cone pulleys and back gears. It is

now more common to drive the primary shafts of such machines at a constant speed and to obtain all changes in speed by means of gears. Such drives are known as constant speed drives. Two examples of this type of drive are shown in Figs. 490 and 491.

Fig. 490 is a diagrammatic representation of the geared head of a LeBlond lathe. The constant speed pulley is fitted with a multiple disk clutch similar to Fig. 478 by means of which shaft A is engaged or disengaged from the rotating pulley. •The clutch is operated by rod R which runs through shaft A to the clutch. Shaft A is splined for the triple sliding gear 1 composed of gears a, b, and c. Gears a', b', and c' are keyed to shaft B, which is splined for the double sliding gear composed of gears d and e. Gears d' and e' are keyed to C, the lathe spindle. Six different spindle speeds are possible. With d in mesh
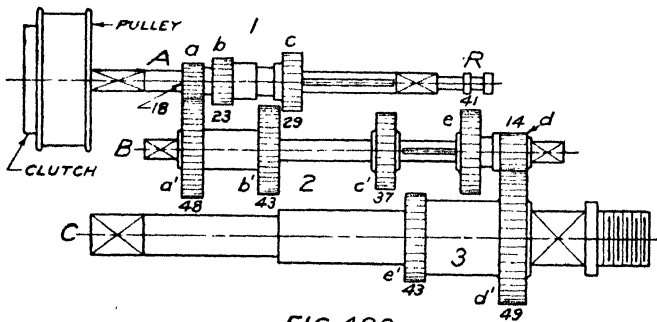
FIG. 490

with d' the three lowest spindle speeds in ascending order are obtained by placing a in mesh with a', next b with b', and then c with c'. The three next speeds in ascending order are obtained in the same manner by first placing e in mesh with e'. Assuming the pulley to rotate at 500 R.P.M., the six possible speeds for the numbers of teeth given in the figure are

53.5, 76.4, 112.0, 178.8, 255.0, and 372.5 R.P.M.

These speeds are approximately in geometric progression.

Fig. 491 is a diagrammatic representation of the geared head of a Reed–Prentice lathe. This head has eight forward and eight reverse speeds of the spindle D. Bevel gear a is attached to and rotates with the constant speed pulley. By means of the reversing mechanism at R, which is similar to the one shown in Fig. 482, shaft A may be caused to rotate in either direction. Each one of gears d, e, f, g, h, and k is fitted with a friction clutch similar to the one shown in Fig. 479. Unlike the head in Fig. 490, all the gears in the train are constantly

in mesh. By means of the double friction clutch at *1*, either gear *d* or gear *e* may be connected to shaft *A*. Gears *d'* and *e'* of the compound gear *2* are keyed to shaft *B*. Either *g* or *f* may be connected to the hollow shaft to which *h* is keyed by means of the double friction·clutch at *3*; and either *h* or *k* may be connected to the lathe spindle *D* by means of the double friction clutch at *5*. Gears *i* and *j* of the compound
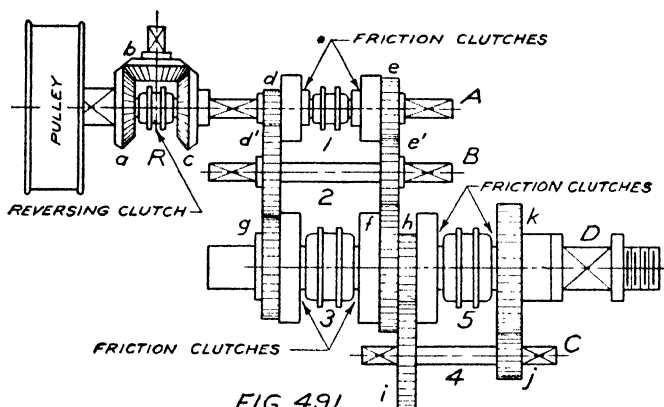
FIG. 491

gear *4* are keyed to shaft *C*. The lowest spindle speed is obtained by engaging gears *k*, *f*, and *d*, motion being transmitted from shaft *A* to *D* through gears *d*, *d'*, *e'*, *f*, *h*, *i*, *j*, and *k*. The next speed higher is obtained by disengaging *f* and engaging *g*. The next two speeds higher are obtained by first disengaging *d* and engaging *e* and then having first *f* and then *g* engaged. Four more speeds are obtained by disengaging *k* and engaging *h* and operating the clutches at *1* and *3* as for the first four speeds. Since friction clutches are used throughout, any speed change can be made without danger while the shafts are in motion. For the train shown in Fig. 490, the clutch connecting the pulley and shaft *A* should be disengaged before making a speed change.
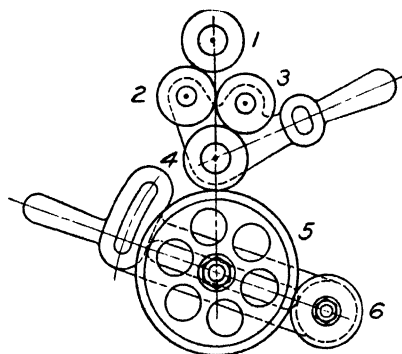
233. Screw-Cutting Train. The cutting of screw threads was explained in Art. 222 in discussing the lathe shown in Fig. 474. In modern lathes the required number of turns of the lead screw is commonly controlled by a gear train containing a cone and tumbler gear

FIG. 492

mechanism similar to that shown in Fig. 485. The earlier lathes of the type shown in Fig. 474 used a train similar to that shown in Fig. 492. This train is a combination of the mechanisms shown in Figs. 481 and 483. If *1* and *4* are of the same size and *1* makes the same number of turns as the lathe spindle, then the number of turns of gear *6*, which is keyed to the lead screw, will depend on the numbers of teeth in gears *4* and *6*. Assuming the lead screw to be a single-thread screw having 4 threads per inch, or a lead of $\frac{1}{4}$ in., the tool carriage would advance 1 in. for every 4 turns of the lead screw. To cut 4 threads per inch, gears *4* and *6* must therefore have the same number of teeth. To cut 9 threads per inch the spindle and gear *4* must make 9 turns while gear *6* and the lead screw make 4 turns; or the ratio of the angular velocities of gears *4* and *6* must be $\dfrac{\omega_{40}}{\omega_{60}} = \dfrac{9}{4}$. This ratio is also inversely as the numbers of teeth in the respective gears, or $\dfrac{\omega_{40}}{\omega_{60}} = \dfrac{N_6}{N_4} = \dfrac{9}{4}$. Hence, to cut n threads per inch with a lead screw having n' threads per inch

$$\frac{N_6}{N_4} = \frac{n}{n'} \quad \text{or} \quad N_6 = N_4\,\frac{n}{n'} \quad . \quad . \quad . \quad . \quad (330)$$

Corresponding values of n and N_6 are given below for $N_4 = 24$ and $n' = 4$:

$n =$	5	6	7	8	9	10	11	$11\frac{1}{2}$	12	13	and	14
$N_6 =$	30	36	42	48	54	60	66	69	72	78	and	84

By placing a 48-tooth gear on the stud with gear *4* the following values of N_6 for cutting 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, and 4 threads per inch are obtained:

$n =$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	and	4
$N_6 =$	24	30	36	42	and	48

Hence a set of 16 change gears, including two 48-tooth gears, are required to cut the numbers of threads per inch given above. The reversing mechanism in the train enables either right- or left-hand threads to be cut.

In some screw-cutting trains of the above type a compound gear is substituted for *5*, the idler gear. Although the screw-cutting trains of lathes of different make vary in type and in details of design and arrangement, the principles of all such trains are the same as the train just described.

Since the constancy of the angular velocity ratio of gears having

involute teeth is not affected by variation of the distances between centers, involute gears only should be used in change gear trains.

234. Automobile Transmission. A typical automobile transmission having three forward and one reverse speed is shown in Fig. 493. The speed changes are secured as in Fig. 490 by the use of sliding gears. Pinion *1* is formed on the end of shaft *A* which is coupled with the crank-shaft of the engine. The end of shaft *A* is hollowed out and bushed to serve as a supporting bearing for the end of shaft *D*. The left end of sliding gear *f* is formed to fit over the end of *1* and acts as a jaw clutch coupling. Shaft *D* is splined for the sliding gears *f* and *g*, and its right end is coupled to a shaft leading to the rear axle drive. Gears *a*, *b*, *c*, and *d* are keyed to counter-shaft *B*, forming the compound

FIG. 493
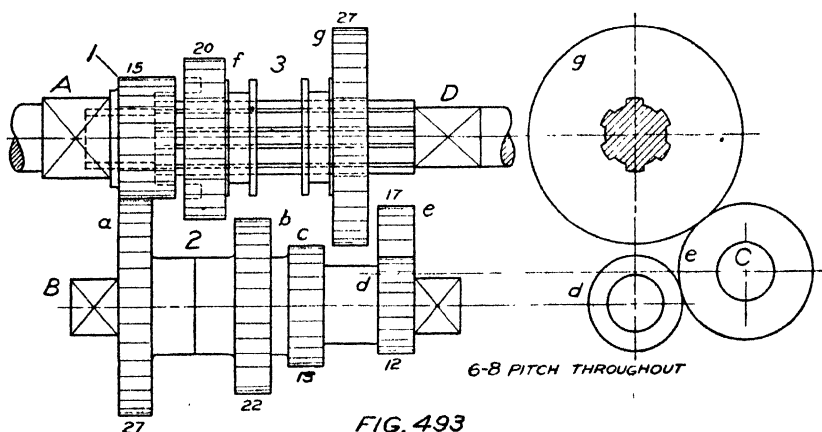
gear *2*. The reverse idler gear *e* is carried by stud *C*, the location of whose axis relative to the axes of *B* and *D* is shown in the end elevation. For low speed forward, gear *g* is shifted to the left into mesh with gear *c*, and *D* is driven by *A* through gears *1*, *a*, *c*, and *g*. For the second speed, gear *f* is shifted to the right into mesh with gear *b*, the drive being through gears *1*, *a*, *b*, and *f*. For the highest speed, gear *f* is shifted to the left over the end of *1*, making the drive from *A* to *D* direct. The reverse speed is secured by shifting *g* to the right into mesh with *e*, the drive being through gears *1*, *a*, *d*, *e*, and *g*. For the numbers of teeth shown the respective ratios of the speed of *D* to that of *A* are $(\frac{15}{27} \times \frac{15}{27}) = 0.309$, $(\frac{15}{27} \times \frac{22}{20}) = 0.611$, and 1.0 for the forward speeds, and $(\frac{15}{27} \times \frac{12}{27}) = 0.247$ for the reverse speed.

235. Cyclic Trains. A mechanism has been defined as a kinematic chain with one of its links fixed. The mechanism obtained from a given

kinematic chain depends, therefore, upon which link of the chain is fixed. This is just as true of a train of wheels as it is of any other train, or chain.

In Fig. 494, O and 2 are mating gears. With 1 as the fixed member, the chain becomes an ordinary gear train such as has been treated in the previous articles. With gear O as the fixed member the chain becomes a cyclic gear train. If 1 is rotated about the fixed center O_{10}, any point on the pitch circle of 2 will describe an epicycloid on O. If O were an internal gear as shown in Fig. 496, then any point on the pitch circle of 2 would describe a hypocycloid on O. Since either or both of these actions may be present, it is convenient to refer to wheel trains with one of the wheels fixed as cyclic trains. Such mechanisms are unique and at times very convenient and useful.
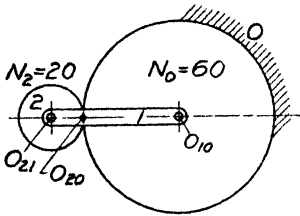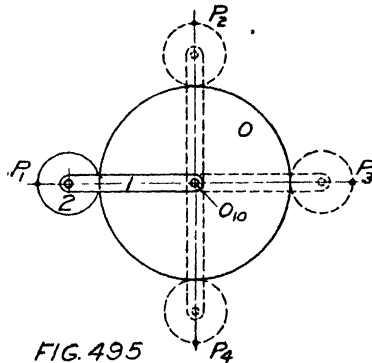
FIG. 494   FIG. 495

In cyclic wheel trains the information usually desired is the ratio, with respect to the fixed member, of the angular velocity of the last wheel of the train to that of the arm which carries it. With O in Fig. 494 as the fixed member, this ratio is evidently $\dfrac{\omega_{20}}{\omega_{10}}$. Now $\omega_{21} = \omega_{20} - \omega_{10}$; hence,

$$\omega_{20} = \omega_{10} + \omega_{21}$$

or,

$$\frac{\omega_{20}}{\omega_{10}} = \frac{\omega_{10} + \omega_{21}}{\omega_{10}} = 1 + \frac{\omega_{21}}{\omega_{10}}$$

But,

$$\omega_{10} = -\omega_{01}; \text{ hence}$$

$$\frac{\omega_{20}}{\omega_{10}} = 1 - \frac{\omega_{21}}{\omega_{01}}. \quad \ldots \ldots \ldots \quad (331)$$

It is to be noted that in equation (331) the ratio of the angular velocities of *2* and *1* relative to *O* as the fixed member is equal to unity minus the ratio of the angular velocities of *2* and *O* relative to *1* as the fixed member. Since with *1* fixed the mechanism is an ordinary wheel train, the ratio $\dfrac{\omega_{21}}{\omega_{01}}$ is evident by inspection. If the angular velocities ω_{21} and ω_{01} are opposite in sense, this ratio is negative; and it is positive if the angular velocities are of the same sense. Unity in the right-hand member of the equation means that *2*, solely because of its motion with *1* about O_{10}, makes one revolution relative to the fixed member for each revolution of *1*. This is shown in Fig. 495 where the four positions P_1, P_2, P_3, and P_4 of a point in *2* are given for one rotation about O_{10} with *O*, *1*, and *2* locked together. Because of this rotation about O_{10}, *2* has obviously made one revolution relative to the fixed member.

The angular velocity ratio $\dfrac{\omega_{20}}{\omega_{10}}$ for the cyclic train shown in Fig. 494 may be expressed in terms of the respective numbers of teeth N_0 and N_2 in gears *O* and *2* by substituting in equation (331). Since with *1* fixed *O* and *2* would rotate in opposite directions, the angular velocity ratio $\dfrac{\omega_{21}}{\omega_{01}}$ is negative; or

$$\frac{\omega_{21}}{\omega_{01}} = -\frac{N_0}{N_2}$$

Substituting this in (331) gives

$$\frac{\omega_{20}}{\omega_{10}} = 1 - \frac{\omega_{21}}{\omega_{01}} = 1 - \left(-\frac{N_0}{N_2}\right) = 1 + \frac{N_0}{N_2} = 1 + \frac{60}{20} = +4$$

That is, as shown by the sign, *2* rotates in the same direction and four times as fast as *1* relative to the fixed member *O*.

For the cyclic train shown in Fig. 496 the angular velocity ratio $\dfrac{\omega_{21}}{\omega_{01}} = +\dfrac{N_0}{N_2}$. Hence,

$$\frac{\omega_{20}}{\omega_{10}} = 1 - \frac{\omega_{21}}{\omega_{01}} = 1 - \frac{N_0}{N_2} = 1 - \frac{60}{20} = -2$$

That is, as shown by the sign, *2* rotates in the opposite direction and two times as fast as *1* relative to the fixed member *O*.

For the cyclic train shown in Fig. 497 the angular velocity ratio

$$\frac{\omega_{21}}{\omega_{01}} = \frac{N_0}{N_3} \times \frac{N_3}{N_2} = + \frac{N_0}{N_2}. \quad \text{Hence,}$$

$$\frac{\omega_{20}}{\omega_{10}} = 1 - \frac{\omega_{21}}{\omega_{01}} = 1 - \frac{N_0}{N_2} = 1 - \frac{60}{20} = -2$$

In other words, this train is the exact equivalent of the train of Fig. 496.

In solving cyclic trains, the two operations indicated by equation (331) may be performed and the results tabulated. The entire train is first locked and turned one revolution in a positive direction about O_{10}; the train is then unlocked, the supporting arm is fixed, and the gear which is to be the fixed member is rotated to its original position by giving it one turn in a negative direction, that is, in a direction opposite to that used for the locked train. By tabulating the turns of each
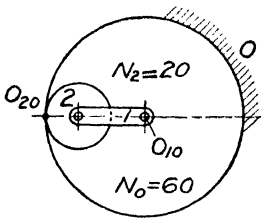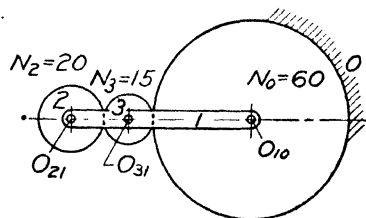
FIG. 496   FIG. 497

member for each of the two manipulations mentioned, the resultant turns of each member of the train are readily determined. Such a tabulation for the train of Fig. 497 is shown in Table 14.

TABLE 14

Operation	Number of Revolutions of Each Member			
	0	*1*	*3*	*2*
Entire train locked and given one positive turn about O_{10}	$+1$	$+1$	$+1$	$+1$
Train unlocked and O given one negative turn with *1* fixed	-1	0	$+\dfrac{N_0}{N_3}$	$-\left(\dfrac{N_0}{N_3} \times \dfrac{N_3}{N_2}\right) = -\dfrac{N_0}{N_2}$
Resultant turns for one positive turn of *1* with O fixed	0	$+1$	$\dfrac{\omega_{30}}{\omega_{10}} = 1 + \dfrac{N_0}{N_3}$ $= +5$	$\dfrac{\omega_{20}}{\omega_{10}} = 1 - \dfrac{N_0}{N_2} = -2$

In dealing with cyclic trains it should be very carefully noted that the sign of an angular velocity ratio is even more important than its numerical value.

236. Ferguson's Paradox. The cyclic train known as Ferguson's paradox is shown in plan and elevation in Fig. 498. The base is fitted with a vertical spindle to which gear O is fastened and about which arm *1* revolves. This arm carries two spindles. Idler *2* rotates about the first spindle carried by arm *1*, and the three separate gears *3*, *4*, and *5* rotate about the second spindle. The three separate gears are so nearly of the same size that they can be driven continuously by the same idler. If the drive of each separate gear is to be without variation of the angular velocity, involute gears must be used.
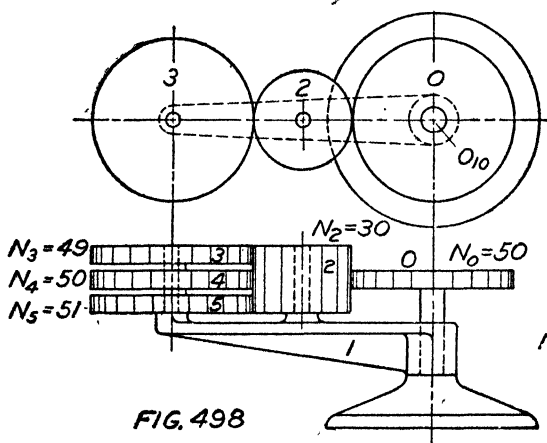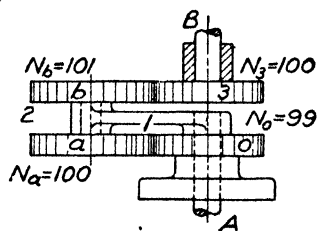
FIG. 498   FIG. 499

The angular velocity ratios desired are $\dfrac{\omega_{30}}{\omega_{10}}$, $\dfrac{\omega_{40}}{\omega_{10}}$, and $\dfrac{\omega_{50}}{\omega_{10}}$. These ratios may be obtained by applying the two manipulations explained in the previous article and tabulating the results as shown in Table 15. In the tabulation the numerical values of the number of teeth in the respective gears have been used.

It is to be noted that for each turn of *1* gear *3* makes $\frac{1}{49}$ of a turn relative to O in a direction opposed to that of *1*, gear *4* is stationary relative to O, and gear *5* makes $\frac{1}{51}$ of a turn relative to O in the same direction as *1*. Thus, paradoxically, two out of three gears driven by the same idler rotate in opposite directions, and the other is stationary relative to the fixed member. This mechanism demonstrates that in a cyclic train the directions of rotation of the gears relative to the fixed member are not determined by the number of axes.

TABLE 15

Operation	Number of Revolutions of Each Member					
	O	*1*	*2*	*3*	*4*	*5*
Entire train locked and given one positive turn about O_{10}	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
Train unlocked and O given one negative turn with 1 fixed	-1	0	$+\dfrac{50}{30}$	$-\left(\dfrac{50}{30}\times\dfrac{30}{49}\right)$	$-\left(\dfrac{50}{30}\times\dfrac{30}{50}\right)$	$-\left(\dfrac{50}{30}\times\dfrac{30}{51}\right)$
Resultant turns for one positive turn of 1 with O fixed	0	$+1$	$\dfrac{\omega_{20}}{\omega_{10}}=\dfrac{8}{3}$	$\dfrac{\omega_{30}}{\omega_{10}}=-\dfrac{1}{49}$	$\dfrac{\omega_{40}}{\omega_{10}}=0$	$\dfrac{\omega_{50}}{\omega_{10}}=+\dfrac{1}{51}$

The above ratios may also be obtained by writing expressions corresponding to equation (331) and making proper substitutions of the numbers of teeth in the gears. Thus,

$$\frac{\omega_{30}}{\omega_{10}} = 1 - \frac{\omega_{31}}{\omega_{01}} = 1 - \frac{N_0}{N_3} = 1 - \frac{50}{49} = -\frac{1}{49}$$

$$\frac{\omega_{40}}{\omega_{10}} = 1 - \frac{\omega_{41}}{\omega_{01}} = 1 - \frac{N_0}{N_4} = 1 - \frac{50}{50} = 0$$

$$\frac{\omega_{50}}{\omega_{10}} = 1 - \frac{\omega_{51}}{\omega_{01}} = 1 - \frac{N_0}{N_5} = 1 - \frac{50}{51} = \frac{1}{51}$$

237. Reverted Cyclic Train. The operation of a cyclic train is not affected by the shape of the carrying arm. In Fig. 497 center O_{21} is on $O_{10}O_{31}$ extended. It is evident that O_{21} could be on any line through O_{31} without affecting the operation of the train. When the axis of the last wheel of a cyclic train coincides with the axis of the fixed wheel, the train is called a *reverted* cyclic train. Such a train is shown in Fig. 499. Since the axes of the driving and driven shafts are coincident, the bearings supporting the shafts may be made integral with the frame to which the fixed gear is fastened. The convenience and adaptability of the mechanism are primarily due to this feature. In the figure, compound gear *2*, which is composed of gears *a* and *b*, is carried by arm *1* which rotates with shaft *A* to which it is keyed. Expressing the

desired angular velocity ratio in accordance with equation (331) and using the numbers of teeth shown, gives,

$$\frac{\omega_{30}}{\omega_{10}} = 1 - \frac{\omega_{31}}{\omega_{01}} = 1 - \left(\frac{N_0}{N_a} \times \frac{N_b}{N_3}\right) = 1 - \left(\frac{99}{100} \times \frac{101}{100}\right) = \frac{1}{10000}$$

That is, gear 3 and shaft B make one revolution for every 10,000 revolutions of arm 1 and shaft A. To obtain this very striking reduction in speed, it is to be noted that the sum of the numbers of teeth on gears b and 3 is slightly greater than the sum for gears O and a. This means that b and 3 must have slightly smaller teeth than O and a if the axes of shafts A and B are to coincide. Hence a standard pitch can be used for only one of the pairs of gears.

For both pairs of gears in Fig. 499 to have the same pitch, $(N_0 + N_a)$ must equal $(N_b + N_3)$. This will be true for $N_0 = 99$, $N_a = 101$, $N_b = 100$, and $N_3 = 100$. This apparently slight change makes, however, a great change in the ratio of transmission, the ratio becoming

$$\frac{\omega_{30}}{\omega_{10}} = 1 - \frac{\omega_{31}}{\omega_{01}} = 1 - \left(\frac{99}{101} \times \frac{100}{100}\right) = \frac{2}{101}$$

That is, shaft B makes 2 revolutions for every 101 revolutions of A.

If, in Fig. 499, A rotates 10,000 times as fast as B, it is to be noted that the torque on shaft B, ignoring friction, will be 10,000 times that on shaft A. Hence, if substantially the same size of tooth is to be used on both pairs of gears, the size should correspond to the maximum torque; otherwise, the size of the teeth on each pair of gears should correspond to the torques on the driving and driven shafts.

In practical applications of the reverted cyclic train, a spider carrying two or more gears equally spaced around O and 3 is generally used instead of a single arm or crank. This is done to balance the mechanism and to distribute the load to be carried over several teeth, thus reducing the size of the teeth and the gears. The use of a spider instead of a crank is illustrated in the applications to follow.

238. Applications of the Reverted Cyclic Train. Fig. 500 illustrates an application of the reverted cyclic train to a hand-operated hoist Similar trains are applied to power hoists. Sprocket S is rotated by pulling on a hand chain which passes over it. The rotation of S is transmitted to shaft A through the automatic sustaining brake R which is keyed to A. Shaft A rotates in the hollow shaft of the hoisting sprocket H. Pinion 3 is keyed to the right-hand end of A and meshes with gears b of the compound gears 2. The compound gears are carried by the spider 1 which is keyed to the hollow shaft of the hoisting sprocket

H. Gears a mesh with the fixed internal gear O. As pinion 3 rotates with A, the action of the compound gears with A and the fixed internal gear O causes the spider and the hoisting sprocket H to rotate. Let the numbers of teeth in the gears be as follows: $N_0 = 54, N_a = 14, N_b = 28$, and $N_3 = 12$. Taking the desired angular velocity ratio as $\dfrac{\omega_{10}}{\omega_{30}}$, then

$$\frac{\omega_{10}}{\omega_{30}} = \frac{\omega_{10}}{\omega_{10} + \omega_{31}} = \frac{1}{1 + \dfrac{\omega_{31}}{\omega_{10}}} = \frac{1}{1 - \dfrac{\omega_{31}}{\omega_{01}}}$$

or,

$$\frac{\omega_{10}}{\omega_{30}} = \frac{1}{1 - \left(-\dfrac{N_0}{N_a} \times \dfrac{N_b}{N_3}\right)} = \frac{1}{1 + \left(\dfrac{54}{14} \times \dfrac{28}{12}\right)} = \frac{1}{10}$$

$N_0 = 54$
$N_a = 14$
$N_b = 28$
$N_3 = 12$
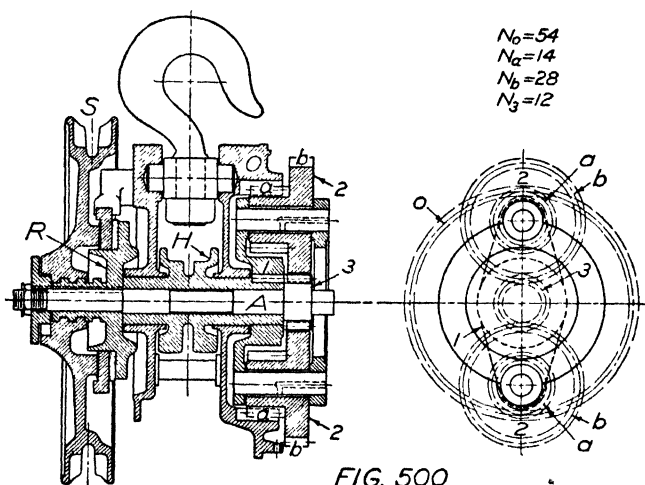
FIG. 500

That is, the hoisting sprocket H rotates in the same direction as the hand-operated sprocket S and makes one revolution for every 10 revolutions of the latter.

Fig. 501 illustrates an application of the reverted cyclic train to a speed-reducing unit. To balance the mechanism and to distribute the load over more teeth, three gears are carried by spider 1 which rotates with shaft B. Pinion 3 is keyed to or is made an integral part of the high revolution shaft A. Gears 2 mesh with the internal gear O which is held by the stationary frame or casing. Either $\dfrac{\omega_{10}}{\omega_{30}}$ or $\dfrac{\omega_{30}}{\omega_{10}}$ may be

regarded as the angular velocity ratio desired. For the numbers of teeth shown in the figure,

$$\frac{\omega_{30}}{\omega_{1c}} = 1 - \frac{\omega_{31}}{\omega_{01}} = 1 - \left(-\frac{N_0}{N_2} \times \frac{N_2}{N_3}\right) = 1 + \left(\frac{115}{46} \times \frac{46}{23}\right) = 6$$

That is, shaft A rotates 6 times as fast as shaft B and in the same direction.

The planetary transmission formerly used in Ford automobiles employed reverted cyclic trains of external gears similar to that shown in Fig. 499. The compound gear contained three instead of two gears;
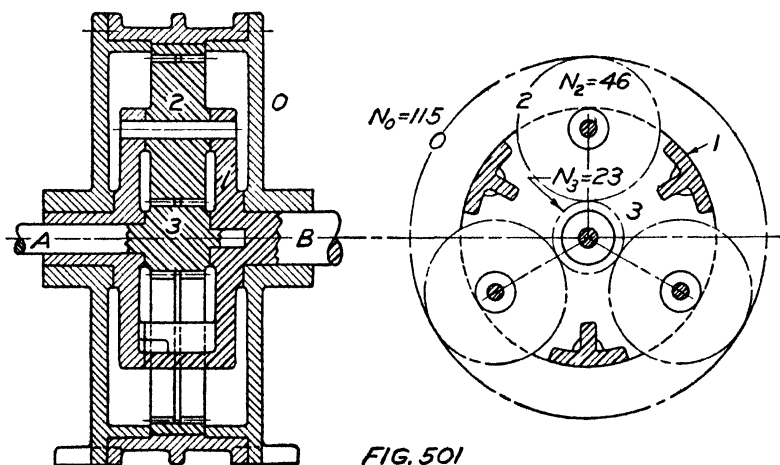
FIG. 501

and two gears, either one of which could be held stationary, were used to secure the second and reverse speeds.

239. Boring Bar Feed Train. In the use of one type of boring bar the work is held stationary and the head which holds the cutting tool is fed axially along while it rotates with the bar. A cyclic feed train is generally used to propel the tool head along the bar. Such bars are sometimes used in lathes. The piece to be bored is fastened to the carriage or bed of the lathe concentric with the boring bar which is mounted between the lathe centers. The bar rotates with the face plate of the lathe and drives its own feed train through gears mounted on the tail stock of the lathe. The cyclic feed train of such a bar is shown in Fig. 502. The boring bar 1 and the fixed gear O, which is fastened to the tail stock, are concentric with the line of centers XX of the lathe. The compound gear 2 is mounted on a pin carried by a lug or arm of the boring bar. Gear 3 is keyed to the feed screw S which

lies in a longitudinal groove in the bar and passes through a threaded lug on the tool head. The direction and magnitude of the feed of the tool head along the bar for each revolution of the bar will depend on the direction and number of revolutions of the feed screw relative to the bar. Hence the desired angular velocity ratio is

$$\frac{\omega_{31}}{\omega_{10}} = -\frac{\omega_{31}}{\omega_{01}} = -\left(\frac{N_0}{N_a} \times \frac{N_b}{N_3}\right)$$

Assuming the head stock to the left and the tail stock to the right, then, with the bar rotating forward, the tool head would have to move along the bar to the right to cut right-hand internal threads. Hence to cut 6 internal right-hand threads per inch with a single-thread lead screw having 8 right-hand threads per inch, the lead screw would have
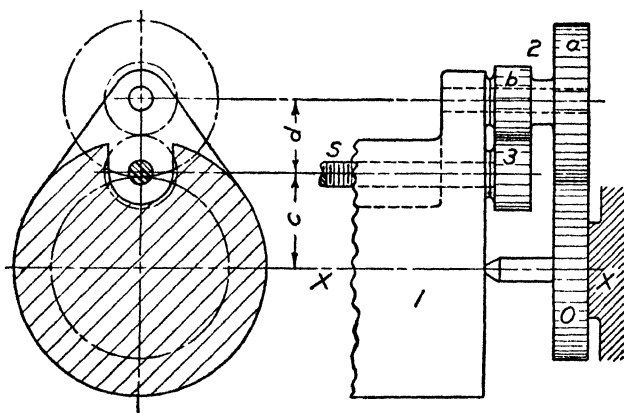
FIG. 502

to make 8 negative turns relative to the bar while the bar made 6 positive turns relative to O; or,

$$\frac{\omega_{31}}{\omega_{10}} = -\left(\frac{N_0}{N_a} \times \frac{N_b}{N_3}\right) = -\frac{8}{6} = -\frac{4}{3}$$

If $N_b = N_3 = 24$, then $\dfrac{N_0}{N_a} = \dfrac{4}{3}$. Assuming the gears to have the same pitch and that $\dfrac{d}{c+d} = \dfrac{3}{7}$, then

$$\frac{N_b + N_3}{N_0 + N_a} = \frac{3}{7} \quad \text{or} \quad N_0 + N_a = \frac{7}{3}(N_b + N_3) = \frac{7}{3}(24 + 24) = 112$$

But $N_0 = \dfrac{4}{3} N_a$; hence,

$$N_a(\tfrac{4}{3} + 1) = 112 \quad \text{or} \quad N_a = \tfrac{3}{7} \times 112 = 48 \quad \text{and} \quad N_0 = 64$$

240. Humpage's Gear. Thus far cyclic trains composed only of spur gears have been treated. As shown in Fig. 503, bevel gears may also be used in the construction of such trains. Though more difficult and expensive to construct than spur gear trains, they are, in general, more compact and are very flexible with respect to the transmission ratio and the direction of motion transmitted. Although four bevel gears are used in the reverted cyclic train shown in Fig. 503, only three gears are necessary to its operation. A reverted cyclic train of spur gears would require at least four gears to obtain the same speed reduc-
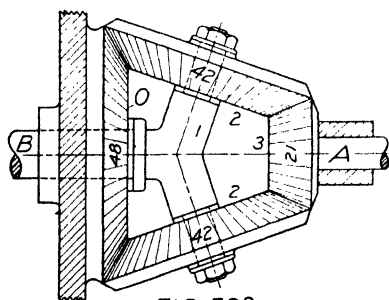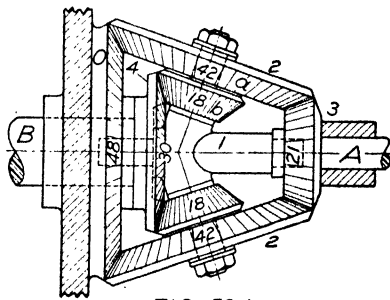
FIG. 503   FIG. 504

tion between the two shafts. In the figure, shaft A, to which gear 3 is attached, is the high revolution shaft, and O is the fixed gear. For balance 1 is fitted with two gears of the same size. The angular velocity ratio desired is between shafts A and B, or between 3 and 1, relative to the fixed member O. For the numbers of teeth shown,

$$\frac{\omega_{30}}{\omega_{10}} = 1 - \frac{\omega_{31}}{\omega_{01}} = 1 - \left(-\frac{N_0}{N_2} \times \frac{N_2}{N_3}\right) = 1 + \frac{N_0}{N_3} = 1 + \frac{48}{21} = \frac{23}{7}$$

That is, shafts A and B rotate in the same direction, A rotating $\tfrac{23}{7}$ times as fast as B.

The compound reverted cyclic train of bevel gears shown in Fig. 504 is called Humpage's gear. This mechanism has been used in machine tools and other machines as a speed-reducing gear. The desired

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That is, shafts *A* and *B* rotat
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tools and other machines as a **456**

QUESTIONS AND PROBLEMS

The questions and problems to follow are so numbered as to be readily associated with the articles to which they relate. Thus 6.1 is the first question or problem on Art. 6. 22.6 the sixth question or problem on Art. 22, 68.3 the third on Art. 68, etc.

CHAPTER I

FUNDAMENTAL CONCEPTIONS

1.1. Define Kinematics. Define Kinematics of Machinery. What has made the development of kinematics as a separate subject possible?

2.1. Name in proper sequence the four steps into which the design of a machine may be divided and explain the purpose of each step and how the steps are related.

3.1. What two kinds of quantities are dealt with in kinematics and how do th differ?

4.1. Define what is meant by a *vector* and explain the terms *origin* and *term,* Explain and illustrate what is meant by a *localized* vector, by a *free* vector. What a tage has the graphical over the purely mathematical method of solving problems?

5.1. Determine and label the vector sum of a vector K running northeast 2 u a vector L running south 1½ units. From the vector sum determined, subtract N running southeast 3 units.

6.1. (a) Define the terms *resultant* and *component* and explain what is m j phrases *composition of vectors* and *resolution of vectors.*

(b) Show graphically how the known motion of a point in a plane at a be resolved into two component motions whose directions are known.

(c) Show graphically how the known motion of a point in a plane any l be resolved into two component motions when the magnitude and direction of one components are known.

6.2. (a) Show graphically how the known motion of a point in plane at any instant can be resolved into two component motions whose magnitudes are known.

(b) Show graphically how the known motion of a point in a plane at any instant can be resolved into two component motions when the magnitude of one component and the direction of the other are known.

6.3. If the components of the motion of a point in a plane are 64 and 48 units in length and the resultant vector is 70 units in length, determine graphically the component vectors.

6.4. Determine the vector components making angles c 30 and 60 degrees on opposite sides of a resultant 450 units long.

6.5. Graphically determine the resultant of three components the first of which is 60 units long and is directed vertically upwards and the second and third are 80 and 100 units long and respectively directed 30 and 75 degrees to the right of the first.

6.6. A vector 40 units in length is to extend upward and to the right at an angle of 60 degrees with the horizontal:

(a) Resolve the vector into two components one of which is to have a length of 30 units and the other is to be directed to the right at an angle of 45 degrees with the horizontal.

(b) Resolve the vector into a horizontal and a vertical component.

(c) Resolve the vector into components 25 and 50 units long.

7.1. Define and illustrate what is meant by the path of a moving point having plane motion and show how the position of the point at any instant may be specified. Define and give examples of rectilinear, curvilinear, uniform, and non-uniform motion.

8.1. Define and illustrate what is meant by the displacement of a moving point and express the displacement in two vectorial equations. How may the direction of the displacement be expressed algebraically, and how by means of the expression can it be shown that the direction of motion of a point at any instant is tangent to the path at the point?

10.1. Define and illustrate angular displacement and express the relation between linear and angular displacement in a vectorial equation. To what in the limit does this relation reduce?

11.1. Define the terms *linear speed* and *linear velocity*. What kind of quantity is speed and what kind is velocity? Derive the mathematical expression for the linear speed of a particle at any instant.

11.2. (a) If a sprinter runs 100 yards in 10.4 seconds, what is his speed in miles per hour. *Ans.* 19.66 mi./hr.

(b) If a train is running at a constant linear speed of 54 miles per hour, express its speed in feet per minute, in feet per second, and in inches per second.
Ans. 4752 ft./min., 79.2 ft./sec., 950.4 in./sec.

A body moves in a straight path at a speed of $V = \frac{8}{27} t^3 + t^2 + 2t$, where V is in feet per second and t is in seconds. How far in feet will the body move from rest in 3 seconds? *Ans.* 112.96 ft.

Define and illustrate what is meant by the angular velocity of a point. In what units may angular velocity be expressed? Derive the differential equation expressing the angular velocity of a moving point or body at any instant.

From the fundamental differential equations for the angular velocity and linear speed of a point moving in a path whose radius of curvature at the point is R, derive the mathematical relation between the angular velocity and linear speed.

13.2. A boy is turning a grindstone 28 inches in diameter at a uniform rate of 30 R.P.M. Determine the angular velocity of the grindstone in radians per second and the linear speed of a point on its periphery in feet per minute. *Ans.* π rad./sec., 220 ft./min.

13.3. An engine with a stroke of 18 inches runs at 210 R.P.M. Determine the linear speed of the crank-pin center in feet per second, the angular velocity of the crank in radians per second, and the mean piston speed in feet per minute.
Ans. 16.5 ft./sec., 22 rad./sec., 630 ft./min.

13.4. A drill manufacturer states that mild steel is drilled to best advantage at a peripheral speed of 120 feet per minute for high speed drills. Determine the number of revolutions per minute that should be used for drills $\frac{1}{2}$, $\frac{7}{8}$, $1\frac{3}{8}$, and $2\frac{1}{4}$ inches in diameter.
Ans. 916.4, 523.6, 333.2, 203.6 R.P.M.

13.5. The difference in the linear speeds of two points on a radial line in the plane of motion of a wheel having an angular velocity of 105 R.P.M. is 3300 inches per minute. How far apart are the two points? *Ans.* 5 in.

14.1. Where, for a point having curvilinear motion, the center of curvature of the curve at the point is not taken as the origin of the radius vector, how are the transverse and radial components of the velocity and the axial components of the velocity expressed?

15.1. Define the linear acceleration of a moving point. To what may the linear acceleration of a moving point be due? Explain what changes in the velocity of a moving point take place in variable rectilinear motion, in uniform curvilinear motion, and in variable curvilinear motion? In what units may the acceleration of a moving point be expressed?

16.1. For uniformly accelerated rectilinear motion derive algebraic expressions for:
(a) the velocity at any instant in terms of the initial velocity, acceleration, and time;
(b) the velocity in terms of the initial velocity, acceleration, and displacement;
(c) the displacement in terms of the initial velocity, acceleration, and time;
(d) the displacement in terms of the initial velocity, the final velocity, and the time.

16.2. For non-uniformly accelerated rectilinear motion derive differential expressions for:
(a) the velocity at any instant in terms of displacement and time;
(b) the acceleration in terms of velocity and time and in terms of displacement and time;
(c) the acceleration in terms of velocity and displacement.

16.3. In starting from rest on a straight horizontal road an automobile is given a constant linear acceleration of $4\frac{1}{2}$ feet per second per second. For how many seconds must it be accelerated to attain a speed of 42 miles per hour and how many feet will it hav passed over? *Ans.* 13.68 sec., 421 ft.

16.4. A suburban train travels 2 miles in a direct line between station stops in 6 r utes. If it is given a constant acceleration along the track for three-fourths of the tance and a constant retardation for the last fourth, determine the acceleration retardation, and the maximum velocity attained.
Ans. 0.2173 ft./sec./sec., 0.6518 ft./sec./sec., 58.67 ft

16.5. In 2.75 feet the speed of a point moving in a rectilinear path uniformly from 10 to 34 inches per second. Determine the time interval in seconds and th tion in inches per second per second. *Ans.* 1.5 sec. and 16 in.

17.1. Define simple harmonic motion and derive algebraic expressions for ment, velocity, and acceleration of a point having simple harmonic motio terms *amplitude*, *period*, and *frequency*, and cite an example of a body m ple harmonic motion.

17.2. Assuming a point to be moving at a uniform rate in a circular path 8 inch diameter at 90 R.P.M., determine the numerical value of the velocity and acceleratio of the projection of the point on a horizontal diameter when the rotational radius of the point is at 45 degrees and when at 90 degrees with the horizontal.
Ans. 2.22 and π ft./sec., 20.9 and 0.00 ft./sec./sec.

18.1. (a) Draw a figure representing a point having variable curvilinear motion, and draw the vector diagram representing the tangential and normal changes of velocity of the point.
(b) Derive the differential equation for the tangential acceleration and the algebraic expression for the normal acceleration of the point. How are these acceleration components directed with respect to the radius of curvature of the path? Write the vector equation and draw the vector diagram for the total acceleration of the point.

18.2. (a) For a point having variable curvilinear motion, tell what is meant by the tangential and by the normal acceleration of the point and tell to what changes these acceleration components are due. How are these acceleration components directed with respect to the radius of curvature of the path? Express the direction of the acceleration in terms of its components.
(b) Write the vectorial and algebraic equations representing the total acceleration of the point and draw the corresponding vector diagram for the acceleration.

18.3. The speed of a point moving in a circular path 21 inches in diameter uniformly increases in one-half revolution from 10 to 34 inches per second. Determine the tangential component of acceleration, the maximum value of the normal component of acceleration, and the corresponding total acceleration of the moving point and the angle which the total acceleration makes with the radius of curvature.

         Ans. 16 in./sec./sec., 110.1 in./sec./sec., 111.25 in./sec./sec., and 8° 16′.

18.4. An automobile in going around a right angle turn of 100-feet radius uniformly decreases in speed from 20 miles to 16 miles per hour. What is the total linear acceleration of the automobile in miles per hour per second at the beginning and also at the end of the retardation period? How is the acceleration at the end of the retardation period directed with respect to the radius of curvature?

         Ans. 5.904 mi./hr./sec., 3.814 mi./hr./sec., and 9° 59′ +.

19.1. (a) What components of acceleration other than the normal and tangential are often found convenient especially in the mathematical analysis of the motion of a point? State the differential equations for these components in terms of the velocity components and time and in terms of the component displacements and time.

     (b) Give the algebraic equations for the total acceleration and for the angle which the total acceleration makes with the Y axis.

20.1. For uniformly accelerated motion of a point in a circular path, derive the algebraic expressions for:

     (a) the angular velocity at any instant in terms of the initial angular velocity, the ular acceleration, and the time;

) the angular velocity in terms of the initial angular velocity, the angular acceleration and the angular displacement;

     the angular displacement in terms of the initial angular velocity, the time, and the acceleration;

     e angular displacement in terms of the initial angular velocity, the final angular nd the time.
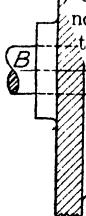
     non-uniformly accelerated motion of a point in a circular path or in a path of curvature at the instantaneous position of the point is known, derive the ressions for:

     ngular acceleration at any instant in terms of the angular velocity and the in terms of the angular displacement and time;

     )) the angular acceleration and displacement in terms of the angular velocity.

20.3. In starting from rest the flywheel of a stationary steam engine attains its full speed of 112 R.P.M. in 45 seconds. Assuming the angular acceleration to be constant, determine its value in radians per second per second and determine the number of revolutions made by the flywheel in coming to full speed.   *Ans.* 0.261 rad./sec./sec., 42 rev.

21.1. For accelerated curvilinear motion of a point, derive the algebraic expression for the tangential acceleration at any instant in terms of the radius of curvature and the angular acceleration. Express the normal acceleration in terms of the angular velocity and radius of curvature and determine the total acceleration in terms of the radius of curvature and the angular velocity and acceleration.

21.2. Determine the value in radians per second per second of the angular acceleration of the rotational radius of the point in problem 18.3.   *Ans.* 1.524 rad./sec./sec.

21.3. Determine the maximum and minimum values of the angular velocity and also the value of the angular acceleration of the automobile in problem 18.4.

       Ans. 0.2933 rad./sec., and 0.2347 rad./sec.; 0.009859 rad./sec./sec.

22.1. Define the relative motion of a point or body. In mechanics and kinematics what relative motion is taken as the absolute motion of a point or body? What is meant by a reference body?

22.2. The driving wheels of a steam locomotive are 5 feet in diameter and the stroke of the piston is 24 inches. Calculate in feet per second the average linear speed of the piston relative to the locomotive frame when the locomotive runs along the track at the rate of 40 miles per hour. Determine the average linear speed in feet per second of the piston relative to the ground on the forward stroke of the piston, also on the back stroke.

Ans. 14.93 ft./sec., 73.60 ft./sec., 43.74 ft./sec.

22.3. On a freight train running forward at 20 miles per hour is a brakeman walking toward the head end of the train at 4 miles per hour. What is the linear speed of the brakeman relative to the track? What would be his speed relative to the track if the train were backing at 20 miles per hour? What would be his speed relative to the track in each case if he were walking toward the rear end of the train at 4 miles per hour?

Ans. 24 mi./hr., − 16 mi./hr., 16 mi./hr., − 24 mi./hr.

22.4. A locomotive running at 30 miles per hour has driving wheels 63 inches and pilot wheels 28 inches in diameter.

(a) What is the angular velocity in revolutions per minute of the pilot wheels relative to the drivers?   *Ans.* 200 R.P.M.

(b) What is the angular velocity ratio of the pilot wheels to drivers?   *Ans.* 2.25.

(c) What is the linear speed of the highest point in the tread of the drivers relative to the corresponding point on the pilot wheels?   *Ans.* Zero.

22.5. A 21-inch loose pulley rotates clockwise at 150 R.P.M., and a 14-inch loose pulley on the same shaft rotates counter-clockwise at 300 R.P.M. Considering counterclockwise rotation as positive:

(a) What is the peripheral speed in feet per second of the 14-inch pulley relative to the 21-inch pulley?   *Ans.* 32.08 ft./sec.

(b) What is the angular velocity in radians per second of the 21-inch relative to the 14-inch pulley?   *Ans.* − 47.14 rad.

(c) What is the ratio of the angular velocity of the 21-inch to the 14-inch pulley?   *Ans.* ...ion vec-

(d) Answer (a), (b), (c) assuming both pulleys to rotate counter-clockwise.
          Ans. 4.58 ft./sec.; − 15.7 rad./sec., ation, the

22.6. An automobile with 30-inch wheels is traveling at 20 miles per hour.

(a) What is the maximum linear velocity in feet per second that any point may have relative to the frame?   *Ans.* 29.33

(b) What is the maximum linear velocity in feet per second that any point may have relative to the ground?   *Ans.* 58.66 ft./sec.

(c) What is the angular velocity in radians per second of a wheel relative to the ground?
          Ans. 23.46 rad./sec.

22.7. A man walks south at 4 miles per hour over a bridge placed at right angles to the river which flows west at 2 miles per hour. Determine by means of vectors the velocity in miles per hour of the man relative to the river and of the river relative to the man.

Ans. southeast 4.47 mi./hr. and northwest 4.47 mi./hr.

22.8. A man is riding a motor-cycle north 45 degrees east in a wind from the west of 20 miles per hour. If to the man the wind appears to be blowing south 30 degrees east, what is the speed of the motor-cycle and what is the velocity of the wind relative to the man?   *Ans.* 17.936 mi./hr. and 14.64 mi./hr.

23.1. Define and cite an example of free motion, of constrained motion, of unconstrained motion.

24.1. Define the three terms, *motion cycle, period,* and *phase,* and cite an example illustrating the terms.

25.1. Define the three terms, *continuous motion, intermittent motion,* and *reciprocating motion,* and cite examples illustrating these different kinds of motion.

27.1. Define what is meant by the plane motion of a body. Explain why the motion of every point in a body having plane motion may be fully represented by the motion of a plane figure. How many kinds of motion are there?

28.1. Define what is meant by plane motion of rotation of a body and explain what is meant by a gauge line for such a motion.

29.1. Define the four terms, *translation, plane translation, plane rectilinear translation,* and *plane curvilinear translation.* State what is true with respect to displacement, velocity, and acceleration for the points of a body having a motion of translation. Cite examples of plane rectilinear and curvilinear translation.

30.1. Illustrate by means of a sketch and explain how any plane motion of a body may be reduced to a combination of translation and rotation.

30.2. What kind of motion has the piston of a steam engine relative to the cylinder; the crank relative to the frame; the connecting-rod relative to the crank; the connecting-rod relative to the cross-head; the connecting-rod relative to the frame?

31.1. Define the two terms, *helical motion* and *regular helical motion.* Cite an example of the latter.

32.1. Define what is meant by spherical motion and cite an example.

33.1. Explain how the two limits of helical motion are rectilinear translation and plane rotation and how plane motion is a limiting case of spherical motion.

CHAPTER II

TRANSMISSION OF MOTION
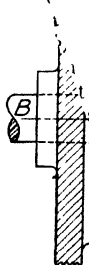
(a) Name and give examples of the three possible kinds of contact between the parts of machines. Define the terms, *kinematic pair, higher pair,* and *lower pair,* concrete illustration of each term.

plain what is meant by the *elements* of a pair. Give concrete examples of a a *sliding pair,* a *screw pair,* and a *spherical pair.* Tell what is meant by and of a *closed pair,* an *unclosed pair.*

why is it usually desirable, if practicable, to substitute lower for higher pairing the construction? Show by means of three sketches of the same simple mechanism an undesirable higher pairing can be improved and how it can be changed to ower pairing without affecting the related motions of the parts. Cite some examples of where higher pairing cannot be avoided. Between what three kinds of surfaces only is it possible to have surface contact?

34.3. By means of a sketch of a cam and its follower show and explain how the operation of the mechanism can be greatly improved by the introduction of an intermediate part. Cite an example of where higher pairing is substituted to advantage for lower pairing and tell why the substitution is effective and desirable.

35.1. Define the term *kinematic link.* Explain what is meant by a *binary link,* by a *ternary link,* by a *quaternary link.* What kind of a link, if the valve gear is neglected, is the frame of a simple steam engine? Of how many and of what kind of elements of pairs is the frame composed? Name the other kinematic links of the engine and the pairing elements in each link. By means of a sketch show how the links of the engine are represented diagrammatically.

36.1. Define the term *chain.* Explain the difference between a *structure,* or *locked chain,* a *kinematic chain,* and an *unconstrained chain.* Make a sketch of each kind of chain.

36.2. The criterion of constraint is written $J + \frac{1}{2}H = \frac{3}{2}N - 2$. Explain the notation in this equation and tell, in obtaining values for J, what equivalents are used for

ternary and quaternary joints. How does the equation indicate whether a chain is locked, constrained, or unconstrained?

36.3. (*a*) Determine by use of the criterion of constraint, $J + \frac{1}{2}H = \frac{3}{2}N - 2$, if the chains shown in Figs. 36.3 (*a*), 36.3 (*b*), and 36.3 (*c*) are locked, constrained, or unconstrained.

(*b*) How many links has the simplest complete kinematic chain? What are kinematic
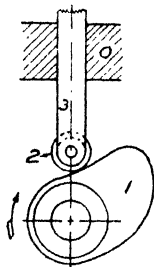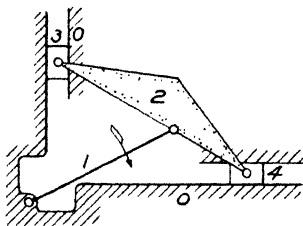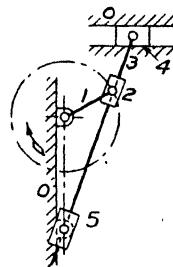
FIG. 36.3 (a)   FIG. 36.3 (b)   FIG. 36.3 (c)

chains called that have fewer links than the basic chain; that have more links than the basic chain?

37.1. What is meant by the term *mechanism*? How many mechanisms may be r from a kinematic chain of *n* links? What term is applied to this process of obtaining ferent mechanisms from the same kinematic chain?

38.1. Make sketches illustrating the various inversions of the slider crank ch name for each of the inversions at least one machine in which the particular inv mechanism is used.

39.1. (*a*) Define the term *machine*. What distinguishes a machine from a or combination of mechanisms? Name some machine elements that are form the mechanisms to be found in machines.

(*b*) In analyzing the relative motions of the parts of a machine, what member regarded as fixed? Cite an example of a machine part whose complete constrain, not due to the material connections. Cite an example showing that the parts of a mac be are not necessarily, to the best advantage, completely constrained.

40.1. State the four different ways in which motion can be transmitted from one body to another and give examples of each method of transmitting motion.

41.1. Define the terms *driver* and *follower*. Make sketches of mechanisms illustrating the three methods of transmitting motion by material connection, indicating the driver and follower in each and their directions of rotation.
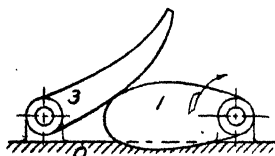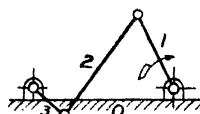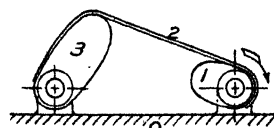
FIG. 42.1 (a)   FIG. 42.1 (b)   FIG. 42.1 (c)

42.1. Define the *line of transmission* and give its location for each of the mechanisms shown in Figs. 42.1 (*a*), 42.1 (*b*), and 42.1 (*c*).

43.1. State the relation of the angular velocities of driver and follower for the three methods of transmitting motion by material connection and prove the relation for Fig. 42.1 (*a*).

43.2. Answer question 43.1 using Fig. 42.1 (*b*).

43.3 Answer question 43.1 using Fig. 42.1 (*c*).

43.4. After drawing in the necessary construction lines, properly letter Figs. 42.1 (*a*), 42.1 (*b*), and 42.1 (*c*), and write for each the two relations of the angular velocities in linear terms.

44.1. State the relation of the angular velocities of driver and follower for the three methods of transmitting motion by material connection, and then state the requirement for a constant ratio of the angular velocities. For what class of direct contact members is this relation of fundamental importance? Where motion is transmitted from driver to follower by a rigid connector, what must be the proportions for constant angular velocity ratio; what for transmission of motion by a flexible connector?

45.1. For the three methods of transmitting motion by material connection, when are the directions of rotation of driver and follower the same; when different? Draw sketches of mechanisms illustrating the three methods of transmission, first for the driver and follower rotating in the same direction and then for rotation in different directions.

46.1. For pure rolling to take place between two direct-contact members, what must be true of the lengths of the contact arcs? For pure rolling what must be true of the contact of the two members relative to a contact point on either one; what must be true for sliding and for combined rolling and sliding?

47.1. For the mechanism shown in Fig. 47.1, assume a linear velocity for point *Q* and and prove how the velocity of sliding between *1* and *2* may be found for the phase

FIG. 47.1   FIG. 47.2

FIG 47.3

FIG. 47.4   FIG. 47.5
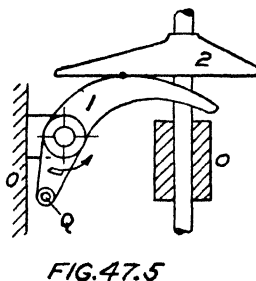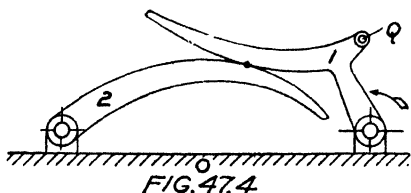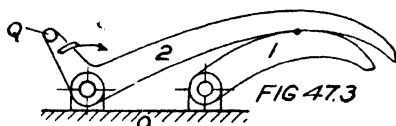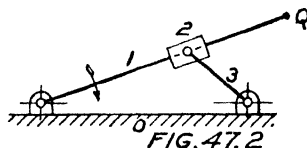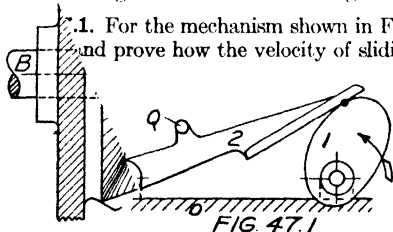
shown by the method of resolution and composition of vectors. Also show how the velocity of sliding may be obtained by the vector subtraction of the velocities of the coincident points of contact.

47.2. Answer question 47.1 using Fig. 47.2.

47.3. Answer question 47.1 using Fig. 47.3.

47.4. Answer question 47.1 using Fig. 47.4.

47.5. Answer question 47.1 using Fig. 47.5.

48.1. State and prove the requirement for pure rolling for direct-contact members. Name three pairs of curves capable of transmitting motion from one to the other with pure rolling action. State and prove the three ways in which the angular velocity ratio may be expressed for direct-contact members having pure rolling action.

49.1. Why are circles or circular arcs rotating about their geometric axes the only class of curves between which there can be pure rolling combined with the transmission of motion at a constant angular velocity ratio?

50.1. Explain what is meant by positive driving. In direct-contact mechanisms, what component only of the motion of the driver can cause motion of the follower? State and prove the criterion for positive driving.

50.2. State the criterion for positive driving and determine, by applying the criterion to the direct-contact mechanisms shown in Figs. 47.1, 47.4, and 47.5, if positive driving exists for the phases shown.

CHAPTER III

ANALYSIS OF PLANE MOTION

51.1 State the five fundamental principles whose application is necessary to an analysis of plane motion.

52.1. How must the velocities of any two points in a body having plane motion relative to another body be related? Assuming the velocities of two points so related, determine and prove the location of the instant center of one body relative to the other. With respect to two bodies, what can be said about their instant center or instant axis?

53.1 and **54.1.** Describe what is meant by *fixed* and *permanent* centers and those that are neither fixed nor permanent, and illustrate the same by using the slide-crank mechanism of a simple engine as an example. Explain and interpret the notation, the notation, the to be used in designating the links and instant centers of a mechanism. What vector

55.1. To locate the instant center of one body relative to another what information only is necessary respecting the velocities of any two points in one of the bodies when the velocities of the points are not perpendicular to the line joining the points; what must be known respecting the velocities where the velocities of the two points are perpendicular to a line joining the points? Why cannot the velocities of any two points in a body relative to some other body be parallel and unequal unless perpendicular to the line joining the points? What is the motion of one body relative to another where the velocities of any two points in one body are parallel and equal?

55.2. Where a locomotive is running on the track without slipping, what is the location of the instant center of any one of its drive wheels relative to the track, and what is the velocity of the highest point on the drive wheel in terms of the velocity of the locomotive? What is the location of the instant center of any drive wheel relative to the track, if, at starting, the drive wheels rotate without advancing the locomotive and train? If there is both rolling and sliding between the drive wheels and rails and the peripheral velocity of the drivers relative to the locomotive frame is greater than the velocity of the locomotive, what would be the location of the instant center of a drive wheel relative to the track and the velocities of the lowest and highest points on the drive wheel in terms of the known velocities?

56.1. Without the aid of Kennedy's Theorem, determine and fully justify the method of determining the locations of the instant centers O_{20} and O_{13} of the kinematic chain shown in Fig. 56.1.
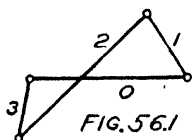
FIG. 56.1
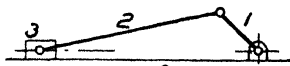
FIG. 56.2

56.2. Without the aid of Kennedy's Theorem, determine and fully justify the method of determining the locations of the instant centers O_{20} and O_{13} of the kinematic chain shown in Fig. 56.2.

57.1. Express, in the form of an equation, the total number of instant centers N for a kinematic chain having n links and then explain how the equation is deduced.

58.1. First state, and then make a suitable sketch and prove, Kennedy's Theorem.

58.2. For the kinematic chain shown in Fig. 56.1, determine the total number of instant centers to be found and then locate all the centers, making use of Kennedy's Theorem. State how and to which centers the theorem was applied.

58.3. Same problem as 58.2 but applied to Fig. 56.2.

59.1. For the mechanism shown in Fig. 59.1 determine the total number of instant centers to be found and then locate all the centers, making use of Kennedy's Theorem. State how and to which centers the theorem was applied.
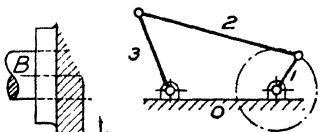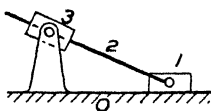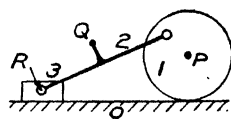
FIG. 59.1   FIG. 59.2   FIG. 59.3
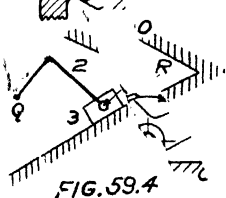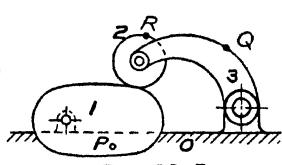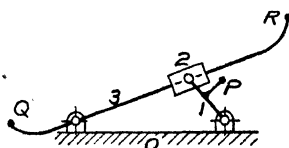
FIG. 59.4   FIG. 59.5   FIG. 59.6

59.2. Same problem as 59.1 but applied to Fig. 59.2.

59.3. Same problem as 59.1 but applied to Fig. 59.3.

59.4. Same problem as 59.1 but applied to Fig. 59.4.

59.5. Same problem as 59.1 but applied to Fig. 59.5.

59.6. Same problem as 59.1 but applied to Fig. 59.6.

60.1. For the incomplete kinematic chain shown in Fig. 47.1 first determine the total number of instant centers to be found and then determine their locations giving a full explanation of the method used.

60.2. Same problem as 60.1 but applied to Fig. 47.4.

60.3. Same problem as 60.1 but applied to Fig. 47.5.

61.1. How does a mechanism differ from a kinematic chain and how many mechanisms can be made from a given chain? State what is meant by inversion and illustrate the same by skectches of an assumed chain. Explain the effect of inversion on the location of the instant centers.

62.1. For the mechanism shown in Fig. 62.1 determine the total number of instant centers to be found, analyze the mechanism, and by the aid of charts and the application of Kennedy's Theorem locate the instant centers. State, for the mechanism, how many determining lines intersect to locate each instant center.

FIG. 62.1

FIG. 62.2   FIG. 62.3

FIG. 62.4

GEARS

FIG. 62.5   FIG. 62.6   FIG. 82.1

PINION & RACK

62.2. Same problem as 62.1 but applied to Fig. 62.2.

62.3. Same problem as 62.1 but applied to Fig. 62.3.

62.4. Same problem as 62.1 but applied to Fig. 62.4.

62.5. Same problem as 62.1 but applied to Fig. 62.5.

62.6. Same problem as 62.1 but applied to Fig. 62.6.

62.7. Same problem as 62.1 but applied to Fig. 62.7.

63.1. What is meant by a fixed centrode, a moving centrode, an axod
in Fig. 56.1 to be the fixed member, show the fixed centrodes of O_{12} and
of the fixed centrode of *2* relative to *0* and the moving centrode of *2*.
of *2* relative to *0*, are the fixed and moving centrodes equivalent to the
What practical difficulties would be involved in making the substitu

64.1. State the two principles used in determining linear vel
instant centers. Assuming for Fig. 56.1 that *0* is the fixed mem
velocity of O_{12} is known, show and explain how, by the method
velocity of O_{23} is graphically determined. Designate the gauge li
were used as pivot points and which point was used as the trans

65.1. For the mechanism shown in Fig. 65.1 assume a line
tive to the fixed member and determine graphically, by th

the corresponding linear velocities of points Q and R. List for each point the pivot and transfer points used in determining its velocity.

65.2. Same problem as 65.1 but applied to Fig. 59.3.

65.3. Same problem as 65.1 but applied to Fig. 59.4.

65.4. Same problem as 65.1 but applied to Fig. 59.6.

65.5. Same problem as 65.1 but applied to Fig. 59.5.
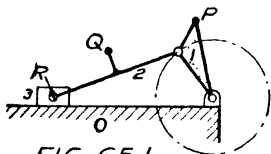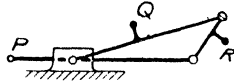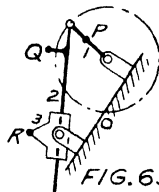
FIG. 65.1   FIG. 65.6   FIG. 65.7

65.6. Same problem as 65.1 but applied to Fig. 65.6.

65.7. Same problem as 65.1 but applied to Fig. 65.7.

65.8. Same problem as 65.1 but applied to Fig. 62.3.

65.9. Same problem as 65.1 but applied to Fig. 62.5.

65.10. For the mechanism shown in Fig. 65.1 assume a linear velocity for point P 'ative to the fixed member and determine graphically, by the parallel line construction, corresponding linear velocities of points Q and R.

66.1. For the mechanism shown in Fig. 62.3 assume a linear velocity for point P relative to the fixed member and determine graphically, by the method of resolution, the corresponding linear velocities of Q and R.

Same problem as 66.1 but applied to Fig. 59.5.

Same problem as 66.1 but applied to Fig. 65.7.

For the mechanism shown in Fig. 62.7 assume a linear velocity for point Q relative to the fixed member and determine graphically, by the method of resolution, the corresponding linear velocities of R and P.

first state and then make a suitable sketch and prove the angular velocity

illustrate, by means of sketches of two simple mechanisms, how the directions of two bodies are related to their pivot and transfer points.

FIG. 59.

the mechanism illustrated in Fig. 59.1 assume an angular velocity vector clockwise rotation of 1 relative to 0 and determine graphically by applying

59.2. Same velocity theorem the angular velocity of 2 relative to 0. From the angular

59.3. Same problem relative to 0, determine graphically the angular velocity of 3 relative to 2.

59.4. Same problem two determinations the pivot and transfer points used; and, assuming rotation as positive, tell whether the angular velocities deter-

59.5. Same problem mined.

59.6. Same problem as 68.1 but applied to Fig. 59.3.

60.1. For the incomplete as 68.1 but applied to Fig. 59.5.

number of instant centers 68.1 but applied to Fig. 59.6.
explanation of the method

60.2. Same problem as 68.1 but applied to Fig. 65.1.

60.3. Same problem as 60.1 but applied to Fig. 65.7.

68.7. For the mechanism in Fig. 59.1 assume an angular velocity vector for clockwise rotation of *1* relative to *0* and determine graphically the corresponding angular velocity of *2* relative to *1*. The determination of ω_{21} is to be done in two steps using two angular velocity ratios and two constructions, treating center O_{20} as not available.

68.8. For the mechanism shown in Fig. 56.1 assume an angular velocity vector for clockwise rotation of *2* relative to *3* and determine graphically the corresponding angular velocity of *3* relative to *0*. The determination of ω_{30} is to be done in two steps using two angular velocity ratios and two constructions, treating center O_{20} as not available.

69.1. What is a graph? Where vector quantities are plotted what only, with respect to the vector quantity, can be shown by the graph. Why does the text use the term velocity instead of speed graph?

70.1. Assuming for the mechanism shown in Fig. 59.1 that link *1* rotates at a uniform rate, plot the radial velocity-space graph of O_{25}, showing and fully justifying the construction for one point on the graph. Fully explain how the velocity scale of the graph would be determined. Plot and explain how the rectified velocity-space graph is obtained from the radial velocity-space graph.

70.2. Make a sketch of the slider-crank mechanism of an engine, assume the uniform velocity of the crank-pin to be represented by the crank length, and plot the velocity-space graph of the cross-head, showing and fully justifying the construction of one point on the graph. Assuming the stroke of the engine to be 15 inches, the R.P.M. of the crank to be 224, and the space scale to be 4 in. = 12 in., determine the velocity scale in feet per second per inch. *Ans.* 5.866 ft./sec./in.

70.3. Make a sketch of the slider-crank mechanism of an engine, assume the uniform velocity of the crank-pin to be represented by the crank length, and plot the polar velocity graph of the cross-head, showing and fully justifying the construction for one point on the graph. Assuming the stroke of the engine to be 18 inches, the R.P.M. of the crank to be 210, and the space scale to be 3 in. = 12 in., determine the velocity scale in second per inch. *Ans.* 7.333 ft./

70.4. Make a sketch of the slider-crank mechanism of an engine, assume velocity of the crank-pin to be represented by the crank length, and plot time graph of the cross-head, showing and fully justifying the construction on the graph. Assuming the stroke of the engine to be 20 inches, the R.P.M. to be 168, and the space-scale to be 3 in. = 12 in., determine the velocity scale second per inch. *Ans.* 5.866 f

70.5. In a distance-time, or space-time, graph the space scale is 1 in. = time scale is 1 in. = k_t sec. Making a suitable sketch, show and prove how nates may be obtained from the space-time graph and to what scale such represent velocities.

71.1. (*a*) In a velocity-space graph, the space scale is 1 in. = k_s ft scale is 1 in. = k_v ft./sec. Making a suitable sketch, show and prov ordinates may be obtained from the velocity-space graph and to what would represent accelerations.

(*b*) Where the velocity-space graph applies to rectilinear moti ordinates represent; where the graph applies to curvilinear moti ordinates represent?

(*c*) What would be the acceleration scale if the respective s graph were 1 in. = k_s = 2 ft. and 1 in. = k_v = 26.4 ft./sec.?

Ans. 1 in. = center line of travel of the

71.2. (*a*) In a velocity-time graph, the time scale is 1 in. is 1 in. = k_v ft./sec. Making a suitable sketch, show and p

FIG. 82.1

..is 6 inches long. Assuming the
..rmine graphically, by applying

may be obtained from the velocity-time graph and to what scale such ordinates would represent accelerations.

(b) Where the velocity-time graph applies to rectilinear motion, what do the derived ordinates represent; where the graph applies to curvilinear motion, what do the derived ordinates represent?

(c) What would be the acceleration scale if the respective scales of the velocity-time graph were 1 in. = k_t = 0.159 sec. and 1 in. = k_v = 26.4 ft./sec. and a constant length of 0.375 inch is used for the time increment in obtaining the acceleration ordinates?

$Ans.$ 1 in. = k_a = 442.8 ft./sec./sec.

71.3. When only is the Klein construction applicable in determining the acceleration of the cross-head of a slider-crank mechanism for all positions of the crank? Make a sketch showing the Klein construction for a position of the crank and explain how the acceleration scale for the construction is determined.

72.1. In Fig. 72.1, for uniform rotation of the crank,

$$V^x{}_{PO} = - \omega R \left(\sin\theta + \frac{c}{2n}\sin 2\theta \right)$$

$$V^y{}_{PO} = \omega R(1 - c)\cos\theta$$

$$A^x{}_{PO} = - \omega^2 R \left(\cos\theta + \frac{c}{n}\cos 2\theta \right)$$

$$A^y{}_{PO} = - \omega^2 R(1 - c)\sin\theta$$

$$\omega_{20} = \frac{\omega\cos\theta}{\sqrt{n^2 - \sin^2\theta}}$$

$$\alpha_{20} = \frac{\omega^2(1 - n^2)\sin\theta}{(n^2 - \sin^2\theta)^{3/2}}$$

$\frac{l}{R}$, and $c = \frac{b}{l}$.

mechanism, assume the stroke of the cross-head to be 18 inches, the R.P.M. to be 210, and the length of the connecting rod to be 45 inches.

FIG. 5

FIG. 72.1

59.2. Same probl...

59.3. Same probl...

59.4. Same proble...

59.5. Same problem.

59.6. Same problem 1 ...se of the proper equations, the angular velocity and angular

60.1. For the incompl...ating rod for the crank in the 120 degree position.

number of instant center $Ans.$ 2.233 rad./sec., 84.1 rad./sec./sec.

explanation of the method ... the proper equations, the X and Y components of the linear

60.2. Same problem as 6... point on the connecting rod for the crank in the 120-degree

60.3. Same problem as 60... ...as. Also determine the velocity and the acceleration of the

...n.

Ans. 13.718 ft./sec., 4.95 ft./sec., 196.0 ft./sec./sec., 188.615 ft./sec./sec., 14.58 ft./sec at 70°-10′ with the vertical and 272.0 ft./sec./sec. at 46°-6′ with the vertical.

72.2. By use of the data and the equations given in 72.1 determine the velocity and acceleration of the cross-head of the engine.

Ans. 12.86 ft./sec., 217.8 ft./sec./sec.

72.3. For the mechanism shown in Fig. 72.3 assume the stroke of the slotted cross-head to be 18 inches, the R.P.M. of the crank to be 210, and determine, by use of the equations given in 72.1, the velocity and acceleration of the cross-head for the crank in the 120-degree position.

FIG. 72.3

Ans. 14.29 ft./sec., 181.5 ft./sec./sec.

CHAPTER IV

VELOCITY AND ACCELERATION VECTOR DIAGRAMS

73.1. (a) The magnitude and direction of the forces acting on the parts of high-speed machinery depend upon what three things? Which one of the three factors may usually ͺ, be neglected? For low-speed machinery what two factors are often neglected in determ͵ mining the magnitude and direction of the forces acting on each part?

(b) Why in a steam engine is the effective force at the cross-head pin during the fi͵ ͵ͷ part of the stroke less than the difference in steam pressure on the head and crank-ᵗᵉᵗ͵ₒₙ sides of the piston? Why are the reactions at the crank-shaft bearings less than w͵ correspond to the net steam pressure on the piston?

(c) Why is it of importance, especially in the design of high-speed machinery͵ₒₙ vec- able to determine the linear acceleration of any point, and the angular acceleratior͵ link of a mechanism?

a͵tion, the

73.2. (a) Suppose that the rod shown in Fig. 73.2, weighing 161 pounds, is͵ᵉͷᵗents? a part of a machine, to have linear but no angular acceleration, and that the linea͵ʰₐₜ vector tion of its center of gravity C is $A_C = 25$ ft./sec./sec. in the direction shown. ͷ force is required to accelerate the rod and what must be its line of action? If͵ᵢₒₙ, $A_{DO} =$

FIG. 73.2   *FIG. 75.1*

FIG. 82.1

of gravity of the rod is midway between centers B and D and the · ated by pulls through these centers, what would be the magniti͵ forces?   *Ans.* $F = 125$ lb. through C in direction of A

(b) Assume the rod in Fig. 73.2 to be 30 inches long, to ha 8 inches about its center of gravity and to have, in addition t͵ angular acceleration of 18 rad./sec./sec.; determine the accel action with respect to C. Assuming the rod actually accele͵ center line of travel of the ters B and D, determine the magnitude and direction of the͵sis 6 inches long. Assuming the 45 degrees with center line DB.   ͵rmine graphically, by applying

Ans. $F = 125$ lb. parallel to and 3.84 in. from A_C, $F_D = 39.87$ lb. and $F_B = 85.13$ lb parallel to F.

74.1. (*a*) What simple relation of the velocities of any two points in the same rigid body is the key to the construction of velocity vector diagrams?

(*b*) What is the velocity image of a link of a mechanism, and how is the image related to the link in shape, size, and position?

(*c*) How may the angular velocity of a link be obtained from a linear dimension of the link and its velocity image?

75.1. (*a*) For the mechanism shown in Fig. 75.1, the known velocity of B relative to the fixed member is represented by a vector equal in length to $O_{10}B$; determine the corresponding velocities of points C, Q, and R by means of the velocity vector diagram, number the velocity images, and draw the velocity vectors from C, Q, and R in the sketch of the mechanism.

(*b*) Represent the phorograph for the phase shown, tell to what scale it must be drawn, and explain how it is obtained from the vector diagram.

75.2. For the mechanism shown in Fig. 62.3 assume a vector to represent the velocity of P relative to the fixed member and determine the corresponding v points Q and R by means of the velocity vector diagram, number the velocit and draw the velocity vectors from Q and R in the sketch of the mechanism.

75.3. Same problem as 75.2 but applied to Fig. 65.7.

75.4. Same problem as 75.2 but applied to Fig. 59.5.

75.5. For the mechanism shown in Fig. 62.7 assume a vector to represent the city of Q relative to the fixed member and determine the corresponding velocit. s P and R by means of the velocity vector diagram, number the velocity images, the velocity vectors from P and R in the sketch of the mechanism.

(*a*) Why should a related and consistant notation be used in the constructi ity and acceleration vector diagrams? In accordance with the text, how, in vector diagram, would the velocity V_{BO} be represented; how, in an acceleratioi gram, would the acceleration A_{BO} be represented?

at is the key to the construction of acceleration vector diagrams? In such a how, by means of a sketch, how the vectors representing A_{BO}, A^n_{BO}, and rked and related.

Show, for Fig. 72.1, how to construct the velocity vector diagram giving the oints B, P, and D. Assuming the space scale to be 1 in. $= k_s$ ft. and the r velocity of the crank-pin to be represented by the crank length, determine the velocity scale of the diagram.

FIG. 5.

59.2. Same pro

59.3. Same prob

59.4. Same proble

59.5. Same problem

59.6. Same problem

60.1. For the incompl number of instant center explanation of the method

60.2. Same problem as 60

60.3. Same problem as 60.

(*b*) Having the velocities and the velocity scale, show how to construct the acceleration vector diagram to a scale of 1 in. $= k_a$ ft./sec./sec.

76.3. Same problem as 76.2 but applied to points P, Q, and R of Fig. 65.1.

76.4. Same problem as 76.2 but applied to points B, C, Q, and R of Fig. 75.1.

77.1. In Fig. 77.1 let the velocity of point P at the instant along its path ST be represented by the vector PQ in inches to a scale of 1 in. $= k_v$ ft./sec., and let the radius of curvature R of the path at point P be represented in inches by OP to a space scale of 1 in. $= k_s$ ft.

of PQ, OP, and the space and velocity scales.

wn normal to OQ, then NP represents A^n_{PO} in magnitude

and direction to a scale of 1 m $= k_a = \dfrac{k_i{}^2}{k_s}$. If k_a and k_s are arbitrarily chosen, what must be the velocity scale for the construction shown?

(c) If PQ is made equal to OP and QN is drawn normal to OQ, to what does the acceleration scale reduce?

78.1. For Fig 72 1 assume the stroke of the cross-head to be 18 inches the R.P.M. of the crank to be 210, the length of the connecting-rod to be 45 inches, the distance b to be 18 inches, and the displacement of the crank to be 120 degrees:

(a) What must be the acceleration scale for a complete graphical solution by means of velocity and acceleration vector diagrams if the constant linear velocity of the crank-pin center B is represented by the crank length to a space scale of 1 m. = ¦ ft.?
$$Ans \quad 1 \text{ m} = k_a = 121 \text{ ft./sec./sec.}$$

(b) By means of vector diagrams determine V_{PO}, V_{DO}, V_{DB}, A_{PO}, A_{DO}, and A'_{DB}.
$$Ans \quad 14\ 58, \ 12\ 86, \text{ and } 8\ 55 \text{ ft /sec}, \text{ and } 272\ 0, \ 217\ 8, \text{ and } 315\ 4 \text{ ft./sec /sec.}$$

(c) Determine the angular velocity and angular acceleration of the connecting-rod.
$$Ans. \quad \omega = 2.233 \text{ rad./sec. and } \alpha_{20} = 84.1 \text{ rad./sec /sec.}$$

79.1. (a) By means of the acceleration vector diagram of problem 78 1 show how to locate the instant center of acceleration of the connecting-rod and explain what it means.

(b) In what way is the instant center of acceleration similar and in what way different from the instant center of velocity?

(c) Show how to determine the angle which the acceleration vector for any point in, the connecting-rod makes with the ray from the instant center of acceleration to the respective point.

80.1. (a) Make a sketch of a slider-crank mechanism and show the Klein construction for determining the acceleration of the cross-head. When only is this construction applicable?

(b) How are the space, velocity, and acceleration scales related?

(c) Prove the Klein construction by making use of the velocity and acceleration vector diagrams.

81.1. (a) For a point moving along a path which has a motion of translation, the acceleration of the moving point is equal to the vector sum of what two components?

(b) If the path along which a point moves has a motion of rotation, to what vector sum is the acceleration of the moving point equal?

(c) Interpret Coriolis' Law as expressed mathematically by the equation, $A_{DO} = A_{CO} +\!\!\!+ A_{DC} +\!\!\!+ 2V_{DC}\omega_C$ and tell how the direction of the compound supplementary acceleration is determined

(d) To facilitate the application of Coriolis' Law, on what member should the curve traced by the describing point be located?

82.1. The outline of the automobile cam shown in Fig. 82.1 is made up of circular arcs and straight lines. The cam is rotating counter-clockwise at 1050 R P.M By applying Coriolis' Law to the equivalent mechanism, determine the linear velocity and acceleration of the follower, or point D, for the phase shown. On determining the equivalent mechanism, treat D, which is common to 2 and 3, as the describing point, and C as the coincident point in 1.

Ans. $V_{DO} = 2.61$ ft./sec. and $A_{DO} = 1308.0$ ft./sec./sec.

FIG. 82.1

82.2. Assume for Fig. 62.7 that the center of rotation of the crank is $13\frac{1}{8}$ inches and point R $29\frac{5}{8}$ inches below the center line of travel of the ram when the crank is 30 degrees with the horizontal and is 6 inches long. Assuming the crank rotating at a uniform speed of 21 R.P.M., determine graphically, by applying

Coriolis' Law, the linear acceleration of point D on the vibrator, point D being the describing point and Q the point on the axis of the crank-pin coincident with D. Determine also the linear velocities and accelerations of points P and R.

Ans. $V_{DO} = 0.8433$, $V_{DQ} = 1.111$, $V_{DR} = 0.8617$, $V_{PO} = 1.283$, and $V_{RO} = 0.341$ ft./sec; $A_{DO} = 0.9357$, $A_{PO} = 1.1454$, and $A_{RO} = 1.629$ ft./sec./sec.

83.1. For Fig. 151 of the text, assume crank *1* to rotate counter-clockwise at 105 R.P.M. and determine, by applying the three-line construction, the linear velocities and accelerations of points G, H, and K; then determine, by applying Coriolis' Law, the linear acceleration of point L.

CHAPTER V

CAMS

84.1. (*a*) Define the term *cam*. Make a sketch of a cam mechanism. In such a mechanism, which is the *driver* and which the *follower*, and what two things determine the shape of the cam? What is meant by an *inverse* cam mechanism? Which type of mechanism is most used, the cam or inverse cam mechanism?

(*b*) Why and where are cam mechanisms used?

85.1. Into what five general classes may cams be divided?

86.1. (*a*) Define the term *disk cam* and make a sketch of a disk cam mechanism. Why are some disk cams called *periphery* cams? What is meant by a *radial* roller follower, by an *offset* roller follower?

(*b*) What is meant by a *positive return* disk cam? Why are some positive return disk cams called *face* cams?
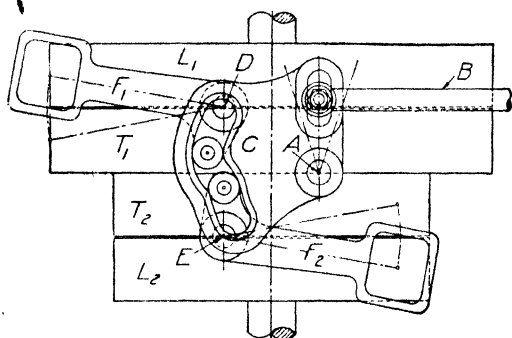
86.2. (*a*) Make a sketch of a disk cam mechanism with a reciprocating flat-faced follower. In what two ways may such a follower be *offset?* What is meant by a *mushroom* follower and what are the advantages of such a follower?

(*b*) Make a sketch of a rotating disk cam having an oscillating roller follower and a sketch of such a cam with an oscillating flat-faced follower?

(*c*) Show, by a sketch, what is meant by a *toe* and *wiper* cam mechanism.

FIG. 86.3

86.3. What kind of a disk cam and followers is shown in Fig. 86.3? State the purpose of the mechanism and explain how it works.

86.4. (*a*) Make a sketch of a positive return cam mechanism having a single disk cam and a reciprocating flat-faced follower. Why is such a cam called a *yoke* cam?

(*b*) Make a sketch of a type of disk cam that is often called an *adjustable* plate cam.

87.1. Make a sketch of a *translation* cam mechanism and define the term *translation* cam.

88.1. (*a*) Make a sketch of a *cylindrical* cam mechanism and define the term *cylindrical* cam.

(*b*) Why are some cylindrical cams called *end* cams? Make a sketch of a positive return cylindrical cam.

(*c*) What is meant by a *drum* or *barrel* cam and where are such cams much used? Why are some cylindrical cams called *dog* cams?

89.1. Make a sketch of a *conical* cam mechanism, tell when such a cam mechanism would be used, and define the term conical cam. How would you make a positive return conical cam?

90.1. Make a sketch of a *spherical* cam mechanism, tell when such a cam mechanism would be used, and define the term spherical cam. How would you make a positive return spherical cam?

91.1. (a) Which two of the five classes of cams are most frequently used in machinery?

(b) Make a sketch of a disk cam with a radial direct contact follower, mark the *pitch point* for the phase drawn, label the *working surface*, determine and indicate the *total displacement*, and tell what is meant by the term *dwell*.

91.2. (a) What expedient is generally used to facilitate the determination of the working surface of a cam for a given displacement?

(b) Show how to determine the working surface of a disk cam for a radial roller follower having given the desired displacement of the follower and the center of rotation of the cam. Fully justify the construction for one position of the follower.

91.3. (a) Make a sketch of a disk cam with a radial roller follower, label the pitch surface and the working surface, and indicate the pitch point and pressure angle for the phase shown.

(b) What, in general, is regarded as a limiting value for the pressure angle? Why should this angle be limited?

(c) Explain why a roller follower is to be preferred to a direct-contact follower.

92.1. A cam follower having recitilinear motion is to be uniformly accelerated from rest during the first third of its total displacement h, is to move at a uniform velocity for the second third, and is to be uniformly retarded to rest during the last third of its displacement.

(a) Assuming the first third of the displacement to take place in 3 equal time periods, show and explain how to plot the corresponding portion of the displacement-time graph; and determine also the value of the acceleration in terms of h and the velocity attained at the end of the period of acceleration in terms of h.

Ans. $A = \frac{2}{27}h/$time interval/time interval, $V = \frac{2}{9}h/$time interval.

(b) Show that the middle third of the displacement at uniform velocity would be covered in one-half the time of the first or last third.

(c) Determine the slope of the graph at the end of the acceleration period and the slope of the uniform velocity portion of the graph. How far from the origin would this latter portion of the graph intersect the X axis if extended downward?

Ans. slopes $= \frac{2}{9}h$. $x = 1\frac{1}{2}$ time intervals.

92.2. Show how to construct a displacement time graph for a reciprocating follower driven by a cam which rotates at a uniform rate. The follower is to move outward with uniform acceleration for $\frac{1}{8}$, uniform velocity for $\frac{1}{4}$, uniform retardation for $\frac{1}{8}$, and is to dwell for $\frac{1}{8}$ revolution of the cam; and the follower is to return with uniform acceleration for $\frac{1}{8}$, uniform retardation for $\frac{1}{8}$, and is to dwell for $\frac{1}{8}$ revolution of the cam.

92.3. Show how to construct a displacement-time graph for a reciprocating follower driven by a cam which rotates at a uniform rate. The follower is to move outward with simple harmonic motion for $\frac{3}{8}$ and is to dwell for $\frac{1}{8}$ revolution of the cam; and the follower is to return with uniform acceleration for $\frac{1}{6}$, uniform retardation for $\frac{1}{6}$, and is to dwell for $\frac{1}{6}$ revolution of the cam.

92.4. Show how to construct a displacement-time graph for a reciprocating follower driven by a cam which rotates at a uniform rate. For $\frac{1}{2}$ its displacement, the follower is to move outward with simple harmonic motion in $\frac{1}{4}$ revolution of the cam, is to dwell for $\frac{1}{8}$, and is to complete its displacement with simple harmonic motion in $\frac{1}{4}$ revolution of the cam; and the follower is to return with simple harmonic motion in $\frac{1}{4}$ and is to dwell for $\frac{1}{12}$ revolution of the cam.

92.5. Show how to construct a displacement-time graph for a reciprocating follower driven by a cam which rotates at a uniform rate. The follower is to move outward with uniform acceleration for 75 degrees, uniform retardation for 60 degrees, and is to dwell for 45 degrees rotation of the cam; and the follower is to return with simple harmonic motion for 120 degrees and is to dwell for 60 degrees rotation of the cam.

93.1. Show how to determine the outline of the working surface of a disk cam which, in rotating at a uniform rate, is to drive a radial roller follower in a 45-degree line in accordance with an assumed displacement scale. Fully justify the construction for one position of the follower.

93.2. For a disk cam rotating at a uniform rate, show how to determine the working surface of the cam for a radial roller follower moving horizontally in accordance with an assumed displacement scale. Fully justify the construction for one position of the follower.

94.1. Show, by means of sketches what must be the relation between the radius of the follower roller and the radius of curvature of the pitch surface of a non-positive and of a positive return disk cam.

95.1. (a) What is meant by the base radius of a radial disk cam? In Fig. 95.1 is shown the base diagram for a radial disk cam rotating at a uniform rate. Assuming for a total displacement h that the cam rotates through an angle β in degrees, determine an expression for the base radius R in terms of h and β and the cam factor f.
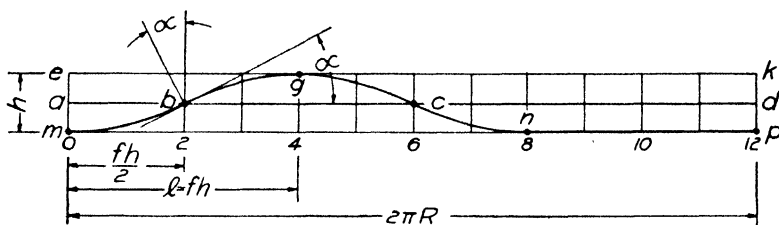
FIG. 95.1

(b) Assuming the follower is uniformly accelerated for the first half and uniformly retarded for the second half of the displacement, determine an expression for the cam factor f in terms of the pressure angle α.

96.1. (a) What are the two methods of designing cams?

(b) Assume that the follower of a radial disk cam is uniformly accelerated for the first half and uniformly retarded for the second half of its displacement; determine expressions for the displacement, velocity, and acceleration of the follower; draw the graphs; and determine the product, fh, of the cam factor and displacement in terms of h and the maximum pressure angle.

96.2. (a) What are the two methods of designing cams?

(b) Assume the follower of a radial disk cam to have simple harmonic motion; determine expressions for the displacement, velocity, and acceleration of the follower; draw the graphs; and determine the product, fh, of the cam factor and displacement in terms of h and the maximum pressure angle.

96.3. (a) The base radius of a disk cam is $R = \dfrac{360fh}{2\pi\beta}$, and the cam factor for a radial follower having uniformly accelerated motion for the first half and uniformly retarded motion for the second half of its displacement is $f = 2/\tan\alpha$. Assuming the cam to rotate through 120 degrees for a desired displacement of 1.5 inches, determine the base radius of the cam for a maximum pressure angle of 30 degrees.

Ans. $R = 2.48$, say $2\frac{1}{2}$ in.

(b) Determine the minimum distance from the pitch point of the follower to the center of the cam, also the minimum radius of the cam for a follower roller diameter of $\frac{3}{4}$ inches.

Ans. $1\frac{3}{4}$ and $1\frac{3}{8}$ in.

96.4. (a) The base radius of a disk cam is $R = \dfrac{360 fh}{2\pi\beta}$; and the cam factor for a radial

follower having simple harmonic motion is $f = \pi \div 2 \tan \alpha$. Assuming the cam to rotate through 150 degrees for a desired displacement of 2 inches, determine the base radius of the cam for a maximum pressure angle of 25 degrees. *Ans.* $R = 2.575$, say $2\frac{9}{16}$ in.

(b) Determine the minimum distance from the pitch point of the follower to the center of the cam, also the minimum radius of the cam for a follower roller diameter of $\frac{3}{4}$ inches.

Ans. $1\frac{9}{16}$ and $1\frac{3}{16}$ in.

96.5. (a) The base radius of a disk cam is $R = \dfrac{360 fh}{2\pi\beta}$, and the cam factor for a radial

follower for a displacement-space graph made up of two equal circular arcs is $f = \dfrac{\sin \alpha}{1 - \cos \alpha}$.

Assuming the cam to rotate through 150 degrees for a desired displacement of 1.25 inches, determine the base radius of the cam for a maximum pressure angle of 25 degrees.

Ans. $R = 2.155$, say $2\frac{1}{8}$ in.

(b) Determine the minimum distance from the pitch point of the follower to the center of the cam, also the minimum radius of the cam for a follower roller diameter of $\frac{3}{4}$ inches.

Ans. $1\frac{1}{2}$ and $1\frac{1}{8}$ in.

97.1. (a) Assuming for problem 96.4 that the cam rotates at a uniform speed of 150 R.P.M., determine the angular velocity ω_r of the rotating vector and the acceleration and retardation of the follower during its total displacement in 6 equal intervals.

Ans. $\omega_r = 6\pi$ rad./sec., $A = 29.61, 25.64, 14.80, 0, 14.8, 25.64$ and 29.61 ft./sec./sec.

(b) Assuming the follower and roller to weigh 4.83 pounds and to move vertically up and down, determine if the weight of the follower and roller is sufficient to keep the roller in contact with the cam. *Ans.* Yes.

(c) What is the vertical component of pressure between the roller and cam at the beginning and at the end of its upward displacement and during the period of dwell?

Ans. 9.27, 0.39, and 4.83 lb.

(d) If the speed of the cam were increased 50 per cent what scale of spring would be required for an initial compression of 0.84 pound and a vertical pressure component between cam and roller of 0.68 pound for the maximum lift position. *Ans.* 2.5 lb./in.

98.1. The exhaust valve of an automobile engine is to have a total displacement, or lift, of $\frac{5}{16}$ of an inch. The valve is to open 45 degrees before the crank reaches its bottom dead center position and is to close 5 degrees after the crank has reached its top dead center position when the engine is cold; and the valve is to dwell in its extreme position for a period corresponding to a rotation of the crank of 30 degrees. To allow for expansion as the engine warms up, there is to be a clearance of about 0.006 of an inch between the cam and follower roller when the engine is cold. A tangent cam with a radial roller follower is to be used, the minimum radius of the cam to be $1\frac{1}{16}$ and the diameter of the follower roller to be $\frac{7}{8}$ of an inch.

(a) Through what angle ϕ would the cam rotate when the engine is cold before the valve would begin to lift? *Ans.* $\phi = 5° - 54'$, say 6°.

(b) Through what angle β does the cam turn during the lift of the valve after the engine has warmed up? *Ans.* $\beta = 56°$.

(c) Determine the maximum radius R_2 of the cam and the radius R_4 of the nose.

Ans. $R_2 = 1.0$ and $R_4 = 0.2911$ in.

(d) Determine the angles δ and $(\beta - \delta)$ turned through by the cam during which the follower roller is in contact respectively with the flank and nose of the cam.

Ans. $\delta = 27°-35'$ and $(\beta - \delta) = 28°-25'$.

98.2. Assuming for problem 98.1 that the maximum speed of the engine is 2100 R.P.M.,

compute the displacements and linear velocities and accelerations of the follower for $\phi = 0°$, 13°–47.5′, and 27°–35′ and for $\theta = 28°$–25′, 14°–12.5′, and 0°.

$Ans.$ $s = 0.00$, 0.0334, 0.1443, 0.1443, 0.2697, and 0.3125 in.
  $V = 0.00$, 2.61, 6.06, 6.06, 3.14, and 0.00 ft./sec.
  $A = 1133.5$, 1307.8, 1976.8, 1279.2, 1362.5 and 1409.5 ft./sec./sec.

98.3. Draw the cam and follower three times full size, using the values found in 98.1, and determine the displacement-time graph of the follower for the phases given in 98.2 and from it determine the velocity and acceleration graphs by graphical differentiation. The values found should agree within 5 per cent with those found in 98.2.

98.4. (a) By using equivalent mechanisms for the cam and follower, graphically determine, by means of vector diagrams, the velocities and accelerations of the follower for the phases given in 98.2. The values found should agree very closely with the computed values.

(b) In what way is the vector superior to the analytical method for determining velocities and accelerations?

99.1. The exhaust valve of an automobile engine is to have a total displacement, or lift, of $\frac{5}{16}$ of an inch. The valve is to open 45 degrees before the crank reaches its bottom dead center position and is to close 5 degrees after the crank has reached its top dead center position when the engine is cold. To allow for expansion as the engine warms up there is to be a clearance angle of the cam shaft of 4.5 degrees when the engine is cold. The minimum radius of the cam is to be $\frac{11}{16}$ and the radius of the nose $\frac{3}{16}$ of an inch.

(a) Determine the maximum radius R_2 of the cam and the angle β turned through by the cam during the lift of the valve after the engine has warmed up.

$Ans.$ $R_2 = 1.0$ in. and $\beta = 62°$.

(b) Determine the radius of curvature R of the flank of the cam.

$Ans.$ $R = 2.4338$ in.

(c) Determine the angle δ turned through by the cam while the follower is in contact with the flank of the cam, also the angle $(\beta - \delta)$ during which there is contact with the nose. $Ans.$ $\delta = 18°$–37.5′ and $(\beta - \delta) = 43°$–22.5′.

99.2. Assuming for problem 99.1 that the maximum speed of the engine is 2100 R.P.M., compute the displacements and linear velocities and accelerations of the follower for $\phi = 0°$, 9°–18.75′, and 18°–37.5′ and for $\theta = 43°$–22′5, 21°–41.25′, and 0°.

$Ans.$ $s = 0.00$, 0.0230, 0.0910, 0.0910, 0.2550, and 0.3125 in.
  $V = 0.00$, 2.589, 5.112, 5.112, 2.752, and 0.00 ft./sec.
  $A = 1760.0$, 1736.8, 1667.8, 866.2, 1107.3, and 1191.7 ft./sec./sec.

99.3. (a) By using the equivalent mechanism for the cam and follower, graphically determine, by means of vector diagrams, the velocities and accelerations of the follower for the phases given in 99.2. The values found should agree very closely with the computed values.

(b) In what way is the vector superior to the analytical method for determining velocities and accelerations?

100.1. Assume a displacement scale and show how to determine the working surface of a disk cam which, while rotating at a uniform rate, is to drive a reciprocating flat-faced follower whose face is at right angles to its stem. Fully justify the construction for one position of the follower and explain how the extent of the follower face is determined. What is it that determines the minimum radius of the cam?

100.2. Assume a displacement scale and show how to determine the working surface of a disk cam which, while rotating at a uniform rate, is to drive a reciprocating flat-faced follower whose face makes an angle of 75 degrees with its stem. Fully justify the construction for one position of the follower and explain how the extent of the follower face is determined. What is it that determines the minimum radius of the cam?

101.1. (a) Make a sketch of a stamp mill cam; state its purpose and explain how it works.

(b) Assume the working surfaces of a stamp mill cam to be involutes and derive an equation for the radius of the base circle of the cam in terms of the displacement h of the follower and angle θ turned through by the cam during involute action.

(c) Explain how to determine the positive displacement of the follower after involute action ceases.

102.1. (a) Assume a displacement scale and show how to determine the working surface of a disk cam which, while rotating at a uniform rate, is to drive an oscillating roller follower. Fully justify the construction for one position of the follower.

(b) Assuming the follower arm to move in the plane of motion of the cam, explain how a suitable shape for the follower arm may be determined.

103.1. Assume a displacement scale and show how to determine the working surface of a disk cam which, while rotating at a uniform rate, is to drive an oscillating flat-faced follower whose face does not pass through the axis of oscillation. Fully justify the construction for one position of the follower and explain how the extent of the follower face is determined.

104.1. What must be true of the to and fro motion of a yoke follower for a single disk cam? Assume a displacement scale and show how to determine the working surface of such a cam to drive a flat-faced yoke follower. How are such cams often referred to?

104.2. What must be true of the to and fro motion of a yoke follower for a single-disk cam? Assume a displacement scale and show how to determine the working surface of such a cam to drive a roller yoke follower. How are such cams often referred to?

105.1. (a) What is meant by positive return single-disk circular arc cams?
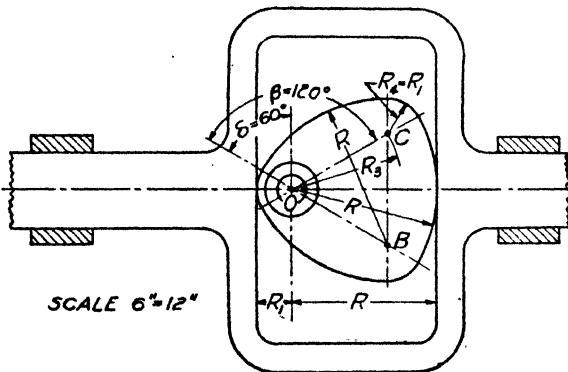
FIG. 105.1

(b) In Fig. 105.1 the outline of the working surface of the cam is made up of circular arcs of two different radii with their centers at the apices of an equilateral triangle. For the follower in contact with the flank of the cam of radius R the displacement is $s = (R - R_1)(1 - \cos \phi) = R_3(1 - \cos \phi)$; and is $s = R_3 \cos \theta$ for the follower in contact with the nose of radius $R_4 = R_1$. Determine expressions for the velocity and acceleration of the follower for follower contact with the flank and for follower contact with the nose of the cam. Through what angle does the cam rotate for each of these contacts, and through what angle does the cam rotate during the dwell at each end of the stroke?

105.2. (a) What is meant by positive return single-disk circular arc cams?

(b) For the cam mechanism shown in Fig. 105.2, determine the value of R_3 in terms of R_1 and α for a given value of the minimum radius R_1 and total displacement h of the follower.
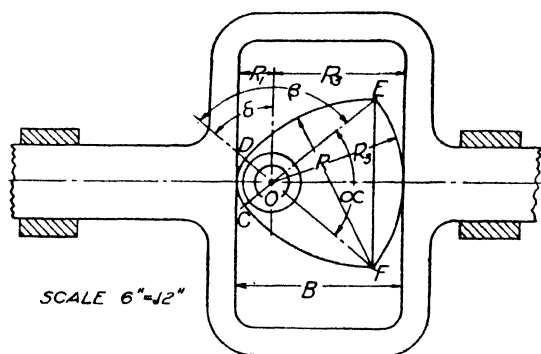
(c) For the follower in contact with the flank of radius R the displacement is $s = R_3(1 - \cos \phi)$; and is $s = R_3 \cos \theta - R_1$ for the follower in contact with the sharp nose of the cam. Determine expressions for the velocity and acceleration of the follower for follower contact with the flank and for follower contact with the nose of the cam. Through what angle does the cam rotate for each of these contacts, and through what angle does the cam rotate during the dwell at each end of the stroke?

SCALE 6"=12"

FIG. 105.2

105.3. (a) For the cam mechanism shown in Fig. 105.3 what must be the eccentric throw for a desired oscillation θ of the follower and a given distance between the centers C and O?

(b) Through what angles does the cam or eccentric rotate for the to and fro oscillation of the follower? Why, for a given oscillation, is it difficult to determine the working surface of a single-disk positive return cam that is not in the form of an eccentric?

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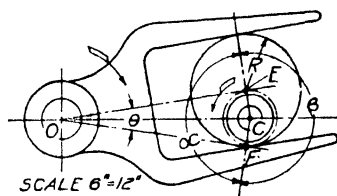
FIG. 105.3

106.1. Assume a displacement scale and show how to determine the working surfaces of a double-disk cam which, while rotating at a uniform rate, is to drive a reciprocating roller yoke follower. Fully justify the construction for one position of the follower.

107.1. (a) What information is it necessary to have and what drawings is it necessary to make to determine the working surfaces of a positive return cylindrical cam?

(b) How should the shape of the follower roller for a positive return cylindrical cam be determined for the most satisfactory action? Why would such a roller be better than a plain cylindrical roller?

107.2. (a) Assume a displacement scale and show how to determine the working surface of a positive return cylindrical cam which, while rotating at a uniform rate, is to drive a roller follower parallel to the axis of the cam. Fully justify the construction for one position of the follower.

(b) Explain what should be the shape of the follower roller for best results.

108.1. (a) What is meant by an inverse cam mechanism?

(b) A roller at the end of an oscillating arm is to drive a translation cam along a line in the same horizontal plane as the mid position of the center line of the oscillating arm. The center of the roller moves in a circular arc path with uniform tangential acceleration for the first half and uniform tangential retardation for the second half of the oscillation of the arm; the corresponding motion of the translation cam is to be simple harmonic. Show how to determine the outlines of the pitch and working surfaces of the slot in the cam.

108.2. In an inverse cam mechanism the driver is a roller reciprocating in a horizontal path and the follower is a slotted plate reciprocating at 15 degrees with the vertical. Assume displacement scales for the driver and follower, show how to determine the working surface of the cam, and indicate the direction of motion of the driver and the follower for the phase shown.

108.3. In an inverse cam mechanism the driver is a roller reciprocating in a horizontal path and the follower is a slotted plate oscillating about a pivot in the line of motion of the driver extended. Assume displacement scales for the driver and follower, show how to determine the working surface of the cam, and indicate the direction of motion of the driver and the follower for the phase shown.

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CHAPTER VI

ROLLING CURVES AND FRICTION GEARING

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109.1. (a) State the requirements for pure rolling between two curves with respect to the point of contact and the relation of the common tangent to the contact radii.

(b) State the four ways in which the angular velocities of a pair of rolling curves are related. What must be the shape of the rolling curves if the angular velocity ratio is to be constant?

(c) State the criterion of positive driving for a pair of rolling curves. What can be said of a pair of circles with respect to pure rolling and positive driving?

110.1. (a) Why is positive driving not assured between two circles or two circular cylinders? If they are to roll together what must be assumed?

(b) Assuming two circles *1* and *2* to be in contact externally, deduce the equation for the radius of one in terms of the angular velocity ratio and distance between centers.

(c) Assuming two circles *1* and *2* to be in contact internally, deduce the equation for the radius of one in terms of the angular velocity ratio and distance between centers.

110.2. (a) Determine the radii of a pair of rolling circles for an angular velocity ratio of $\frac{4}{3}$ and a distance between centers of 7 inches, the circles to rotate in opposite directions.

Ans. 3 and 4 in.

(b) What would be the respective radii if the circles are to rotate in the same direction?

Ans. 21 and 28 in.

110.3. (a) Determine the radii of a pair of rolling circles for an angular velocity ratio of $\frac{5}{3}$ and a distance between centers of $10\frac{1}{2}$ inches, the circles to rotate in the same direction.

Ans. 7 and $17\frac{1}{2}$ in.

(b) What would be the respective radii if the circles are to rotate in opposite directions?

Ans. 3 and $7\frac{1}{2}$ in.

111.1. (a) Having given the minor and major axes of an ellipse, explain how the foci are located. How are the radii vectors to any point on an ellipse related to the tangent to the ellipse at the said point?

(b) Prove that pure rolling is possible between two equal ellipses whose axes are a distance apart equal to the major axes of the ellipses.

111.2. (a) Point out, for two equal ellipses, the critical phases during a rotation of 180 degrees when the conditions for positive driving do not obtain. Why cannot one ellipse drive another equal ellipse through more than 180 degrees?

(b) To what linkage, or system of links, do two equal ellipses correspond? By what two means may continuous positive driving between two equal ellipses be secured?

111.3. Assuming for a pair of rolling ellipses that the distance between the centers of rotation is to be 7 inches and that the angular velocity ratio is to vary from $\frac{5}{2}$ to $\frac{2}{5}$, determine the major and minor axes and the distance between the foci of the ellipses.

Ans. 7, 6.325, and 3 in.

111.4. Assuming for a pair of rolling ellipses that the distance between the centers of rotation is to be 10 inches and that the angular velocity ratio is to vary from $\frac{3}{4}$ to $\frac{2}{3}$, determine the major and minor axes and the distance between the foci of the ellipses.

Ans. 10, 9.798, and 2 in.

111.5. (a) Show how to lay out a pair of oscillating sectors of ellipses for a distance of 15 inches between the centers of rotation and for a change in the angular velocity ratio of driver to follower from $\frac{2}{4}$ to $\frac{2}{3}$ while the driving sector rotates through an angle of 75 degrees.

(b) What must be true if sectors of ellipses are to oscillate through equal angles?

111.6. Show how to lay out a pair of oscillating sectors of ellipses for a distance of 15 inches between the centers of rotation and for a change in the angular velocity ratio from $\frac{2}{4}$ to $\frac{1}{2}$ while the driver and follower each oscillate through an angle of 75 degrees.

112.1. The polar equation of the logarithmic spiral is $a\theta = \log_e R$. Prove that for any given value of a the angle between the tangent and the radius vector is the same at all points in the spiral.

112.2. (a) The polar equation of the logarithmic spiral is $a\theta = \log_e R$. Prove that the difference in the lengths of the radii vectors is the same for all arcs of equal length.

(b) Explain how it follows that pure rolling is possible between two similar spirals rotating about their foci.

112.3. (a) Two sectors of similar logarithmic spirals in oscillating about axes 6 inches apart are to transmit motion from one to the other at an angular velocity ratio of driver to follower varying from $\frac{2}{1}$ to $\frac{1}{3}$ while the driver rotates through an angle of 75 degrees. Determine the radii vectors of the sectors and by use of the equations $a\theta = \log_e R$ and $\tan \phi = \dfrac{1}{a}$ determine the angle of oscillation of the follower.

Ans. $R'_1 = 2$, $R'_2 = 4$, $R''_1 = 4.5$, and $R''_2 = 1.5$ in., $\theta_2 = 90°\text{-}37.4'$.

(b) Having the angular displacement and terminal radii vectors of each spiral, explain how the rolling curves are plotted.

112.4. (a) Having the equation $a\theta = \log_e R$, show that the radius vector R bisecting the angle between two known radii vectors R' and R'' is equal to the square root of the product of the known radii vectors.

(b) Two sectors of similar logarithmic spirals in oscillating about axes 6 inches apart are to transmit motion from one to the other at an angular velocity ratio varying from $\frac{3}{4}$ to $\frac{1}{2}$ while driver and follower each rotate through 60 degrees. Determine the terminal radii vectors of the sectors and plot the rolling curves.

Ans. R'_1 and $R''_2 = 2$ in., R''_1 and $R'_2 = 4$ in.

113.1. (a) Show how to construct the rolling curves of a pair of sectors and to obtain the angular displacement of the follower, having given the successive angular displacements of the driver, the distance between centers, and the terminal and intermediate angular velocity ratios which have been independently chosen.

(b) Why is this method for rolling sectors not applicable to the construction of lobed wheels?

113.2. (a) Show how to construct a pair of rolling sectors for a given distance between centers where the total angular displacement of the follower and the total and intermediate angular displacements of the driver have been independently chosen together with the terminal and intermediate angular velocity ratios.

(b) Why is this method of construction applicable to lobed wheels?

114.1. (a) Give the most familiar example of a pair of uni-lobed wheels.

(b) What must be true if sectors of equal ellipses or sectors of similar logarithmic spirals are to roll together through equal angles.

(c) How may uni-, bi-, and tri-lobed wheels be derived from equal ellipses and similar logarithmic spirals?

114.2. (a) What must be true if sectors of equal ellipses or sectors of similar logarithmic spirals are to roll together through equal angles?

(b) Explain how you would derive a pair of unsymmetrical uni-lobed wheels, also a pair of symmetrical tri-lobed wheels, from similar logarithmic spirals.

114.3. (a) What must be true if sectors of equal ellipses or sectors of similar logarithmic spirals are to roll together through equal angles?

(b) For equal arcs, how are the radii vectors of a logarithmic spiral related?

(c) Explain how it is possible to derive from similar logarithmic spirals a train of lobed wheels having one, two, and three lobes.

114.4. By the use of a pair of rolling sectors explain the process known as the *contraction* or *expansion* of angles. To what well-known pair of rolling curves may this process be applied in deriving pairs of wheels having an unequal number of lobes?

115.1. What are the pitch surfaces and the relation of the axes of *spur* gears, of *bevel* gears, of *helical* gears, of *hyperboloidal* gears?

117.1. Using a combination of circular right cylinders and cones prove that pure rolling is possible between two cones of the same slant height.

117.2. (a) For axes intersecting at an angle of 60 degrees show how to determine the frusta of a pair of cones which in rolling together will transmit motion from one shaft to the other at an angular velocity ratio of $\frac{3}{2}$, and prove the construction.

(b) For the same angular velocity ratio and relation of the shafts as in (a), show what changes in the mechanism are necessary to reverse the rotation of the driven shaft.

118.1. How may a hyperboloid be generated by a straight line rotating about an axis? Why is the resulting figure known as an hyperboloid of revolution? What is the gorge circle of such a figure? What is the shape of the contact line of a pair of rolling hyperboloids and what is the distance between their axes of rotation?

119.1. (a) Why cannot rolling surfaces whose transverse sections are not circular be used in friction gearing? Why are surfaces of circular section suitable?

(b) When power is transmitted from one shaft to another by means of friction wheels, what resistances must the friction in the tangent plane be sufficient to overcome? What is the value of this friction in terms of the coefficient of friction and the normal pressure at contact?

(c) Why are the contact surfaces of a pair of friction wheels often made of different materials? With respect to the driver and follower, explain how the two different materials should be used.

119.2. (a) By means of a pair of circular right cylinders whose axes are 14 inches apart, motion is to be transmitted from the driver to the driven shaft at an angular velocity ratio of $\frac{4}{3}$; determine the diameters of the wheels assuming no slip. *Ans.* 16 and 12 in.

(b) The driver, which is covered with leather, rotates at 280 R.P.M. The driven wheel is cast-iron; the coefficient of friction is 0.30; and the wheels are 4 inches wide. Determine the horse-power transmitted assuming the wheels pressed together with a force of 600 pounds. *Ans.* 4.8 H.P.

119.3. A pair of friction wheels is to be used to transmit 6 horse-power between parallel shafts 16 inches apart. The leather-covered driver rotates at 350 and the cast-iron follower at 210 R.P.M. Assuming a coefficient of friction of 0.30 and an allowable pressure per inch of face of 150 pounds, determine the normal pressure between the wheels and their widths of face. *Ans.* 600 lb. and 4 in.

119.4. (a) Under what conditions might friction wheels instead of gears be used to advantage? Where they are more suitable what advantages have they?

(b) Where are toothed gears to be preferred to friction wheels? Give some examples of services where friction wheels could not be used in place of toothed gears.

120.1. (*a*) Make a suitable sketch and determine the relation between the radial pressure *P* in the plane of the axes of a pair of V-grooved friction wheels and the pressure *N* normal to the contact surfaces. What would be the relation between *P* and *N* if the tangential components in the plane of the axes were considered only 25 per cent effective?

(*b*) Having the relation between *P* and *N*, determine an expression for the friction normal to the plane of the axes.

120.2. A pair of cast-iron V-grooved friction wheels having 45-degree grooves is to be used to transmit 6 horse-power between parallel shafts 16 inches apart, the driver rotating at 350 and the follower at 210 R.P.M. Assuming a coefficient of friction of 0.20 determine the radial force in the plane of the axes with which the wheels must be held together.

Ans. 386 lb.

120.3. Assuming the wheels in problem 120.2 to have one 45-degree wedge and groove of 2-inch working depth, determine by what per cent the secondary sliding or grinding action is greater than if the two wheels had four 45-degree wedges and grooves of a working depth of $\frac{1}{2}$ inch.   *Ans.* 300 per cent.

121.1. (*a*) What is the purpose of friction speed variators?

(*b*) Make a sketch of the Evans friction cone pulleys for parallel shafts and explain how the angular velocity ratio is varied; also sketch the Sellers' feed disks and explain how the angular velocity ratio is varied. Is secondary slippage present in these variators?

121.2. (*a*) Make a sketch of a friction speed variator for shafts whose axes coincide and explain how the angular velocity ratio is varied.

(*b*) Make a sketch of a friction speed variator for shafts at right angles and explain how the angular velocity ratio is varied and the direction of rotation of the driven wheel reversed.

CHAPTER VII

STRAIGHT AND HELICAL SPUR GEARS

122.1. (*a*) What are the pitch surfaces and the relation of the axes of *spur* gears?

(*b*) What is meant by the *pitch* element of a gear tooth?

(*c*) What is meant by a *straight* spur gear, by a *curved* spur gear, by a *helical* spur gear? Make a sketch showing the axes, the pitch surfaces, and the pitch elements of the teeth of a pair of straight spur gears, also of a pair of helical spur gears.

122.2. (*a*) What are the pitch surfaces and the relation of the axes of *bevel* gears?

(*b*) What is meant by the *pitch* element of a gear tooth?

(*c*) What is meant by a *straight* bevel gear, by a *curved* bevel gear, by a *spiral* bevel gear, by a *skew* bevel gear? Make a sketch showing the axes, the pitch surfaces, and the pitch elements of the teeth of a pair of straight bevel gears, also of a pair of spiral bevel gears.

122.3. (*a*) What are the pitch surfaces and the relation of the axes of *helical* gears, of *hyperboloidal* gears?

(*b*) What is meant by the *pitch* element of a gear tooth?

(*c*) Make a sketch showing the axes, the pitch surfaces, and the pitch elements of the teeth of a pair of helical gears.

122.4. Classify in tabular form the different kinds of gears, giving the name, the relation of the axes, the pitch surfaces, the pitch elements of the teeth, the kind of teeth, and the nature of the tooth contact for each kind of gear.

123.1. (*a*) How long have toothed gears been used, about how many years ago was the theory of correct tooth shapes first enunciated, and for about how many years have machines been available for cutting gear teeth?

(*b*) When and by whom was the theory of correct tooth shapes first deduced? When and by whom was the theory of cycloidal tooth outlines first put on a practical basis?

When and by whom were involute tooth outlines and proportions put on a practical basis?

(c) When and by whom was the first power-driven, precision, gear-cutting machine built? When and by whom was the generating method of cutting gears invented?

124.1. By means of a labeled sketch illustrate the following terms: *pitch diameter, pitch circle, addendum circle, dedendum circle, circular pitch, addendum, dedendum, whole depth, working depth, clearance, face, flank, point,* and *root.*

124.2. (a) Express the circular and the diametral pitches in terms of the pitch diameter and number of teeth and deduce the relation between them.

(b) Define the terms *blacklash, pinion,* and *rack.*

125.1. (a) How may mating gear teeth be regarded and why are they used?

(b) State the fundamental law of gear tooth profiles for the positive transmission of motion equivalent to the rolling of the pitch curves.

126.1. Assume a gear with teeth having straight line profiles and by application of the fundamental law of gear tooth profiles show how to determine graphically the path of the point of contact.

126.2. Assume a gear with teeth having straight line profiles and by application of the fundamental law of gear tooth profiles show how to determine graphically the conjugate rack tooth profile.

126.3. Assume a gear with teeth having straight line profiles and by application of the fundamental law of gear tooth profiles show how to determine graphically the conjugate gear tooth profile.

126.4. (a) By the use of a pair of pitch surfaces, a metal template of an arbitrary tooth profile, and a piece of paper, show and explain how the conjugate tooth profile may be graphically determined.

(b) How does this method suggest how the given tooth acting as a cutter could generate its conjugate?

127.1. Explain and prove how a point on the generator G in Fig. 127.1 may be used to describe gear tooth profiles for the pair of pitch curves *1* and *2* that will transmit motion from one to the other equivalent to the rolling of the pitch curves.

128.1. (a) Why is the circle the only generating curve that has been used to describe gear tooth profiles by rolling on the pitch circles?

(b) Define the terms *epicycloid, hypocycloid,* and *cycloid,* show how the first two curves are generated, and explain a simpler method of generating these curves which avoids the necessity of rolling the describing circles.

FIG. 127.1

128.2. (a) Tell what is meant by the involute of a circle, show how it is generated, and explain a simpler method of plotting the curve which avoids the necessity of rolling the generating line.

(b) What is meant by the base pitch?

(c) Briefly explain an approximate method of drawing involute tooth profiles.
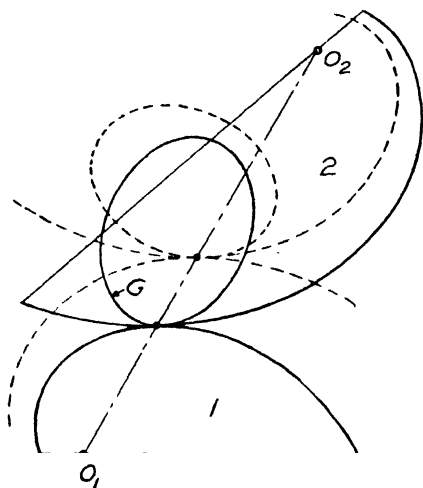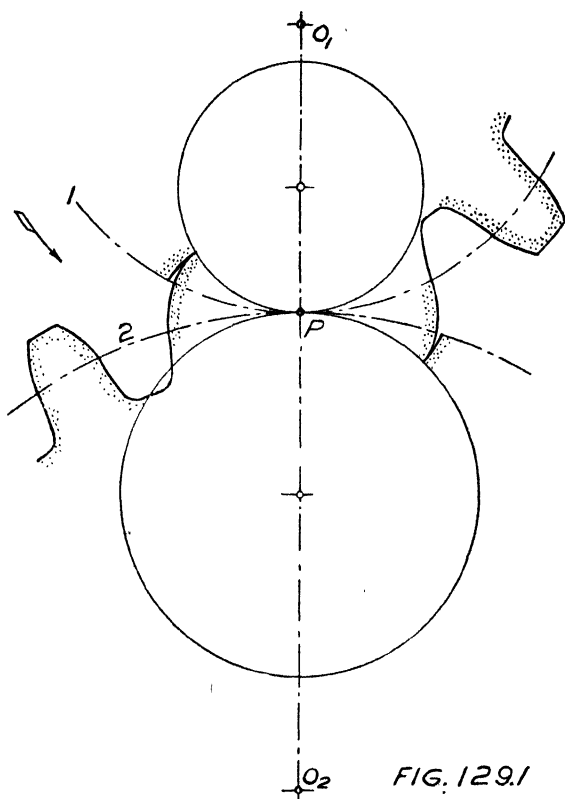
FIG. 129.1

129.1. (a) Make a sketch similar to Fig. 129.1, state the fundamental law of gear tooth profiles, and prove that by means of cycloidal tooth profiles motion equivalent to the rolling of the pitch circles *1* and *2* can be transmitted from one gear to the other.

(b) What must be done if driving is to continue beyond *P*?
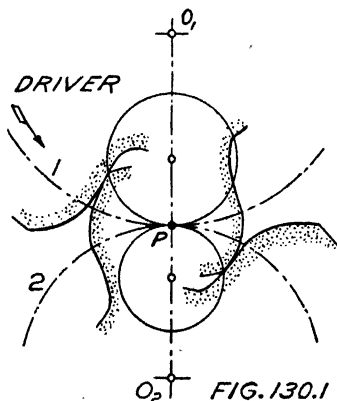
129.2. (a) By use of Fig. 129.1 explain what is done to make it possible for one gear to drive the other in either direction.

(b) How are the points at the beginning and end of action determined and what is the path of the point of contact?

(c) What is required of a pair of gears for positive continuous driving?

130.1. (a) By the use of Fig. 130.1 explain how the path of the point of contact is obtained and show what pitch arcs are of the same length as the path of contact.

(b) Designate the arcs of action and the arcs of approach and recess, also the angles of action, approach, and recess.

FIG. 130.1

130.2. (a) What requirement must be fulfilled to secure continuous driving? What is meant by the contact ratio, what value of this ratio is desirable, and what value should be regarded as a minimum for the quiet and efficient transmission of any appreciable amount of power? For any given circular pitch and pitch diameter, what two things fix the maximum length of teeth and maximum arc of action possible? How does the strength of the teeth limit one of these factors?

130.3. (a) By the use of Fig. 130.1 show what is meant by the *angle of obliquity* and tell how it varies during action. Why is this angle often referred to as the *pressure angle*? What is the effect of the angle of obliquity on the pressure between the teeth and at the bearings?

(b) What is the effect of the friction of sliding between teeth on the obliquity of action of the pressure? How may the smoother action during recess than during approach be accounted for?

130.4. For an assumed pitch line velocity of the gears in Fig. 130.1 show how to obtain the velocity of sliding between the teeth at the beginning of contact. Explain how this velocity of sliding varies during approach and recess.

131.1. (a) State and explain the major requirements for interchangeability of cycloidal spur gears of the same pitch. Name two other requirements for interchangeability.

(b) What is it that determines the diameter of the describing circle of interchangeable cycloidal gears? As a compromise what diameter is usually used?

*(c) What can be said of the path of the point of contact of interchangeable gears?

132.1. (a) What diameter of describing circle was adopted by the William Sellers Company for their system of interchangeable cycloidal gears? Although assuming the smallest pinion to have 12 teeth why did the Brown & Sharpe Company adopt a describing circle diameter equal to the radius of the 15-tooth pinion?

(b) Why are interchangeable gears referred to as equal addendum gears?

(c) About what is the contact ratio and maximum angle of obliquity for a pair of Sellers' 12-tooth pinions?

(d) What is the smallest pair of pinions that would drive continuously for a ratio of describing circle to pitch diameter of 0.50, for a ratio of describing circle to pitch diameter of 0.625?

133.1. Make a sketch of a cycloidal rack and pinion, and, assuming interchangeability, show how the teeth of the rack are described. The rack may be regarded as what kind of gear?

134.1. (a) What is the difference, with respect to the contact of the pitch surfaces, between a pair of internal and a pair of external gears?

(b) How are the tooth profiles of internal cycloidal gears described? How, with respect to faces and flanks, does the action of internal gears differ from that of external gears?

(c) Why must the pitch and describing circles of internal gears be properly related?

134.2. (a) Explain what is meant by saying that the faces of the pinion teeth will not touch or foul the faces of the internal gear teeth if the *intermediate* describing circle of the pinion tooth faces is smaller than the *intermediate* describing circle of the gear tooth faces?

(b) That the active and inactive faces of pinion and gear may not touch or foul $(R_1 + R_4)$ must be less than $(R_2 - R_3)$, where R_1 and R_2 are the respective radii of pinion and gear and R_3 and R_4 are the respective radii of the describing circles of the gear and pinion tooth faces. What does this rule reduce to if the describing circles 3 and 4 are made the same size? What, accordingly, would be the limiting describing circle radius for a pinion and gear whose respective pitch radii are 4 and 8 inches? *Ans.* 2.0 in.

135.1. (a) Make a sketch and show that a pair of pin gears is basically a pair of cycloidal gears for which a single describing circle is used. Of what diameter must the describing circle be made?

(b) Explain how the outlines of the teeth to act with the pins are determined. Why, if possible, should the pin gear be the follower?

(c) Why for continuous driving, assuming the pin gear to be the follower, must a pin be beyond the pitch point before the preceding pin goes out of action?

136.1. (a) Make a sketch similar to Fig. 136.1, state the fundamental law of gear tooth profiles, and prove that with involute tooth profiles motion equivalent to the rolling of the pitch circles *1'* and *2'* can be transmitted from one gear to the other.
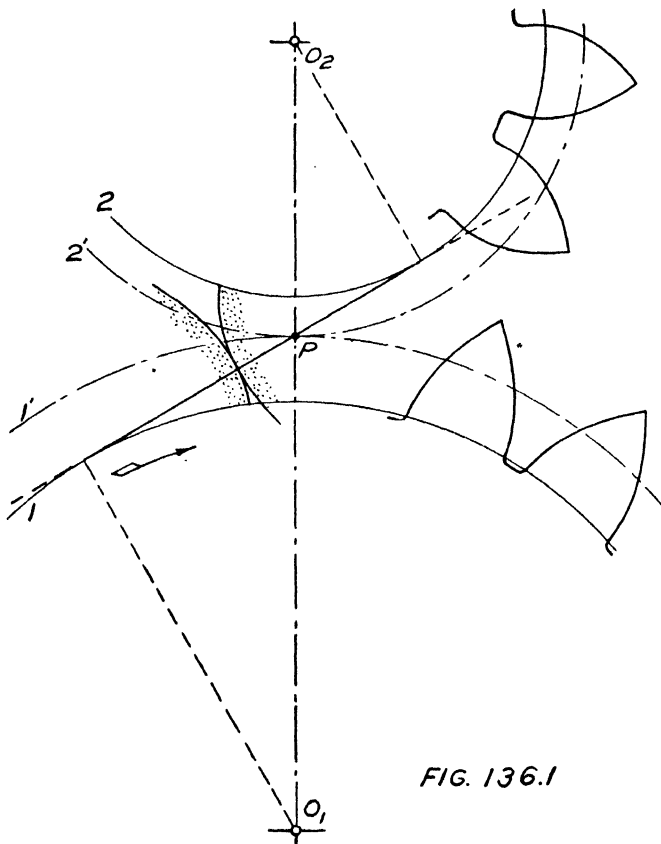
FIG. 136.1

(b) What is the path of the point of contact and how is its length determined?

137.1. (a) By elaborating Fig. 136.1 somewhat, explain how the path of the point of contact is determined.

(b) Determine the arcs of action and the arcs of approach and recess, also the angles of action, approach, and recess.

137.2. (a) Explain what is meant by the *angle of obliquity*. Why is this angle often referred to as the *pressure angle*? What is the effect of the angle of obliquity on the pressure between the teeth and at the bearings?

(b) What is the effect of the friction of sliding between teeth on the obliquity of action of the pressure? How may the smoother action during recess than during approach be accounted for?

137.3. (a) Define the term *base* pitch and determine the value of the base pitch in terms of the circular pitch and angle of obliquity, using Fig. 136.1 if necessary.

(b) What requirement must be fulfilled to secure continuous driving? What is meant by the contact ratio, what value of this ratio is desirable, and what value should be regarded as a minimum for the quiet and efficient transmission of any appreciable amount of power? What fixes the maximum length of teeth and value of the contact ratio for any given pitch, pitch diameters, and angle of obliquity?

138.1. (a) What is one of the most valuable properties of involute gear tooth profiles? Explain the practical value of this property.

(b) By adding to Fig. 136.1 prove that gears *1* and *2* may be pulled apart without affecting the angular velocity ratio or its constancy. Upon what only does the angular velocity ratio of involute gears depend?

(c) When two involute gears are pulled apart, what and how are certain elements of the gears changed? What is it that determines how far a pair of gears may be drawn apart?

139.1 (a) What are the requirements for involute gears of the same pitch to be interchangeable?

(b) What considerations fix the angle of obliquity for interchangeability of involute gears?

140.1 Prove that the tooth profiles of an involute rack are straight lines making an angle of 90 degrees minus the angle of obliquity with the pitch line of the rack.

141.1 (a) Show how to determine the maximum addendum of a rack that will mate with a pinion without interference? What happens if the rack is made with a greater addendum and what is the effect on the pinion teeth if they are cut to mate with the rack teeth?

(b) Explain why the contact ratio is greater for a pinion and rack than for a pinion and gear, also explain why a pinion that will mesh with a rack without interference will mesh without interference with any gear having an equal or greater number of teeth than itself.

141.2. (a) For a rack and pinion, where the addendum of the rack equals k/p'_c, derive the relation $N = \dfrac{2k}{\sin^2 \alpha}$ and $k = \dfrac{N \sin^2 \alpha}{2}$, and interpret the equations.

141.3. (a) What is the smallest number of teeth on a pinion that will mate with a rack without interference for an angle of obliquity of $14\frac{1}{2}$ degrees and a value of k equal to unity? If k is to equal unity, what should the angle of obliquity be for the rack to mate with a 12-tooth pinion without interference?

Ans. $N = 32, \alpha = 24° 5.85'$.

(b) What should be the value of k if a 12-tooth pinion is to mate with a 20-degree rack without interference?   *Ans.* $k = 0.7018$.

142.1. (a) What is the value of the contact ratio for a pair of 12-tooth pinions having $22\frac{1}{2}$ degree involute teeth and an addendum of $0.875/p'_c$? Would the two pinions be suitable for the quiet and efficient transmission of power?   *Ans.* 1.218.

(b) If one or both of a pair of gears had more than 12 teeth would the contact ratio be greater or less than for a pair of 12-tooth pinions?

142.2. (a) What is the value of the contact ratio for a pair of 14-tooth pinions having 20-degree involute teeth and an addendum of $0.80/p'_c$? Would the two pinions be suitable for the quiet and efficient transmission of power?   *Ans.* 1.217.

(b) If one or both of a pair of gears had more than 14 teeth, would the contact ratio be greater or less than for a pair of 14-tooth pinions?

143.1. (a) On a sketch of the basic rack, give the proportions of the addendum and dedendum of the $22\frac{1}{2}$-degree interchangeable involute system of gear teeth proposed by the A.S.M.E.

(b) What is the smallest pinion that will mate with the rack without interference?

(c) What is the contact ratio for a pair of 12-tooth pinions? What is the smallest pair of pinions that will give continuous driving?

143.2. (a) On a sketch of the basic rack, give the proportions of the addendum and dedendum of the 20-degree interchangeable involute system of gear teeth approved by the A.S.A.

(b) What is the smallest pinion that will mesh with the rack without interference?

(c) What is the contact ratio for a pair of 14-tooth pinions? What is the smallest pair of pinions that will give continuous driving?

143.3. (a) What is the smallest pinion that will mesh with a full-depth $14\frac{1}{2}$-degree involute rack without interference?

(b) Give the proportions of the addendum and dedendum for the $14\frac{1}{2}$-degree interchangeable composite system of gear teeth approved by the A.S.A.

(c) Make a sketch and fully explain how the basic rack for the $14\frac{1}{2}$-degree composite system was derived from the 12-tooth pinion.

143.4. What is peculiar about the 20-degree standard involute stub-tooth system of the Fellows Gear Shaper Company? Give one of their pitch designations and explain what it means.

144.1. (a) Explain why the farther a portion of an involute is from the origin the less sensitive it is as to form. Why, as a result, should the active portions of gear-tooth profiles be as far removed as possible from the origin?

(b) By the use of two equal base circles and their involutes fully explain what is meant by *specific sliding*? Why should the specific sliding between gear-tooth profiles be kept as low as possible?

144.2. For a .pair of 22-tooth pinions having 20-degree involute stub teeth of unity diametral pitch, the respective minimum and maximum radii of curvature of the teeth are 1.8118 and 5.7026 inches. Determine the specific sliding on the addendum and dedendum of the teeth.   *Ans.* 0.681 and -2.135.

144.3. For a pair of mating gears having 12 and 30 20-degree involute stub teeth of unity diametral pitch, the respective minimum and maximum radii of curvature are 0.1682 and 3.8014 inches for the pinion teeth and 3.381 and 7.0142 inches for the gear teeth. Determine the specific sliding on the addendum and dedendum of the pinion teeth and the specific sliding on the addendum and dedendum of the gear teeth.

Ans. 0.644, -15.7, 0.94, -1.813.

145.1. (a) Although some of the gears in such machines as lathes, boring mills, automobiles, etc., are interchangeable as to pitch, form, and number of teeth, why are they not in fact interchangeable?

(b) What are the only advantages of cutting gear teeth to an interchangeable system, and what are the disadvantages?

145.2. (a) How otherwise than as non-interchangeable involute gears are such gears referred to?

(b) State two advantages that may be secured by designing gear teeth on a non-interchangeable basis? By what two methods may these advantages be secured and what are the two methods called?

146.1. (a) In the range cutter method for non-interchangeable gears what two things are varied to decrease the sensitiveness of the active profiles and to decrease the wear by reducing the specific sliding?

(b) In the orderly solution outlined by Earle Buckingham how many different generating cutters for each pitch are necessary to cut gears having 10 or more teeth? How many different angles of obliquity and different addenda are employed for each set of generating cutters?

(c) What advantage has the range cutter method over the *variable center distance method*?

147.1. (a) In the variable center distance method for non-interchangeable gears, how many generating cutters are necessary for each nominal pitch? What kind of generating cutters may be used?

(b) In this method what four things are varied to decrease the sensitiveness of the active profiles and to decrease the wear by decreasing the specific sliding?

(c) What is the problem in designing gears to be cut on a variable center distance basis?

147.2. A pair of gears to have 24 and 54 teeth and a nominal diametral pitch of unity are to be generated in accordance with the variable center distance method outlined in Art. 147. Expressing linear dimensions in inches, determine the following for the pinion and gear:

		Pinion	Gear
		—————Ans.—————	
		Pinion	Gear
Root radius	H	11.1662	25.800
Pitch radius	R	12.1062	27.2389
Distance between centers	C	39.3451	
External radius	E	13.3469	27.9807
Circular pitch	p_c	3.1694	3.1694
Tooth thickness	t	1.7170	1.4524
Whole depth of teeth	h	2.1807	2.1807
Addendum	a	1.2407	0.7418
Clearance	c	0.1982	0.1982
Angle of obliquity	α	16° 20′	

147.3. The pair of non-interchangeable gears of problem 147.2 are to be compared with a pair of standard 20-degree stub-tooth gears having 24 and 54 teeth and a diametral pitch of unity. Expressing linear dimensions in inches, determine, for the purpose, the following:

	20° Standard Stub Tooth Gears	14½° Basic Rack Variable Center Distance
Pinion	*————Ans.————*	
Max. radius of curvature of active profile	6.0570	6.3860
Min. radius of curvature of active profile	1.9760	1.8064
Active profile outside pitch circle	0.8000	1.2407
Active profile inside pitch circle	0.5519	0.2476
Total height of active profile	1.3519	1.4883
Specific sliding on addendum of pinion	+0.4650	+0.6675
Specific sliding on dedendum of pinion	−1.5550	−1.3030
Gear		
Max. radius of curvature of active profile	11.3623	9.3560
Min. radius of curvature of active profile	7.2817	4.7764
Active profile outside of pitch circle	0.8000	0.7418
Active profile inside of pitch circle	0.6042	0.4399
Total height of active profile	1.4042	1.1817
Specific sliding on addendum of gear	+0.6080	+0.5655
Specific sliding on dedendum of gear	−1.8700	−2.0060
Contact ratio for pair of gears	1.382	1.537

148.1. (a) What is the purpose of toothed gears?

(b) Name the principal factors involved in the strength of gear teeth.

(c) What assumptions were made by Lewis with respect to the load carried by a gear tooth and how did he arrive at the weakest section of a tooth? The Lewis equation is $W = Sp_cby$; explain the notation of this equation.

148.2. Assume the two pair of gears of problem 147.3 to be cast of steel, to be carefully cut, and to be subjected to a steady load. On the assumption that the gears are 10 inches wide and that the pinion of each pair rotates at 420 R.P.M., determine the safe load and horse-power for each pair.

> Ans. $W = 26{,}050$ lb. and H.P. $= 2084$ for equal add. gears.
> $W = 24{,}450$ lb. and H.P. $= 1975$ for unequal add. gears.

148.3. For durability in operation the maximum stress at contact should not exceed the elastic limit. Determine for the gears of problem 148.2 the safe tooth load and horse-power for a maximum stress at contact of 30,000 lb./sq. in.

> Ans. $W = 4880$ lb. and H.P. $= 390.5$ for the equal add. gears.
> $W = 4080$ lb. and H.P. $= 329.0$ for the unequal add. gears.

148.4. (a) For the least vibration and noise, what should be true of the tooth profiles and their spacing?

(b) What range of sound vibrations per second are audible to the human ear?

(c) When will gears and their casings act as resonators? How may the resonance of gears and gear casings often be reduced? When must consonance be considered and what must be true to secure harmonious sounds?

149.1 (a) What advantage has the involute over the cycloidal form of tooth in the use of formed cutters; what advantages in the use of generating cutters?

(b) Why cannot the sensitive portions of cycloidal tooth forms be eliminated? In what way is this a disadvantage? In what way is it not, as for involute teeth, a disadvantage?

(c) What other advantage has the involute over the cycloidal form of tooth?

(d) Explain the advantages of the cycloidal over the involute form of tooth for external gears.

150.1. (a) What five advantages have internal over external involute gears?

(b) What kind of interference are internal involute gears subject to that external involute gears are not? To avoid such interference how many more teeth should the gear have than the pinion for 20-degree teeth having an addendum of 0.75 divided by the diametral pitch?

(c) By means of a sketch show that a pair of internal gears would have a longer path of contact than a pair of external gears of the same size and tooth proportions. Explain why the internal gear would have teeth of stronger form than the corresponding external gear.

150.2. (a) Why would the specific sliding and wear be less for a pair of internal than for a pair of external involute gears of the same size and tooth proportions?

(b) Why would a pair of internal involute gears lubricate better than a pair of external involute gears of the same size and tooth proportions? Why would the maximum compressive stress at contact be less for the former than for the latter pair of gears?

(c) Which permits of the longer path of the point of contact, a pair of involute or a pair of cycloidal internal gears of the same pitch radii and working depth of teeth?

151.1 What is the Williams' tooth form for internal gears? What are the two advantages of this form of tooth? How is the proposed system for internal gears at a serious disadvantage?

152.1. (a) Why are the teeth of *curved* spur gears almost invariably made in regular helical form? What are such gears called? What is meant by a *herringbone* gear and why are such gears used?

(b) How may the involute tooth surfaces of straight spur gears be regarded as generated, how the involute tooth surfaces of helical spur gears? How does it follow that the contact line of the mating teeth of a pair of helical spur gears is a straight diagonal line and that the tooth sections are involute in outline only in the plane of rotation?

152.2. (a) What is meant by a stepped spur gear? What must be true for a stepped gear to be quieter and smoother in action than a straight spur gear? Why, for cut gears, are not stepped gears used instead of helical spur gears?

(b) Where and for what reason are cast, stepped spur gears sometimes used?

(c) How may a helical spur gear be regarded as a limiting case of a stepped spur gear?

152.3. (a) Why are helical spur gears quieter and smoother in action than straight spur gears?

(b) Why is it possible with helical spur gears to secure continuous driving by the use of pinions that would be too small for continuous driving with straight spur gears?

(c) How, with helical gears, is it theoretically possible to get pure rolling and constant angular velocity ratio combined with positive driving? Why is this possibility of no practical importance?

152.4. (a) Make a sketch of the pitch surface of a helical spur gear showing the pitch elements of adjacent teeth, define the normal circular pitch, and express it in terms of the circular pitch and helix angle. Deduce the relation between the normal diametral pitch and the diametral pitch and helix angle of the gear.

(b) Although helical spur and helical gears appear alike and are cut in the same way, explain why they are differently classified and how they differ in action.

152.5. (a) Treating the force and components in the tangential plane only, show that the load per inch of length of tooth is the same for helical as for straight spur gears. Account for the fact that helical spur gear teeth are actually stronger than the teeth of straight spur gears of the same pitch and width of face.

152.6. (a) How, for helical spur gears, should the circular pitch, width of face, and helix angle be related? What minimum value should the width of face of herringbone gears exceed? Why should large helix angles be confined to high speeds and light loads?

(b) Upon what, according to the American Gear Manufacturers' Association, should the design of helical spur gears be based?

153.1. (a) What is the purpose of non-circular spur gears? Name the most familiar example of a pair of non-circular spur gears.

(b) What would be required in the way of formed cutters to cut the tooth spaces of a pair of non-circular gears with precision? Why have not machines been developed for generating the teeth of non-circular gears?

(c) How may approximately correct cycloidal tooth spaces be cut in non-circular gears?

(d) How may approximately correct involute tooth spaces be cut in non-circular gears?

CHAPTER VIII

CUTTING OF STRAIGHT AND HELICAL SPUR GEARS

154.1. (a) Of what material were the first toothed wheels made and how were the first metallic gears made?

(b) What are the two methods of producing cast gears? Where and when are cast gears still used? Why are pattern molded gears made with blacklash?

154.2. (a) For what two reasons are gears machine molded instead of being pattern molded?

(b) By the use of suitable sketches explain the process of machine molding a gear. Why are machine molded gears more accurate than pattern molded gears?

154.3. (*a*) How are gears die cast? To about what melting temperature is this process limited? What metals and alloys are generally used and what non-metallic materials are so molded? Why are machine parts die cast?

(*b*) Explain the principle of rolling metallic gears to form. What kinds of gears were made by this process and what material was used?

155.1. (*a*) What can be said of gear-cutting machines as contrasted with other kinds of machine tools? What two types of cutting tools do gear-cutting machines use? What is the outstanding difference in the use of these cutting tools?

(*b*) What are the three principal types of gear-cutting machines?

156.1. (*a*) Make a suitable sketch and fully explain the method of cutting a straight spur gear by the use of a formed planing tool. How, with such a tool, would a helical spur gear be cut?

(*b*) What are the objections to the use of a formed planing tool in cutting gears?

156.2. (*a*) Make a suitable sketch and fully explain the method of cutting a straight spur gear by the use of a formed rotating cutter of the earliest type. How, with such a tool, would a helical spur gear be cut?

(*b*) In what two ways is a formed rotating cutter superior to a formed planing tool?

156.3. (*a*) When was the Brown type of formed, rotating cutter invented? In wh‧ ‧ way is it superior to the earliest type of rotating cutter and in what way s‧ formed planing tool?

(*b*) Make a suitable sketch and fully explain the method of cutting a straight spu‧ gear by the use of a Brown type of cutter. When only will the method result in correc‧ shaped tooth spaces?

156.4. (*a*) Make a suitable sketch and‧‧‧‧‧ explain the method of cutting ‧ spur gear by the use of a Brown type of cutter. ‧‧‧‧

(*b*) Why, by using Brown cutters made for straight‧‧, ‧r gears, is it imposs‧ correctly shaped tooth spaces in helical spur gears, it be‧‧ assumed that the the space will correspond to the shape of the cutter?

156.5. (*a*) Determine, in terms of the helix angle and actual number of tee‧ formative number of teeth to which the shape of the tooth spaces of a helical gear ap‧ mately correspond, the radius of curvature at the end of the minor axis of an ellipse t $\rho = R/\cos^2\phi$.

(*b*) Why will not a cutter selected for the formative number of teeth cut a tooth space in a helical spur gear corresponding in shape to itself? Would a cutter selected for a higher or lower number of teeth cut a space more closely corresponding to the formative number of teeth?

157.1. (*a*) What kind and shape of cutting tool is used in the template method of cutting gear teeth?

(*b*) Make a suitable sketch and fully explain the template method of cutting straight spur gears. How, by this method, would a helical spur gear be cut?

(*c*) What is it that determines when the template method would be used?

158.1. (*a*) In the shaping method of generating gear teeth what two types of planing tools are used?

(*b*) Make a suitable sketch and fully explain the method of generating the teeth of a straight spur gear by using a cutter of basic rack form.

158.2. (*a*) In the shaping method of generating gear teeth what two types of planing tools are used?

(*b*) Make a suitable sketch and fully explain the method of generating the teeth of a helical spur gear by using a cutter of basic rack form.

158.3. (*a*) In the shaping method of generating gear teeth what two types of planing tools are used?

(b) Make a suitable sketch and fully explain the method of generating the teeth of a straight spur gear by using a pinion shaped cutter

158.4. (a) In the shaping method of generating gear teeth what two types of planing tools are used?

(b) Make a suitable sketch and fully explain the method of generating the teeth of a helical spur gear by using a pinion shaped cutter

159.1. Make a sketch of a triple-thread screw and of the triangle showing the relation of the pitch elements of the screw Define the terms *lead*, *axial pitch*, and *lead angle* Express the tangent of the lead angle in terms of the lead and pitch radius of the screw, the axial pitch in terms of the lead and number of threads, and the normal pitch of the screw in terms of the axial pitch and a function of the lead angle

159.2. (a) What is a worm? To properly drive a straight spur gear by such a screw-like member, what must be true of the form of the threads and what two pitches must be equal? Deduce the relation between the axial pitch of the worm, the circular pitch of the gear, and the lead angle of the worm

(b) Of what material are hobs made, how are they made, and for what purposes are they used?

159.3. (a) Make a suitable sketch and fully explain how the teeth of a straight spur gear are generated by the hobbing process At what angular velocity ratio must the hob and gear blank be driven? Is this ratio affected by the rate of feed of the hob?

(b) In what way is the hobbing method superior to the shaping method of generating gear teeth?

159.4. Make a sketch of the developed pitch surface of a right-hand helical gear and show that for $\frac{N_2}{N_1}$ re view of a right-hand hob the blank must be driven at more than one revolution for n the peripheral, the coincide rate the teeth of the helical spur gear Show that the angular velocity and sliding must be

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pp **159.5.** r of helical spur gears?

make a sketch showing the pitch surfaces and the pitch element
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spur right-hand helical gears, assume the direction of rotation of one,
and the
explain ding direction of rotation of the other?
ne d

assume (a) Make a sketch showing the pitch surface and the pitch el
the
erly re teeth of a helical gear and determine the relation between the circu
mal
and between the circular and normal diametral pitches.

thread) Make a sketch of the developed pitch surface and lead triangle of a al gear
determine the number of teeth in terms of the pitch radius, normal diame al pitch,
helix angle, and express the lead in terms of the pitch radius and helix angle.

173.2. (a) How may the tooth surfaces of an involute helical gear be considered as rated?

(b) For a helix angle ϕ and an angle of obliquity α in the plane of rotation, determine radius of the base cylinder of a helical gear and the tangent of the helix angle ϕ' of the cylinder as a function of α and ϕ. How is the angle ϕ' related to the method of generation mentioned in (a)?

174.1. (a) Make a plan view of the pitch surfaces of a pair of helical gears, and from the linear velocities of the coincident points in contact determine the angular velocity ratio of the gears in terms of the pitch radii and helix angles.

(b) Assuming the pinion to have 16 teeth of 4 diametral pitch, determine N_2 and the pitch radii of pinion and gear.   *Ans.* $N_2 = 24$, $R_1 = 2$ in., $R_2 = 3$ in.

166.1. (a) Explain why a pair of bevel gears will run more smoothly than a pair of spur gears of the same pitch and number of teeth?

(b) Explain the action between bevel gear teeth. What condition must be fulfilled for positive driving and how may the contact ratio of a pair of bevel gears be approximated?

167.1. (a) State the requirements for interchangeability of bevel gears.

(b) Why is it not found desirable to make interchangeable bevel gears?

168.1. (a) What is meant by a *straight* bevel gear, a *curved* bevel gear, a *spiral* bevel gear, and a *skew* bevel gear? Is there any difference in the action of the teeth of such bevel gears?

(b) Make a sketch of a right-hand spiral bevel pinion and gear showing the pitch surfaces and pitch elements of the teeth. What are the advantages of spiral over straight bevel gears and why do the former run more quietly than the latter?

168.2. Make a sketch of a right-hand spiral bevel pinion and gear showing the pitch surfaces and pitch elements of the teeth, and explain how the direction of the thrust along the pinion shaft is determined for clockwise rotation of the pinion and then how determined for counter-clockwise rotation, the pinion to be the driver.

169.1. (a) Make a sketch similar to Fig. 169.1 (a) and (b) and fully explain how the
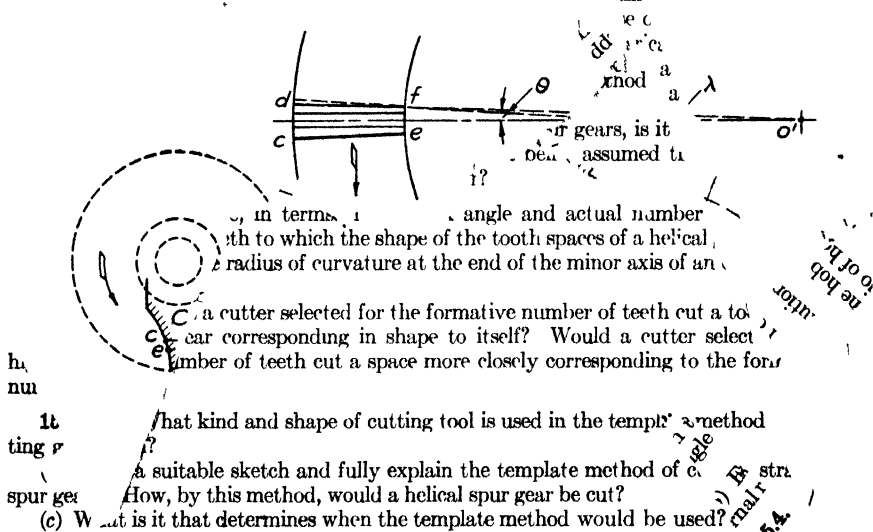
gears, is it assumed

in terms angle and actual number th to which the shape of the tooth spaces of a helical radius of curvature at the end of the minor axis of an a cutter selected for the formative number of teeth cut a tooth ear corresponding in shape to itself? Would a cutter select mber of teeth cut a space more closely corresponding to the form

What kind and shape of cutting tool is used in the template method? a suitable sketch and fully explain the template method of cut straight spur gear How, by this method, would a helical spur gear be cut?

(c) What is it that determines when the template method would be used?

158.1. (a) In the shaping method of generating gear teeth what two types of tools are used?

(b) Make a suitable sketch and fully explain the method of generating the teeth straight spur gear by using a cutter of basic rack form.

158.2. (a) In the shaping method of generating gear teeth what two types of planing tools are used?

(b) Make a suitable sketch and fully explain the method of generating the teeth of helical spur gear by using a cutter of basic rack form.

158.3. (a) In the shaping method of generating gear teeth what two types of planing tools are used?
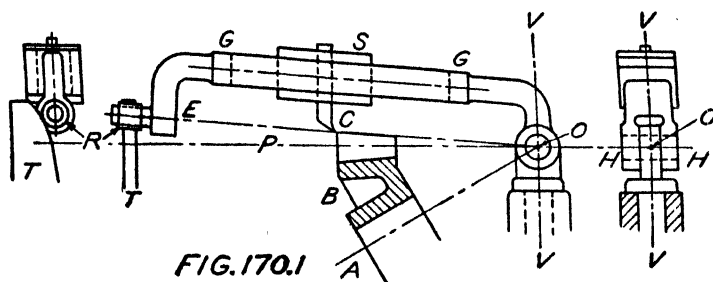
FIG. 170.1

171.1 Make suitable sketches and explain the Gleason method of generating the teeth of straight bevel gears by the shaping process.

171.2. Make suitable sketches and explain how the teeth of spiral bevel gears of the Gleason type are generated by the shaping method. Make a separate sketch of the type of cutter used.

CHAPTER X

HELICAL AND HYPERBOLOIDAL GEARS

CUTTING OF HELICAL GEARS

172.1. (a) What are the pitch surfaces and the relation of the axes of helical gears? For what purpose are such gears used?

(b) Make a plan view of the pitch surfaces of a pair of helical gears whose axes are at an angle θ, assume peripheral velocities of the pitch cylinders and show how to determine the helices through the coincident points of contact and the corresponding velocity components of rolling and sliding.

(c) In order that the contemplated sliding may take place, what must be true of the teeth for the pitch cylinders?

172.2. (a) How may the involute teeth of helical spur and helical gears be considered as generated?

(b) How, with respect to tooth action and otherwise, does a pair of helical gears differ from a pair of helical spur gears?

(c) Make a sketch showing the pitch surfaces and the pitch elements of the teeth of a pair of right-hand helical gears, assume the direction of rotation of one, and indicate the corresponding direction of rotation of the other?

173.1. (a) Make a sketch showing the pitch surface and the pitch elements of the adjacent teeth of a helical gear and determine the relation between the circular and normal pitches and between the circular and normal diametral pitches.

(b) Make a sketch of the developed pitch surface and lead triangle of a helical gear and determine the number of teeth in terms of the pitch radius, normal diametral pitch, and helix angle, and express the lead in terms of the pitch radius and helix angle.

173.2. (a) How may the tooth surfaces of an involute helical gear be considered as generated?

(b) For a helix angle ϕ and an angle of obliquity α in the plane of rotation, determine the radius of the base cylinder of a helical gear and the tangent of the helix angle ϕ' of the base cylinder as a function of α and ϕ. How is the angle ϕ' related to the method of generation mentioned in (a)?

174.1. (a) Make a plan view of the pitch surfaces of a pair of helical gears, and from the linear velocities of the coincident points in contact determine the angular velocity ratio of the gears in terms of the pitch radii and helix angles.

(b) Express the number of teeth on each gear in terms of the pitch radius, the normal diametral pitch, and the helix angle. Express the distance between centers in terms of the numbers of teeth, the normal diametral pitch, and the helix angles of the gears.

175.1. (a) What is the distinguishing difference between a worm and worm-wheel pair and a pair of regular helical gears?

(b) If a worm is to drive involute spur and helical gears what kind of teeth must the worm have and how may they be considered to be generated?

(c) Why are worms seldom used to drive straight spur or helical gears?

(d) For a worm and worm wheel to be conjugate, what must be true of the hob and worm?

175.2. (a) Make a sketch of a double-thread right-hand worm and its lead triangle and express the tangent of the lead angle in terms of the lead and pitch radius of the worm, the axial pitch of the worm in terms of the lead and number of teeth or threads of the worm, and the normal pitch in terms of the axial pitch and lead angle of the worm.

(b) The angular velocity ratio of a pair of helical gears was shown to be

$$\frac{\omega_1}{\omega_2} = \frac{R_2 \cos \phi_2}{R_1 \cos \phi_1}.$$

Assuming the axes at 90 degrees, show that the angular velocity ratio of a worm and worm wheel is independent of the pitch radius of the worm. Why, however, should the pitch radius of the worm be chosen with due regard to the axial pitch and number of threads?

175.3. (a) The tangent of the lead angle of a worm having an axial pitch of 1 inch and a pitch diameter of 2.5 inches is 0.38197; determine the lead angle and the number of threads on the worm.   *Ans.* $\lambda = 20° 54.5'$, $N_1 = 3$.

(b) What would be the angular velocity ratio and the distance between centers if the worm mated with a worm wheel having 60 teeth?   *Ans.* $\frac{\omega_1}{\omega_2} = \frac{20}{1}$, $C = 10.7993$ in.

175.4. (a) For a worm and worm wheel, about what lead angle gives the highest efficiency?

(b) Why are wear and efficiency related? What is the range of the lead angle for least wear, for wear that would not be serious, and for wear that would not be destructive? What are the corresponding ranges of the helix angle? How may these ranges be applied to the design of regular helical gears?

176.1. Shafts at an angle of 60 degrees are to be driven at an angular velocity ratio of 2 by a pair of helical gears having a normal diametral pitch of 4. Assuming the helix angles of the gears to be equal and the center distance to be approximately 6.5 inches, determine the number of teeth, the pitch radii, and the center distance for the gears.
Ans $N_1 = 15$, $N_2 = 30$, $\phi_1 = \phi_2 = 30°$, $R_1 = 2.165$ in., $R_2 = 4.330$ in., $C = 6.495$ in.

176.2. Shafts at an angle of 75 degrees and exactly 5.5 inches apart are to be driven at an angular velocity ratio of 1.5 by a pair of helical gears having a normal diametral pitch of 4. Determine the number of teeth, the helix angles, and the pitch radii of the gears.
Ans. $N_1 = 14$, $N_2 = 21$, $\phi_1 = 38° 39'$, $\phi_2 = 36° 21'$, $R_1 = 2.24077$ in., $R_2 = 3.2592$ in.

177.1. Show and prove that the relations between the dimensions of a pair of helical gears may be represented graphically for an angle between shafts of θ, an angular velocity ratio of driver to follower of ω_1/ω_2, a distance between centers of C, and a desired helix angle for one of the pair of gears.

177.2. Having given θ, ω_1/ω_2, p'_n, and C, show how to determine graphically the values of N_1 and N_2, D_1 and D_2, and ϕ_1 and ϕ_2 for a pair of helical gears. What check should be applied to the graphical solution?

178.1. (a) Why is there no difference in the cutting of a helical gear and the cutting of a helical spur gear?

(b) How are single- and double-thread hobs fluted? How are hobs having more than two threads fluted?

(c) Make a suitable sketch and fully explain the hobbing of a worm wheel for non-intersecting axis at 90 degrees.

179.1. (a) What are the pitch surfaces and the relation of the axes of hyperboloidal gears?

(b) By what two kinds of gears may motion be transmitted at a constant angular velocity ratio between shafts that are at an angle but do not intersect?

(c) Why are hyperboloidal gears better than helical gears for the transmission of power? What advantage have they over spiral bevel gears?

CHAPTER XI
LINKWORK AND MISCELLANEOUS MECHANISMS

180.1. What is meant by the term *linkwork*? To what kinds of mechanisms does the term apply? To what three conversions of motion may such mechanisms be applied?

181.1 Make a sketch of a four-link mechanism for circular reciprocation of driver and follower, dot in the three limiting positions of the mechanism, and explain how the three positions were determined. Explain what is meant by dead point positions and why the reciprocation of the follower should be such as to avoid dead point positions. How are the lengths of the links related in such a mechanism, and what must be the relation for complete rotation of the driver?

181.2. Make a sketch of a Corliss valve and driving mechanism with the steam edge of valve and port coincident and explain the operation of the mechanism. What characteristics of the motion of the valve on its seat are sought in proportioning the links? For the case represented, show how to find the angular velocity ratio of valve to driver by the instant center method and explain how this ratio varies during a complete cycle.

182.1. Make a sketch of a four-link mechanism for rotation of the driver and circular reciprocation of the follower. Why would the motion of the follower be incompletely constrained if $(l_2 - l_1) = (l_0 - l_3)$ or if $(l_2 + l_1) = (l_0 + l_3)$? What must be the relation of the lengths of the links for complete rotation of the driver and complete constrainment of the follower? Show how to find for such a mechanism the angular velocity ratio of the follower to driver by the method of instant centers.

183.1. Make a sketch of a four-link mechanism for continuous rotation of driver and follower. Why would not the length relations $(l_2 + l_1) = (l_3 + l_0)$ and $l_2 = (l_3 - l_0 + l_1)$ be satisfactory for such a mechanism? How must the lengths of the links be related? Show how to find for such a mechanism the angular velocity ratio of follower to driver by the method of instant centers. What is such a four-link mechanism called?

183.2. If a drag link is used to connect two cranks the axes of whose shafts coincide, what is the angular velocity ratio of the two cranks? In what way might such an arrangement be superior to a shaft with a center crank? What effect has misalignment of the shafts on the angular velocity ratio of the cranks?

183.3. Why in a drag-link mechanism will not rotation of the driver cause continuous rotation of the follower when the cranks are of equal length and the length of the drive rod is equal to the distance between the centers of rotation? Sketch and explain how, for these proportions, complete constrainment of the follower is secured in the case of locomotive drive wheels? Show, by means of sketches, two ways of

connecting a driver and two followers to secure complete constrainment of the followers. How must the crank centers and centers of rotation be spaced and which crank should be the driver?

184.1. Make a sketch of a mechanism having four binary links and three turning and one sliding pair for converting rotation into circular reciprocation. Explain what changes are necessary to transform such a mechanism into the common slider-crank chain.

185.1. Make a sketch of the common slider-crank mechanism having a connecting-rod of finite length and explain how to find the position of the cross-head for any crank position and how to find the crank positions for any cross-head position. What is meant by "the angularity of the connecting-rod"? Express the distance from the mid-stroke position of the cross-head to its position corresponding to either 90-degree position of the crank in terms of the radius R of the crank and length l of the connecting-rod. To what is this distance equal for a rod of infinite length?

185.2. Make a sketch of the common slider-crank mechanism having a rod of finite length l and crank radius R and determine an expression in terms of l and R for the distance from the line of travel of the cross-head pin center to the center of the crank-pin when the linear velocity of the cross-head first equals that of the crank-pin center. Compute the distance for $l = 60$ and $R = 12$ inches. *Ans.* 11.168 in.

185.3. For the common slider-crank mechanism having a rod of finite length l and crank radius R, the acceleration of the cross-head is

$$A = -\omega^2 R \left(\cos \theta + \frac{R}{l} \cos 2\theta \right)$$

where ω is the constant angular velocity of the crank in radians per second and θ the angular displacement of the crank. By use of the equation, determine an expression for the displacement of the crank at the instant the linear velocity of the cross-head is a maximum. Determine the angle in degrees and minutes for $l = 60$ and $R = 12$ inches, and determine also the angle in degrees and minutes when the rod is perpendicular to the crank. *Ans.* 79° 16′ 21″ and 78° 41′ 24″.

185.4. Make a sketch of a slider-crank mechanism with an offset crank and explain how to determine the stroke of the cross-head. Assuming the mechanism to be used as a quick return motion and the crank to rotate at a uniform rate, show how to determine the path of travel of the cross-head and the offset of the crank for a time ratio of forward to return motion of 1.4/1 and for a connecting-rod length of three times the crank length.

186.1. Make a sketch of a double slider-crank mechanism having a connecting-rod in the form of a ternary link whose length is equal to twice that of the crank. Explain what addition will make this mechanism into an elliptic trammel and prove that the pencil point will trace an ellipse.

186.2. Make a sketch of an Oldham coupling and explain what inversion of the double slider-crank mechanism it represents. Make a suitable sketch and prove that the inverted mechanism may be used in the form of an Oldham coupling to transmit motion between parallel shafts at a constant angular velocity ratio.

187.1. What name has been given to the slider-crank mechanism having a rod of infinite length? Make a sketch of such a mechanism and prove that the cross-head will have simple harmonic motion for uniform rotation of the crank. What advantages and disadvantages has this mechanism as applied to steam fire engine pumps?

188.1. (a) Make a sketch of a shaft, eccentric, eccentric-rod, and sliding block and show that the mechanism is the exact equivalent of the common slider-crank mechanism having a crank length equal in length to the distance between the centers

of shaft and eccentric. Explain what is meant by the eccentricity, or throw, of an eccentric.

(b) Where and why are eccentrics used in preference to cranks? Why are eccentrics not used in place of large throw cranks?

189.1. If L is the distance from the center of oscillation of a beam to the center line of travel of a piston or ram whose stroke is $2S$, show that for minimum angularity of the connecting-rod the length of the beam from the center of oscillation should be $OG = L + \dfrac{S^2}{4L}$. Determine OG in feet and inches for a stroke of 8 feet and for L equal to 12 feet.   *Ans.* 12 ft. 4 in.

190.1. A bell crank is to transmit motion in the ratio of 2 to 1, respectively, along two lines OA and OB intersecting at 120 degrees; show how to determine graphically the center of oscillation and the arms of the bell crank. Explain how the equation given in 189.1 would be used in determining the radii of the arms.

191.1. (a) Explain why quick return mechanisms are used in shapers and slotters.

(b) Explain the operation of the quick return mechanism shown in Fig. 62.1 and show how to determine the time ratio of the cutting to the return stroke. What practical method is used for changing the length of stroke, and what effect has changing the length of stroke on the time ratio?

(c) Assuming the driving link to rotate at a uniform rate show how to determine the linear velocity of the ram by the method of instant centers.

191.2. Same problem as 191.1 but applied to Fig. 62.7.

191.3. Same problem as 191.1 but applied to Fig. 36.3 (c).

191.4. Same problem as 191.1 but applied to Fig. 191.4.
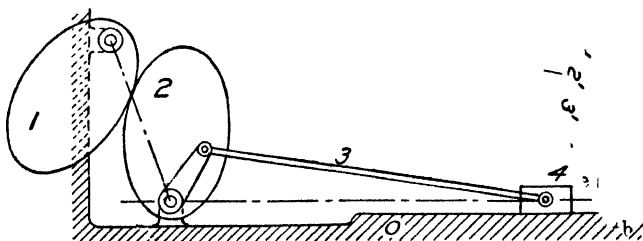
FIG. 191.4

191.5. Same problem as 191.1 but applied to a sketch of the drag link, or Mac Cord, quick return mechanism.

192.1. The application of the floating lever shown in Fig. 192.1 is taken from what field of engineering? Why is lever *3* called a floating lever? Explain what takes place when the operating, or control, lever *1* is moved to the left; when it is moved to the right. Name some machines in which the floating, or differential, lever is used.

193.1. Explain the operation of the variable stroke mechanism shown in Fig. 193.1. How may the stroke of *8* be reversed?

194.1. (a) What are the requirements for perfect side pivot automobile steering? Why, for the front wheels turned as shown in Fig. 194.1, would there be no tendency to skid? By the use of the figure derive the necessary relation between α and β in terms of the pivot distance P and length L of the wheel base.

(b) Make a sketch of the Ackermann automobile steering mechanism with the front wheels turned, the straight-ahead phase of the mechanism being shown by heavy dotted lines. Why are not perfect steering conditions possible for all phases of this mechanism? What should be kept in mind in proportioning the mechanism?

FIG. 192.1

FIG. 193.1
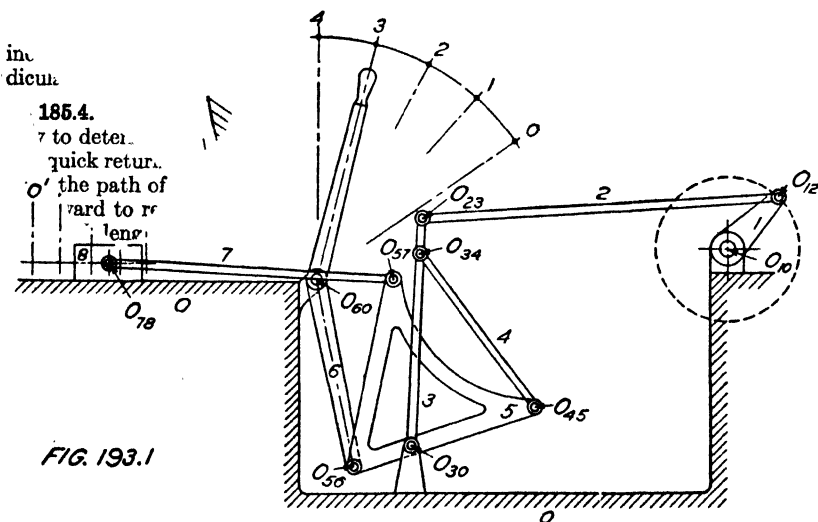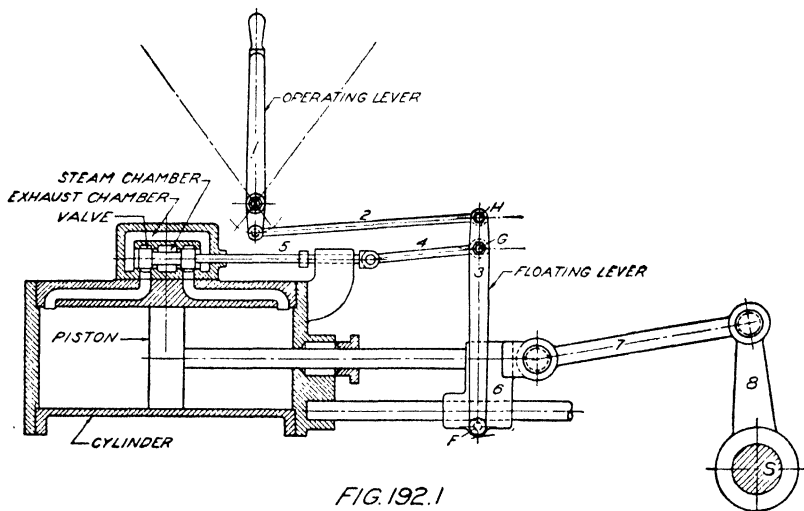
195.1. Make a sketch of Watt's straight-line mechanism. For what purpose did Watt use this mechanism? What should be the relation of the links for maximum length of the straight-line motion, and how, for best results, should the point P on the connecting link be located? Is the motion of P exact or approximate?

195.2. Make a sketch of the essential parts of an engine indicator and describe how and for what purpose it is used.

195.3. Make a diagrammatic sketch of the straight-line mechanism of an engine indicator and explain how the proportions of the links are arrived at for straight-line motion of the pencil. Is the motion of the pencil limited or unlimited, and is it exact or approximate?

195.4. Make a sketch of Roberts straight-line mechanism and state the length of the connecting link in terms of the distance between the fixed centers of rotation of the side links. Is the path of the tracing point P exact or approximate? The condition giving the minimum length of the side links for the tracing point at one of the fixed centers occurs when the connecting link and a side link are in line. Prove, for this condition, that the minimum length of the side links must be 0.593 times the distance between the fixed centers of rotation.

FIG. 194.1
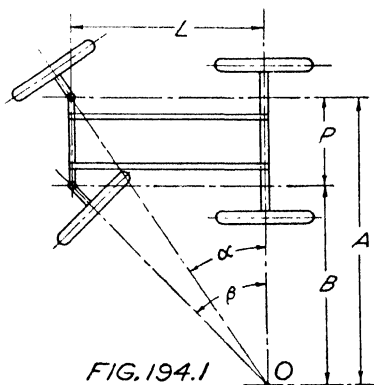
195.5. Make a sketch of Tchebicheff's straight-line mechanism. Is the path of the tracing point exact or approximate? Prove that the length of the side links must be equal to 1.25 times the distance between the fixed centers of rotation for P to lie in the extension of the mid position of the connecting link when the connecting link and either side link are vertical.

195.6. Make a sketch of Paucellier's straight-line mechanism and prove that the motion of the tracing point is exact.

196.1. (a) What is the purpose of a pantograph and where are such mechanisms used?

(b) Make a sketch of a pantograph and explain how the mechanism would be used to enlarge a drawing.

196.2. (a) What is the purpose of a pantograph and where are such mechanisms used?

(b) Make a sketch of a pantograph, properly locate two points in the mechanism, and prove that the two points will trace exactly similar paths.
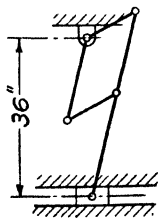
FIG. 196.3

196.3. The pantograph shown in Fig. 196.3 is to be used as an indicator reducing motion on an engine having a stroke of 28 inches. Show how to determine the point of attachment of the indicator drum cord and the necessary addition to the mechanism to take an indicator card 3.5 inches long. In what direction must the cord run from its point of attachment?

197.1. (a) Make a sketch of Hooke's coupling, or universal joint.

(b) For two shafts intersecting at an angle β and connected by a universal joint, the angular velocity ratio of follower to driver for any displacement θ of the driver is

$$\frac{\omega_n}{\omega_m} = \frac{\cos \beta}{1 - \sin^2 \theta \sin^2 \beta}$$

or in terms of β and the angular displacement ϕ of the follower,

$$\frac{\omega_n}{\omega_m} = \frac{1 - \cos^2\phi\,\sin^2\beta}{\cos\beta}$$

For what values of either θ or ϕ is the angular velocity ratio of follower to driver a maximum; for what values a minimum?

(c) Make a sketch and explain how, by the use of an intermediate shaft and two couplings, the angular velocity ratio of follower to driver may be made constant and equal to unity.

197.2. Two shafts intersecting at an angle β are fitted with a universal joint; make a suitable sketch and derive the relation of the angular displacement ϕ of the follower in terms of β and the angular displacement θ of the driver.

197.3. (a) Two shafts intersecting at 60 degrees are connected by a universal coupling. By use of the proper equation from problem 197.1, determine the angular velocity ratio of follower to driver for a displacement of the driver fork of 45 degrees from the vertical to the plane of the shafts. *Ans.* 0.80.

(b) Having found the angular velocity ratio, determine the angular displacement of the follower. *Ans.* 26° 34′.

198.1. (a) For what purpose are intermittent gears used?
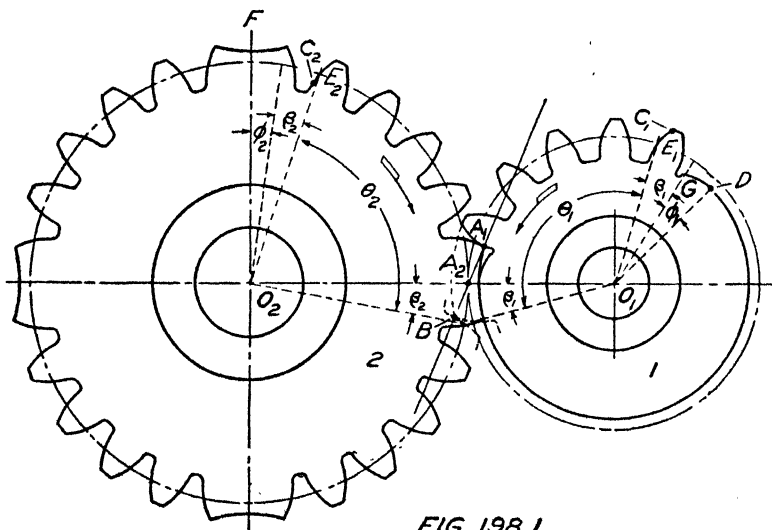
FIG. 198.1

(b) Explain the operation of the pair of intermittent gears shown in Fig. 198.1 and state the ratio of the period of dwell to the period of motion of the follower.

(c) Why, except at low speeds, is the action of intermittent gears unsatisfactory?

199.1. (a) For what purpose are Geneva wheels used, and why, for this purpose, are they much superior to intermittent gears? Name some applications of Geneva wheels.

(b) Explain the operation and features of construction of the pair of Geneva wheels shown in Fig. 199.1. What is the ratio of the period of dwell to the period of motion of the follower?
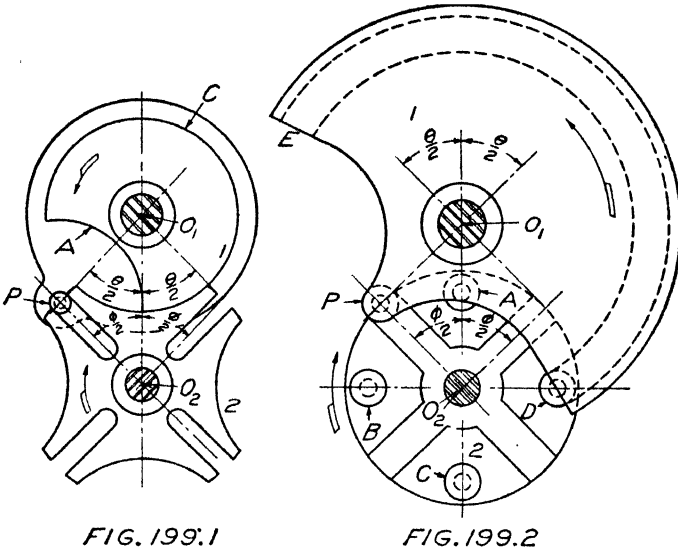
FIG. 199.1   FIG. 199.2

199.2. Same as problem 199.1 but applied to Fig. 199.2.

200.1. (*a*) For what purpose are ratchet and pawl mechanisms used?

(*b*) Make a sketch of a toothed ratchet wheel and pawl mechanism and explain its operation.

(*c*) How must the ratchet teeth and pawl be proportioned to prevent the pawl from being thrown out of action?

(*d*) Why is it sometimes necessary to use a locking pawl? What expedients may be employed to reduce the amount of overhaul?

200.2. (*a*) For what purpose are ratchet and pawl mechanisms used?

(*b*) Show by means of a sketch what is meant by a double-acting ratchet and state what advantage it has over a single-acting ratchet.

(*c*) Make a sketch of a friction ratchet and pawl mechanism. What advantages has such a ratchet mechanism over mechanisms using toothed ratchet wheels?

201.1. (*a*) What is the difference in purpose between a ratchet and an escapement mechanism? Where are escapements used?

(*b*) Make a sketch of an escapement mechanism and explain its operation. For a pendulum control what action must there be between the escape wheel and anchor pallet at each release of the wheel?

CHAPTER XII

BELT, ROPE, AND CHAIN TRANSMISSION

202.1. (*a*) What is the average thickness of single-ply leather belting? How are double-ply and triple-ply leather belts made and about what is the average thickness of double-ply belts of ordinary widths?

(*b*) How are rubber belts made and to what does the ply correspond? Where are rubber preferred to leather belts?

(*c*) How are cotton belts made? What is meant by balata belting? How is camel's hair belting made?

203.1. (*a*) For the continuous driving of one pulley by another by means of a belt, prove that, neglecting slip, the angular velocities of driver and follower are inversely as their diameters plus the thickness of the belt. Why is the effect of the thickness of the belt often neglected?

(*b*) When are belts used in preference to other means of transmitting power? What is the maximum distance between shaft centers for the usual belt widths and pulley diameters; what is usually regarded as the minimum distance between centers when belt tighteners are not used? When is belt instead of chain transmission used?

204.1. (*a*) .Make a sketch of a pulley and belt and prove that the turning moment on the pulley is equal to the radius of the pulley times the difference in the pulls on the tight and slack sides of the belt.

(*b*) Determine an expression for the horse-power that may be transmitted by a belt in terms of the total effective pull of the belt and the diameter and revolutions of the pulley.

(*c*) Name the principal factors upon which the effective belt pull depends.

204.2. A 35-inch pulley rotating at 240 R.P.M. is to transmit 32 horse-power; determine the width of a double-ply leather belt for an effective pull per inch of width of 80 pounds.   *Ans.* 6 in.

205.1. Make a suitable sketch and fully explain how a belt is shifted. Of what do belt-shifting mechanisms consist and how should they be placed?

206.1. (*a*) Make a suitable sketch and fully explain why crowning is effective in keeping a belt from running off a pulley.

(*b*) How in general are pulleys crowned? What is a very common value for the height of crown? When for a leather belt the crowning of the pulley is varied with the belt speed, would low or high speed call for the greater crowning?

207.1. Make a suitable sketch, including the belt shifter, and fully explain how and why tight and loose pulleys are used. For such a drive, explain which of the pulleys should be crowned.

208.1. Make a suitable reference sketch and derive an expression for the length of a crossed belt in terms of the pulley radii and the distance between axes.

208.2. Make a suitable reference sketch and derive an expression for the length of an open belt in terms of the pulley radii and the distance between axes.

209.1. Make a sketch of a pair of equal cone pulleys; let R_1, R_2, etc., be the respective radii of the smallest to the largest step of the driving cone, and r_1, r_2, etc., be the respective radii of the largest to the smallest step of the driven cone; let ω be the constant angular velocity of the driver and ω_1, ω_2, etc., be the angular velocities of the driven cone for the belt on the respective steps 1, 2, etc. Assuming the speeds of the driven cone to be in geometric progression and k to be the geometric ratio, derive the expression $\dfrac{\omega_n}{\omega_1} = k^{n-1}$, where ω_n is the angular velocity of the driven cone for the belt on the nth step. Assuming the cones equal, derive the expression

$$\frac{r_1}{R_1} = \sqrt[4]{k^{n-1}}.$$

209.2. The speeds of the driven cone of a pair of equal cone pulleys are to be in geometric progression; the progression ratio k is not to be less than 1.25; and the maximum and minimum angular velocities, ω_n and ω_1, of the driven cone are to be 180 and 60. Having given $\dfrac{\omega_n}{\omega_1} = k^{n-1}$ and $\dfrac{r_1}{R_1} = \sqrt{k^{n-1}}$, determine the number of steps of the cones and the radii of their smallest and largest steps, the smallest radius

to be not less than $2\frac{1}{2}$ nor more than $2\frac{3}{4}$ inches; also determine the value of k and the speeds of the driven cone.

Ans. $n = 6$; $R_1 = 2\frac{5}{8}$ in., $r_1 = 4\frac{17}{32}$ in., $k = 1.2457$, R.P.M. $= 60$, 74.74, 93.11, 116.1, 144.5, 180.0.

209.3. The length of a crossed belt is,

$$L = 2\sqrt{C^2 - (\overline{R + r})^2} + (R + r)\pi + 2(R + r)\sin^{-1}\left(\frac{R + r}{C}\right)$$

and

$$\frac{\omega_x}{\omega} = \frac{R_x}{r_x} = \frac{R_1}{r_1}k^{x-1},$$

where k is the progression ratio, ω the constant angular velocity of the driving cone and ω_x the angular velocity of the driven cone with the belt on step x. Determine the relation between R_x, r_x, R_1, and r_1 for any given distance C between centers, and then determine an expression for r_x in terms of R_1, r_1, k, and x.

209.4. (a) Why are cone pulleys usually designed for open instead of crossed belts?

(b) By inspection of the equation which follows for the length of an open belt, demonstrate that the radii of the steps of a pair of cone pulleys are not independent of the distance between centers.

$$L = 2\sqrt{C^2 - (R - r)^2} + (R + r)\pi + 2(R - r)\sin^{-1}\left(\frac{R - r}{C}\right)$$

(c) What is the most practical way of determining the radii of the steps of a pair of cone pulleys for an open belt and a given distance between centers? What practically are the only errors involved in the use of this method?

210.1. (a) Make a sketch showing the approximate shape of a pair of speed cones for a crossed belt and then for an open belt for speeds in geometric progression for equal increments of shift of the belt. How do the pairs differ in appearance?

(b) Why may a pair of frusta of equal right cones with straight sides be used for crossed belts? Will the speeds change in geometric progression for equal increments of shift of the belt? Under what conditions may such a pair of cones be used with open belts?

(c) What advantage have speed cones over stepped cones and what objections can be urged against the use of speed cones?

211.1. (a) Make a sketch representing the essential parts of a Reeves variable-speed belt transmission and explain its operation. How is the belt constructed and how does its manner of transmitting power differ from that of an ordinary belt?

(b) For about what range of capacities are Reeves transmissions made and for about what range of the ratio of the maximum to the minimum revolutions of the driven shaft?

212.1. (a) What principle must be followed in designing a belt drive to transmit power between non-parallel shafts? Starting with two pulleys in the same plane on parallel shafts, explain how one shaft may be swung out of parallelism with the other shaft without the belt running off, the shafts to remain in parallel planes. What name is given to a belt drive where the shafts are at right angles but not in the same plane?

(b) What must be done to transmit motion in either direction between pulleys on non-parallel shafts by means of a belt?

212.2. Two shafts at right angles but not in the same plane are to be connected by a belt. By means of a two-view orthographic sketch show how this can be done without the use of guide pulleys.

213.1. (*a*) State the principle to be applied and make a two-view sketch of a belt drive with shafts intersecting at 90 degrees that will operate in either direction, angular velocity ratio $\frac{1}{2}$.

(*b*) Make a sketch of a belt drive between non-parallel shafts where the use of a guide pulley is necessary to drive in one direction. What would be necessary to make the drive operate in either direction?

213.2. By means of a two-view orthographic sketch show how two pulleys unequal in diameter on shafts intersecting at right angles can be connected by a belt using two guide pulleys located on a third shaft.

214.1. (*a*) What is the function of belt tighteners? How does the use of belt tighteners affect the distance between centers that may be successfully used?

(*b*) Make a sketch of an adjustable, automatic belt tightener employing a weighted idler pulley, and explain its operation. On which side of the belt should the idler be placed?

214.2. (*a*) What is the function of belt tighteners? How does the use of belt tighteners affect the distance between centers that may be successfully used?

(*b*) Make a sketch of a self-adjusting, automatic belt tightener employing a gear and pinion, and explain its operation.

215.1. (*a*) Make a sketch of a V-belt drive. Why, for short center distances, has this form of drive displaced many flat belt, chain, and gear drives? In what way are V-belt superior to rope drives?

(*b*) Explain in what two ways V-belts may be kept taut.

216.1. (*a*) What are the names of the two systems of rope drive? About when was each system first applied? What is the usual power range of rope transmissions?

(*b*) Make a sketch of a multiple rope drive and state the advantages of the multiple over the continuous system.

(*c*) About what is the limiting distance between centers for fibrous rope used without idlers? What kind of rope is used for extreme center distances?

216.2. (*a*) What are the names of the two systems of rope drive? About when was each system first applied? What is the usual power range of rope transmissions?

(*b*) Make a sketch of a continuous rope drive and state the advantages of the continuous over the multiple system.

(*c*) About what is the limiting distance between centers for fibrous rope used without idlers? What kind of rope is used for extreme center distances?

216.3. (*a*) What can be said in favor of fibrous ripe drives? When are fibrous rope drives cheaper than rubber or leather belt drives?

(*b*) Why should not the pitch diameter of fibrous rope sheaves be less than 36 times the diameter of the rope? What should be true of the sheave grooves?

(*c*) Make sketches of the types of grooves for the transmission sheaves of the multiple and continuous systems, also sketches of the grooves for the guide and tension sheaves of these systems. Why, for the same size and number of ropes, are transmission sheaves wider for the multiple than for the continuous system? What groove angles are common?

(*d*) Make sketches of the grooves for wire rope hoisting and transmission sheaves. For what two reasons are the bottoms of the grooves of transmission sheaves filled with rubber or leather?

217.1. Make a sketch of the kind of chain generally used in this country for hoisting purposes. Of what materials are such chains made? What is meant by *coil* chain; by *pitch* chain?

218.1. (*a*) When are transmission chains used instead of gears or flat belts?

(*b*) Make sketches of the Ewart detachable chain and the interlocking pintle

chain, and state of what material these chains are made and with what kind of sprockets they are used. Give the maximum velocity of transmission, the number of teeth on the smallest sprocket that should be used, and the maximum ratio of the sprockets for such chain.

218.2. (*a*) When are transmission chains used instead of gears or flat belts?

(*b*) Make sketches of the block and the roller chain and state with what kind of sprockets they are used. Why are block chains made in only a few small sizes? Of what materials are the parts of roller chains made? For single-strand roller chain what range of velocity of transmission for general purposes should not be exceeded and what minimum size of sprocket and range of the maximum ratio of sprockets are recommended for such chain?

(*c*) Why, with a sprocket of given diameter, may a given power be transmitted at a higher rotative speed with multiple- than with single-strand chain? What mechanical efficiency is possible with roller chain?

218.3. (*a*) When are transmission chains used instead of gears or flat belts?

(*b*) Make sketches of the Renold and of the Morse chain, and tell how and of what materials they are made. What sustained efficiency of transmission is possible with these chains, and what range of velocity of transmission is recommended as a maximum? What is the minimum size of sprocket that should be used and what range of ratio of sprockets should not be exceeded? Why should the chain if possible have an even number of links?

219.1. (*a*) Make a diagrammatic sketch of a chain transmission, and show, by application of the angular velocity theorem, that motion cannot be transmitted from one shaft to another at a constant angular velocity ratio. In the demonstration in the text, by what percentage did the angular velocity ratio of follower to driver vary below and above that corresponding to the numbers of teeth in the sprockets? By what means is this variation of the angular velocity ratio greatly reduced and chain transmission made applicable to many purposes?

(*b*) Name some applications where chain could not be substituted for gear transmission.

220.1. (*a*) Show by means of a sketch how to determine theoretically correct outlines of the teeth of sprockets for roller chain, assuming the chain and sprocket teeth to have the same pitch.

(*b*) Why, in operation, does the pitch of a chain increase? What expedient was first applied to permit this change of pitch, and how was the action between the chain and sprockets affected? What would be the result if the pitch of the sprocket teeth were originally made greater than that of the chain?

220.2. (*a*) What expedient did Hans Renold apply to roller chain sprocket teeth to prevent defective action as the chain stretched?

(*b*) Show by means of a sketch the Renold form of tooth space for roller chain sprockets. What action causes the chain to move radially outward in contact with the sprocket teeth as the chain stretches because of wear? How is the obliquity of action determined for a sprocket of N teeth?

(*c*) While the Renold and Diamond roller chain sprocket teeth are based on the same principle in what way do they differ in action?

221.1. (*a*) By means of a suitable sketch show how the teeth of the sprockets for toothed, or silent, chain are determined by the links of the chain. Why, as they stretch, do such chains move radially outward in contact with the sprocket teeth? What is the obliquity of action for 60-degree links, and what would be the tooth angle for a sprocket having 12 teeth?

(*b*) What is the primary difference between the Renold and Morse chains?

CHAPTER XIII

TRAINS OF MECHANISM

222.1. (a) What is meant by the term *prime mover*? Name several examples of prime movers and give the source of energy of each.

(b) What may be done with the mechanical energy delivered by a prime mover before it is consumed in doing useful work? Illustrate by an example.

222.2. (a) Give a definition of a machine that would cover all kinds of machines.

(b) Name some kinematic and machine elements that are used for the moving parts of machines. What are properly proportioned and related combinations of such parts called?

222.3. (a) By means of a sketch illustrate the following parts of a metal working lathe: head stock, tail stock, bed, cone pulley, back gears, spindle, face plate, live and dead centers, tool post, cross-slide, tool carriage, feed rod, and lead screw.

(b) Explain how the work may be held and rotated in a lathe.

222.4. (a) Describe how a piece of work mounted between centers in a lathe is turned (machined), giving the motion of the work and the motion of the cutting tool.

(b) Describe how a piece of work to be bored would be mounted in a lathe and describe how it would be machined.

222.5. Describe how a piece of work to be threaded externally would be mounted in a lathe and describe how the screw threads would be cut.

222.6. (a) What other kinematic and machine elements besides gears and other wheels are found in trains of mechanism? Where in machines are such elements generally found? Usually what kind of motion has the driving element of a machine and what kind of elements figure largely in the transmission of mechanical energy to and through a machine?

(b) What is meant by a mixed wheel train, what by a gear train? What is the difference between an ordinary wheel train and a cyclic wheel train?

223.1. (a) For what purpose are sliding gears used? Make a sketch of a sliding gear and explain how it operates. What is meant by a compound gear?

(b) What is meant by a feather, or spline, and how is it used? What is meant by a splined shaft and when are such shafts used?

224.1. (a) Make a sketch of an assembly of a double jaw clutch, two gears, and a compound gear and explain how the mechanism is used.

(b) Where rotation is to be continuously in the same direction, how may the teeth of a jaw clutch be shaped to facilitate engagement and disengagement? What precaution should be observed in operating jaw clutches?

224.2. What decided advantage has the friction over the jaw clutch? Make a sketch of a cone clutch and explain its operation.

224.3. What decided advantage has the friction over the jaw clutch? Make a sketch of a disk clutch and explain its operation.

224.4. What decided advantage has the friction over the jaw clutch? Make a sketch of an expanding ring clutch and explain its operation.

225.1. First state and then, by using a sketch, demonstrate the purpose for which idler gears are used.

226.1. Make a sketch of the spur gear reversing mechanism as used in lathes and explain its operation.

226.2. Make a sketch of a bevel gear reversing mechanism and explain its operation.

227.1. Make a sketch of the kind of change gear mechanism used on the earlier metal working lathes and explain how and for what purpose it was used.

228.1. Make a sketch of a speed change mechanism using a gear cone and a snap key and explain its operation.

228.2. Make a sketch of a speed change mechanism using a cone gear and tumbler gear and explain its operation. Where are such mechanisms used?

229.1. (a) What is meant by the value of a wheel train?
(b) State in words the value of a wheel train.
(c) Suitably number and letter a sketch of a gear train and prove that the value of the train is as stated in (b).

229.2. (a) For the same overall ratio, what advantages has a train over a single pair of wheels?
(b) If in a gear train the diametral pitch of the high revolution pinion is just adequate, why cannot the same pitch be used throughout the train?

229.3. A circular saw is to run at 3000 R.P.M. while the line shaft runs at 150 R.P.M. A 32-inch pulley is used on the line shaft and a 5-inch pulley on the saw arbor. What commercial sizes of pulleys should be used on the countershaft for the angular velocity ratios of the two changes to be approximately equal?
Ans. 7 and 22 in.

229.4. A 10-inch diameter pulley on a drive shaft is connected by a belt to a 36-inch diameter pulley on an intermediate shaft to which is keyed an 18-tooth bevel pinion meshing with a 60-tooth bevel gear on a worm shaft. The worm has a triple thread and drives a 54-tooth worm wheel. Determine the R.P.M. of the worm wheel for 432 R.P.M. of the drive shaft. *Ans.* 2.0.

229.5. The cutting speed of a planer is to be 33 feet per minute. The driving gear engaging with the rack of the planer platen has a pitch diameter of 21 inches and is keyed to a shaft carrying a 40-tooth worm wheel driven by a double-thread worm on a shaft which carries a 24-inch diameter pulley connected by a belt to a 16-inch diameter counter-shaft pulley. At what R.P.M. should the counter-shaft run to give the desired cutting speed? *Ans.* 180 R.P.M.

229.6. The driving wheel of a hoisting mechanism is 36 inches in diameter. Keyed to the same shaft is a 15-tooth pinion meshing with a 36-tooth gear on a worm shaft. A single thread worm drives a 50-tooth worm wheel keyed to the shaft of a winding drum 12 inches in diameter. What is the ratio of the peripheral speeds of the driving wheel to the winding drum? *Ans.* 360.

230.1. (a) When, in a train of external gears, will the first and last shafts rotate in the same direction, when in opposite directions?
(b) If the worm of the train shown in Fig. 488 is left-hand, would shafts *1* and *4* rotate in the same or in opposite directions? Explain in detail how the relative rotation of the shafts was determined.

231.1. (a) Make a sketch of a cone pulley with back gears and explain how the back gears are thrown in and out of mesh and how the different speeds are obtained.
(b) By what two expedients may the number of possible speeds be increased?

231.2. (a) The steps of two equal cone pulleys for a machine tool drive are 12, 9¾, 7½, and 5 inches in diameter. The pinion attached to the cone pulley has 26 teeth and meshes with a back gear having 90 teeth which is rigidly connected to a pinion of 30 teeth meshing with an 86-tooth gear on the spindle. Assuming the

R.P.M. of the counter-shaft to be 120, determine the possible revolutions per minute of the spindle with and without the back gears.

Ans. 5.05, 9.30, 15.72, 29.02, 50, 92.31, 156, and 288 R.P.M.

231.3. (*a*) Determine the progression ratio for the spindle speeds of problem 231.2 to be in geometric progression. Are the actual speeds practically in geometric progression?   *Ans.* $k = 1.782$.

(*b*) Assuming the number of spindle speeds of problem 231.2 to be doubled by using two counter-shaft speeds, determine the second counter-shaft speed for approximate geometric progression starting with 5.05 as the minimum spindle speed.

Ans. $k = 1.335$, R.P.M. 160.

231.4. (*a*) Determine the progression ratio for the spindle speeds of problem 231.2 to be in geometric progression. Are the actual speeds practically in geometric progression?   *Ans.* $k = 1.782$.

(*b*) Twelve spindle speeds in geometric progression from 5.05 to 288 R.P.M. may be obtained by employing a compound sliding gear in the back gear train. Determine the progression ratio, counter-shaft R.P.M., and diameters of the cone pulley steps assuming 12 inches as diameter of largest step.

Ans. $1.444+$, 166, and $6\frac{29}{32}$, $8\frac{19}{32}$, $10\frac{5}{16}$ and 12 in.

232.1. (*a*) What type of speed change drive has displaced the older drive using a cone pulley and back gears? What is the newer drive called?
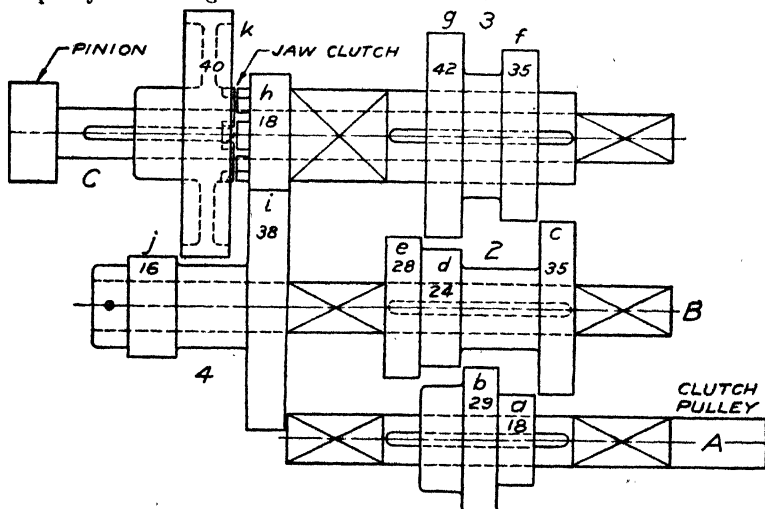
FIG. 232.1

(*b*) The primary speed change drive of a vertical boring mill is shown in Fig. 232.1. Shaft *A* is belt driven at 225 R.P.M. The compound gear *1*, composed of gears *a* and *b*, slides on shaft *A*; the triple gear *2*, composed of gears *c*, *d*, and *e*, is keyed to shaft *B*; the compound gear *3*, composed of gears *f* and *g*, slides on the sleeve to which pinion *h* is attached; compound gear *4*, composed of gears *i* and *j*, is free to rotate on shaft *B*; and gear *k* slides on shaft *C*. In accordance with the numbers of teeth shown, determine the eight possible speeds of shaft *C*.

Ans. 14.62, 21.92, 34.34, 51.51, 77.14, 115.71, 181.25, and 271.87.

232.2. The gears and shafts of a speed change box are shown in Fig. 232.2. Gears *a* and *b* are keyed to shaft *A*; the compound gear *1*, composed of gears *c* and *d*,

and the compound gear 2, composed of gears e and f, slide on shaft B; and gears g and h are keyed to shaft C. In accordance with the numbers of teeth shown, deter-
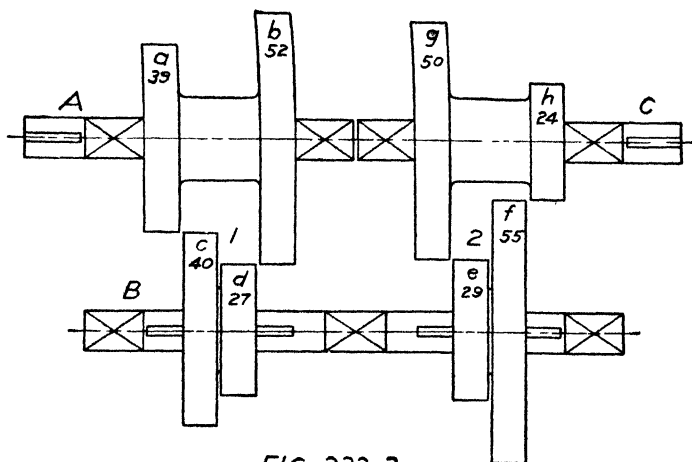
FIG. 232.2

mine the four possible values of the angular velocity ratio of A to C. What would be the progression ratio if the speeds were in geometric progression?

Ans. 0.2265, 0.4475, 0 896, and 1.767, $k = 1.983$.

233.1. The single-thread lead screw of a lathe has 4 threads per inch. The change gears have 24, 30, 36, 42, 48, 48, 54, 60, 66, 69, 72, 78, and 84 teeth. If the stud gear and the lathe spindle rotate at the same angular velocity, what gears should be used on the stud and on the lead screw to cut 3, 4, 9, and 11 threads per inch?

Ans. 48 and 36, 48 and 48, 24 and 54, and 24 and 66.

234.1. An automobile transmission is shown in Fig. 234 1. Explain how the
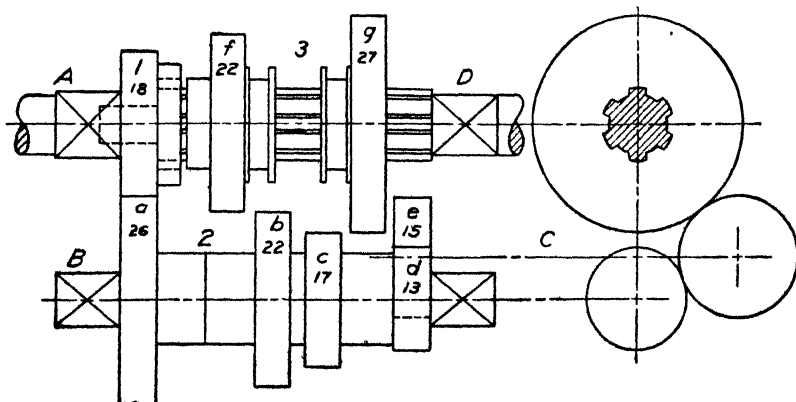
FIG. 234.1

three forward and one reverse speeds are obtained and determine the angular velocity ratios of shaft A to D for these speeds. *Ans.* 1.00, 1.444, 2.294, and 3.00.

235.1. (*a*) What is meant by the term *cyclic train* and how is the term *cyclic* justified?

(*b*) For the cyclic train shown in Fig. 235.1 develop the equation for obtaining the angular velocity ratio of gear *2* to arm *1* each relative to gear *0*. Explain how the directional relation is determined.
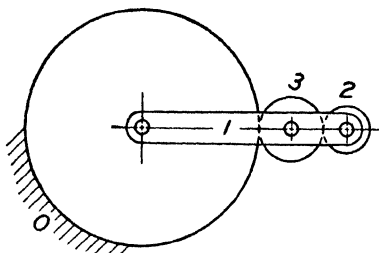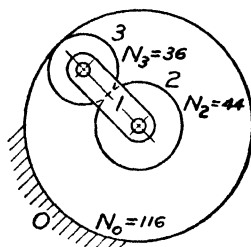
FIG. 235.1   FIG. 235.3

235.2. For Fig. 235.1 assume the respective numbers of teeth of gears *0*, *2*, and *3* to be 75, 15, and 20 and show how the angular velocity ratios ω_{20}/ω_{10} and ω_{30}/ω_{10} can be determined by tabulation. If arm *1* is rotating counter-clockwise, state and explain in what directions gears *2* and *3* are rotating.   *Ans.* -4.0 and $+4.75$.

235.3. For the cyclic train of Fig. 235.3, determine, for the numbers of teeth shown, the numerical values of the angular velocity ratios ω_{20}/ω_{10}, ω_{30}/ω_{10}, ω_{21}/ω_{1}, ω_{20}/ω_{31}, and ω_{23}/ω_{10} and state the directional relations of rotation.

$$Ans.\ +\frac{40}{11},\ -\frac{20}{9},\ -\frac{9}{11},\ -\frac{360}{319},\ +\frac{580}{99}.$$
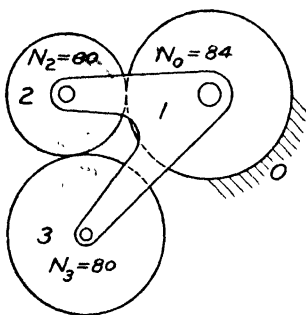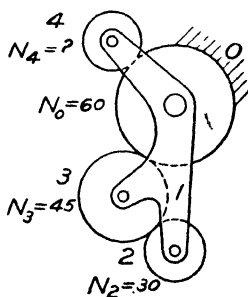
FIG. 235.4   FIG. 235.5

235.4. In Fig. 235.4 arm *1* is keyed to a shaft whose axis coincides with the axis of the fixed gear *0*. Using the numbers of gear teeth given in the figure, determine the number of revolutions of the shaft to cause gear *3* to make one revolution relative to the fixed member and state whether *3* and the shaft rotate in the same or opposite directions.   *Ans.* -20.

235.5. For the cyclic train shown in Fig. 235.5 determine the number of teeth for gear *4* so that gear *4* will make three times as many revolutions as gear *2* relative to the fixed gear but in the opposite direction.   *Ans.* $N_4 = 30$.

236.1. Make a sketch of the mechanism known as Ferguson's Paradox and by means of assumed numbers of teeth for the gears demonstrate what is paradoxical about the mechanism. Explain why involute gears must be used.

237.1. (a) What is meant by a reverted cyclic train and what are the advantages of cyclic trains of such form?

(b) Make a sketch of a reverted cyclic gear train and assign such a number of teeth to each gear that the driving arm will rotate 10,000 times as fast as the driven gear or shaft. Why can not such an extreme ratio be obtained by the use of standard pitch gears?

237.2. In the reverted cylic train of Fig. 237.2 arm *1* rotates with shaft *A*, gears *a* and *b* are keyed to the same stud forming the compound gear *2*, and gear *3* is keyed to shaft *B*. How many revolutions will shaft *B* make for 30 revolutions of shaft *A*. Will *A* and *B* rotate in the same or in opposite directions? *Ans.* −1.347.
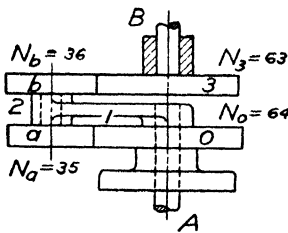
FIG. 237. 2
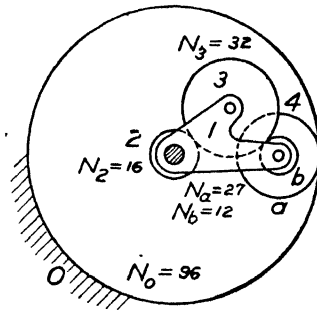
FIG. 237. 3

237.3. In the reverted cyclic train of Fig. 237.3 arm *1*, which is keyed to a rotating shaft, carries gear *3* and the compound gear *4* composed of gears *a* and *b*; gear *2* is keyed to a shaft in line with the shaft to which *1* is keyed. How many revolutions relative to the fixed gear *0* does arm *1* make for 36 revolutions of *2*? Do *1* and *2* rotate in the same or in opposite directions? *Ans.* −21.6.

238.1. Make a two-view sketch of a hand-operated hoist in which a reverted cyclic train containing a stationary internal gear is employed, and explain the operation of the hoist. For what reasons are two rotating arms used instead of one?

238.2. Make a two-view sketch of a reducing gear unit in which a reverted cyclic train containing a stationary internal gear is employed, and explain the operation of the unit. For what reasons are two or more rotating arms used instead of one?

238.3. In the reverted cyclic hoisting mechanism shown in Fig. 238.3 gear *3* is keyed to the drive shaft and the rotating arm *1* is keyed to the same shaft as the hoisting drum. What is the angular velocity ratio of the driving shaft to the drum relative to the fixed internal gear *0*? If the drum is to be turned clockwise in what direction must the driving shaft turn? *Ans.* +12.047.
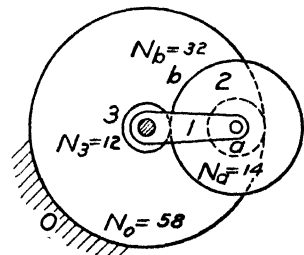
FIG. 238. 3

239.1. In Fig. 239.1 is shown a cyclic boring bar feed train. The bar and gear *1* which is fastened to it rotate about axis *AA*; compound gear *2*, composed of gears *a* and *b*, rotates about axis *BB*; and compound gear *3*, composed of gears *c* and *d*, rotates about axis *AA*. As the bar rotates gear *4* and lead screw *S* are driven by the train of gears *1*, *a*, *b*, *c*, and *d*. Assuming the
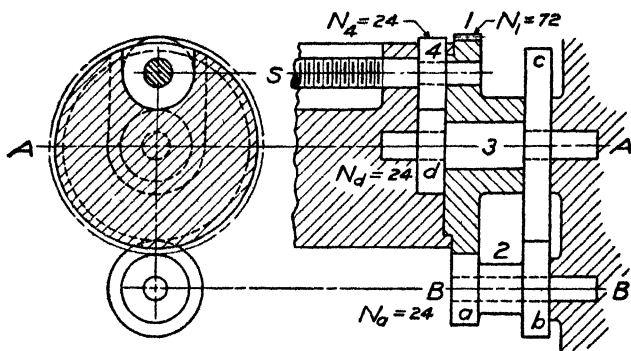
FIG. 239.1

bar to rotate forward and the lead screw to have 4 right-hand threads per inch, determine the number of teeth for gears b and c to cut 8 internal right-hand threads per inch. $Ans.$ $N_b = 32$ and $N_c = 64$.

240.1. (a) Make a sketch of a simple reverted cyclic train of bevel gears, explain its action, and derive an expression for the angular velocity ratio of the driving to the driven shaft.

(b) What advantages have bevel gear over spur gear cyclic trains? Why are the latter more frequently used?

240.2. Explain the action of Humpage's reverted cyclic train of bevel gears, using Fig. 240.2, and determine the angular velocity ratio of shaft A to B for the numbers of teeth given. Do shafts A and B rotate in the same or in opposite directions? $Ans.$ -70.0.
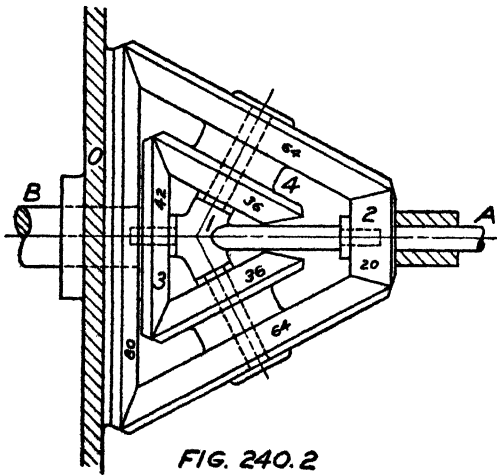
FIG. 240.2

241.1. What is the cyclic equalizing gear used in the axle drives of automobiles called? Make a sketch of such a train and explain its purpose and how it operates. When does it act as a cyclic train and when does it not? What is the relation between the angular velocities of the wheel axles and the ring gear relative to the housing?

242.1. Make a diagrammatic sketch of a cyclic differential hoisting mechanism, explain its operation and derive an expression for the angular velocity ratio of the driving shaft to the hoisting drum. State and explain what kind of gears must be used.

242.2. Make a diagrammatic sketch of a Weston differential chain hoist, explain its operation, and derive an expression for the distance the load is hoisted for one turn of the compound sprocket.

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